Greiner Schäfer

QUANTUM CHROMODYNAMICS

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"QUANTUM CHROMODYNAMICS"

With a Foreword by
D. A. Bromley

With 146 Figures,
and 62 Worked Examples and Exercises

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Foreword to Earlier Series Editions

More than a generation of German-speaking students around the world have worked their way to an understanding and appreciation of the power and beauty of modern theoretical physics – with mathematics, the most fundamental of sciences – using Walter Greiner's textbooks as their guide.

The idea of developing a coherent, complete presentation of an entire field of science in a series of closely related textbooks is not a new one. Many older physicists remember with real pleasure their sense of adventure and discovery as they worked their ways through the classic series by Sommerfeld, by Planck and by Landau and Lifshitz. From the students' viewpoint, there are a great many obvious advantages to be gained through use of consistent notation, logical ordering of topics and coherence of presentation; beyond this, the complete coverage of the science provides a unique opportunity for the author to convey his personal enthusiasm and love for his subject.

The present five-volume set, *Theoretical Physics*, is in fact only that part of the complete set of textbooks developed by Greiner and his students that presents the quantum theory. I have long urged him to make the remaining volumes on classical mechanics and dynamics, on electromagnetism, on nuclear and particle physics, and on special topics available to an English-speaking audience as well, and we can hope for these companion volumes covering all of theoretical physics some time in the future.

What makes Greiner's volumes of particular value to the student and professor alike is their completeness. Greiner avoids the all too common "it follows that . . . " which conceals several pages of mathematical manipulation and confounds the student. He does not hesitate to include experimental data to illuminate or illustrate a theoretical point and these data, like the theoretical content, have been kept up to date and topical through frequent revision and expansion of the lecture notes upon which these volumes are based.

Moreover, Greiner greatly increases the value of his presentation by including something like one hundred completely worked examples in each volume. Nothing is of greater importance to the student than seeing, in detail, how the theoretical concepts and tools under study are applied to actual problems of interest to a working physicist. And, finally, Greiner adds brief biographical sketches to each chapter covering the people responsible for the development of the theoretical ideas and/or the experimental data presented. It was Auguste Comte (1798–1857) in his *Positive Philosophy* who noted, "To understand a science it is necessary to know its history". This is all too often forgotten in modern physics teaching and the bridges that Greiner builds to the pioneering figures of our science upon whose work we build are welcome ones.

Greiner's lectures, which underlie these volumes, are internationally noted for their clarity, their completeness and for the effort that he has devoted to making physics an integral whole; his enthusiasm for his science is contagious and shines through almost every page.

These volumes represent only a part of a unique and Herculean effort to make all of theoretical physics accessible to the interested student. Beyond that, they are of enormous value to the professional physicist and to all others working with quantum phenomena. Again and again the reader will find that, after dipping into a particular volume to review a specific topic, he will end up browsing, caught up by often fascinating new insights and developments with which he had not previously been familiar.

Having used a number of Greiner's volumes in their original German in my teaching and research at Yale, I welcome these new and revised English translations and would recommend them enthusiastically to anyone searching for a coherent overview of physics.

Yale University
New Haven, CT, USA
1989

D. Allan Bromley
Henry Ford II Professor of Physics

Preface

Quantum Chromodynamics (QCD) was formulated more than 25 years ago and has since been a very active field of research. To calculate hadronic properties from QCD, however, is such a formidable task that progress has been slow and many problems are still not solved today. The equations to be solved are highly non-linear, one is dealing with formulas for infinitely many relativistic particles, and one is dealing with a quantum field theory requiring careful renormalization. Thus QCD presents one of the most challenging problems of theoretical physics. To avoid the extreme complexity of real QCD many phenomenological 'QCD-inspired' models have been developed, often with astonishing success.

Presently, however, this situation is changing. The continuous slow progress in computational and analytic techniques has now reached a point where real QCD calculations have become feasible for a rapidly increasing number of problems in hadron physics. Together with the much improved experimental situation this should trigger an enormous boost for hadron physics in the years ahead.

In this book we try to give a pedestrian approach to QCD, stressing the practitioner's point of view. We start with an elementary introduction and review Quantum Electrodynamics (Chap. 2). Chapter 3 analyzes lepton-nucleon scattering, which offers so far the most successful approach to learning about QCD. Also the MIT bag model is introduced, which serves as an illustrative example for a QCD oriented model. In Chap. 4 we then turn to QCD proper and focus on the momentum dependence of its coupling constant, which is the most important property of QCD. We demonstrate explicitly how QCD is renormalized and how the running coupling constant is obtained. Chapter 5 is devoted to the application of QCD to the description of lepton-hadron scattering: the Gribov-Lipatov-Altarelli-Parisi equations are derived, model calculations for the structure functions are presented, the very active and promising field of polarized hadron reactions is sketched, and the fundamental techniques of operator product expansion and renormalization group analyses are discussed. Chapter 6 presents applications to hadron-hadron reactions, most notably, the Drell-Yan process. The techniques discussed allow us to present a currently much debated problem, namely small-x physics. Chapter 7 is then devoted to non-perturbative approaches to QCD, namely lattice gauge calculations and sumrule techniques. Our presentation ends with some remarks on the QCD ground state and its modification at high temperature and/or baryon density (the so-called quark gluon plasma) in Chap. 8.

We have tried to give a pedagogical introduction to the concepts and techniques of QCD. Much had to be left out and we often avoided more formal considerations for the sake of better readability.

In writing this book we profited substantially from a number of existing textbooks, most notably J.J.R. Aitchison and A.J.G. Hey: 'Gauge Theories in Particle Physics', O. Nachtmann: 'Elementarteilchenphysik', B. Müller: 'The Physics of the Quark-Gluon Plasma', P. Becher, M. Boehm and H. Joos: 'Eichtheorien', J. Collins: 'Renormalization', R.D. Field: 'Application of Perturbative QCD', and M. Creutz: 'Quarks, Gluons and Lattices', and several review articles, especially: L.V. Gribov, E.M. Levin and M.G. Ryskin: 'Semihand processes in QCD', Phys. Rep. 100 (1983) 1, L.S. Reinders, H. Rubinstein, S. Yazaki: 'Hadron properties from QCD sumrules', Phys. Rep. 127 (1985) 1.

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Special thanks go to Mrs. A. Steidel who drew the figures, to R. Mattiello who supervised the editing process, and to Mark Seymour who did a great job as copy-editor.

We hope that our readers will enjoy this book.

Frankfurt am Main,
September 1994

Walter Greiner
Andreas Schäfer

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1. The Introduction of Quarks

1.1 The Hadron Spectrum

About 70 years ago, only a small number of "elementary particles", thought to be the basic building blocks of matter, were known: the proton, the electron, and the photon as the quantum of radiation. All these particles are stable (the neutron is stable only in nuclear matter, the free neutron decays by beta decay: $\mathrm{n} \rightarrow \mathrm{p}+\mathrm{e}^{-}+\bar{\nu}$). Owing to the availability of large accelerators, this picture of a few elementary particles has profoundly changed: today, the standard reference "Review of Particle Properties"[1] lists more than 100 particles. The number is still growing as the energies and luminosities of accelerators are increased.

The symmetries known from classical and quantum mechanics can be utilized to classify the "elementary-particle zoo". The simplest baryons are p and n; the simplest leptons e^{-} and μ^{-}. Obviously there are many other particles that must be classified as baryons or leptons.

The symmetries are linked to conserved quantum numbers such as the baryon number B, isospin T with z-component T_3, strangeness S, hypercharge $Y = B+S$, charge $Q = T_3+Y/2$, spin I with z-component I_z, parity π, and charge conjugation parity π_{c}. Conservation laws for such quantum numbers manifest themselves by the absence of certain processes. For example, the hydrogen atom does not decay into two photons: $\mathrm{e}^{-} + \mathrm{p} \rightarrow \gamma + \gamma$, although this process is not forbidden either by energy–momentum conservation or by charge conservation. Since our world is built mainly out of hydrogen, we know from our existence that there must be at least one other conservation law that is as fundamental as charge conservation. The nonexistence of the decays $\mathrm{n} \rightarrow \mathrm{p} + \mathrm{e}^{-}$ and $\mathrm{n} \rightarrow \gamma + \gamma$ also indicates the presence of a new quantum number. The proton and neutron are given a baryonic charge $B = 1$, the electron $B = 0$. Similarly the electron is assigned leptonic charge $L = 1$, the nucleons $L = 0$. From the principle of simplicity it appears very unsatisfactory to regard all observed particles as elementary. To give an impression of the huge number of hadrons known today, we have collected together the baryon resonances in Fig. 1.1. The data are taken from the "Review of Particle Properties". Particles for which there is only weak evidence or for which the spin I and internal parity P have not been determined have been left out. Note that each state represents a full multiplet. The number of members in a multiplet is $N = 2T + I$ with isospin T. Thus the 13 Δ resonances shown correspond to a total of 52 different baryons.

When looking at these particle spectra, one immediately recognizes the similarity to atomic or nuclear spectra. One would like, for example, to classify the

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[1] See *Review of Particle Properties*: Phys. Rev. **D 45** (1992).

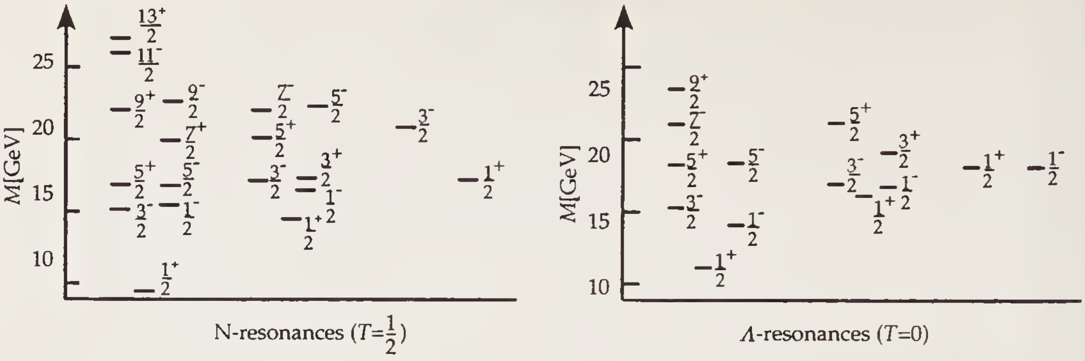


Fig. 1.1. The mass spectra of baryons. Plotted are the average masses of the multiplets. For example, the state $N_{5/2^+}$ at 1.68 MeV stands for two particles, one protonlike and one neutronlike, both with spin $\frac{5}{2}$ and positive internal parity. The figure contains 140 particle states in total

nucleon resonances (N resonances) in analogy to the levels of a hydrogen atom. The $\frac{1}{2}^+$ ground state (i.e., the ordinary proton and neutron) would then correspond to the $1s_{\frac{1}{2}}$ state, the states $\frac{3}{2}^-$, $\frac{1}{2}^-$, and $\frac{1}{2}^+$ at approximately 1.5 GeV to the hydrogen levels $2p_{\frac{3}{2}}$, $2p_{\frac{1}{2}}$, and $2s_{\frac{1}{2}}$, the states $\frac{5}{2}^+$, $\frac{3}{2}^+$, $\frac{3}{2}^-$, $\frac{1}{2}^-$, $\frac{1}{2}^+$ to the sublevels of the third main shell $3d_{\frac{5}{2}}$, $3d_{\frac{3}{2}}$, $3p_{\frac{3}{2}}$, $3p_{\frac{1}{2}}$, $3s_{\frac{1}{2}}$, and so on.

Although one should not take this analogy too seriously, it clearly shows that a model, in which the baryons are built from spin-$\frac{1}{2}$ particles almost automatically leads to the states depicted in Fig. 1.1. The quality of any such model is measured by its ability to predict the correct energies. We shall discuss specific models in Sect. 3.1.

We therefore interpret the particle spectra in Fig. 1.1 as strong evidence that the baryons are composed of several more fundamental particles and that most of the observable baryon resonances are excitations of a few ground states. In this way the excited states $\frac{3}{2}^-$ and $\frac{1}{2}^-$ are reached from the nucleon ground state N(938 MeV) $\frac{1}{2}^+$ by increasing the angular momentum of one postulated component particle by one. ($\frac{1}{2}^+$ can be coupled with 1^- to give $\frac{1}{2}^-$ or $\frac{3}{2}^-$.) As the energy of the baryon resonances increases with higher spin (i.e., total angular momentum of all component particles), one can deduce that all relative orbital angular momenta vanish in the ground states.

To investigate this idea further, one must solve a purely combinatorial problem: How many component particles (called *quarks* in the following) are needed, and what properties are required for them to correctly describe the ground states of the hadron spectrum? It turns out that the existence of several quarks must be postulated. The quantum numbers given in Table 1.1 must be given to them.

The three light quarks u, d, s can be identified with the three states in the fundamental representation of SU(3). This is initially a purely formal act. It gains importance only as one shows that the *branching ratios* of particle reactions and the *mass differences* between stable baryons show –at least approximately– the same symmetries. This means that the so-called *flavor* SU(3) can be interpreted as the symmetry group of a more fundamental interaction.

Hadrons are therefore constructed as flavor SU(3) states. As in addition the spin of the quarks must be taken into account, the total symmetry group becomes SU(3)×SU(2). As an example we give the decomposition of the neutron into quark states:

Table 1.1. Quarkcharge (Q), isospin (T, T_3), and strangeness (S)

	Q	T	T_3	S
u	2/3	1/2	1/2	0
d	−1/3	1/2	−1/2	0
s	−1/3	0	0	−1
c	2/3	0	0	0
t	2/3	0	0	0
b	−1/3	0	0	0

$$\begin{aligned}|\mathrm{n}\uparrow\rangle = \frac{1}{\sqrt{18}}\Big(&2|\mathrm{d}\uparrow\rangle|\mathrm{d}\uparrow\rangle|\mathrm{u}\downarrow\rangle - |\mathrm{d}\uparrow\rangle|\mathrm{d}\downarrow\rangle|\mathrm{u}\uparrow\rangle - |\mathrm{d}\downarrow\rangle|\mathrm{d}\uparrow\rangle|\mathrm{u}\uparrow\rangle\\ &- |\mathrm{d}\uparrow\rangle|\mathrm{u}\uparrow\rangle|\mathrm{d}\downarrow\rangle + 2|\mathrm{d}\uparrow\rangle|\mathrm{u}\downarrow\rangle|\mathrm{d}\uparrow\rangle - |\mathrm{d}\downarrow\rangle|\mathrm{u}\uparrow\rangle|\mathrm{d}\uparrow\rangle\\ &- |\mathrm{u}\uparrow\rangle|\mathrm{d}\uparrow\rangle|\mathrm{d}\downarrow\rangle - |\mathrm{u}\uparrow\rangle|\mathrm{d}\downarrow\rangle|\mathrm{d}\uparrow\rangle + 2|\mathrm{u}\downarrow\rangle|\mathrm{d}\uparrow\rangle|\mathrm{d}\uparrow\rangle\Big) \quad . \end{aligned} \tag{1.1}$$

Particularly interesting for the topic of this volume are the corresponding decompositions of the states Ω^-, Δ^{++}, and Δ^-:

$$\begin{aligned}|\Omega^-\rangle &= |\mathrm{s}\uparrow\rangle|\mathrm{s}\uparrow\rangle|\mathrm{s}\uparrow\rangle \quad ,\\ |\Delta^{++}\rangle &= |\mathrm{u}\uparrow\rangle|\mathrm{u}\uparrow\rangle|\mathrm{u}\uparrow\rangle \quad ,\\ |\Delta^-\rangle &= |\mathrm{d}\uparrow\rangle|\mathrm{d}\uparrow\rangle|\mathrm{d}\uparrow\rangle \quad .\end{aligned} \tag{1.2}$$

To obtain the spin quantum numbers of hadrons, one must assume that the quarks have spin $\frac{1}{2}$. This poses a problem: spin-$\frac{1}{2}$ particles should obey Fermi statistics, i.e., no two quarks can occupy the same state. So the three quarks in Ω^-, Δ^{++}, and Δ^- must differ in at least one quantum number, as we shall discuss in Chap. 4. Before proceeding to the composition of baryons from quarks, we shall first repeat the most important properties of the symmetry groups SU(2) and SU(3).

SU(2) and SU(3) are special cases of the group SU(N), the special unitary group in N dimensions. Any unitary square matrix $\hat{U}$ with N rows and N columns can be written as

$$\hat{U} = \mathrm{e}^{\mathrm{i}\hat{H}} \quad , \tag{1.3}$$

where $\hat{H}$ is a Hermitian matrix. The matrices $\hat{U}$ form the group SU(N) of unitary matrices in N dimensions. $\hat{H}$ is Hermitian, i.e.,

$$\hat{H}^*_{ij} = \hat{H}_{ji} \quad . \tag{1.4}$$

Of the N^2 complex parameters (elements of the matrices), N^2 real parameters for $\hat{H}$ and hence for $\hat{U}$ remain owing to the auxiliary conditions (1.4). Since $\hat{U}$ is unitary, i.e. $\hat{U}^\dagger\hat{U} = 1$, det $\hat{U}^\dagger$det $\hat{U}$ = (det $\hat{U})^*$ det $\hat{U} = 1$ and thus

$$\left|\det \hat{U}\right| = 1 \quad . \tag{1.5}$$

Owing to (1.4), $\mathrm{tr}\{\hat{H}\} = \alpha$ (α real) and

$$\det \hat{U} = \det\left(\mathrm{e}^{\mathrm{i}\hat{H}}\right) = \mathrm{e}^{\mathrm{i}\mathrm{tr}\hat{H}} = \mathrm{e}^{\mathrm{i}\alpha} \quad . \tag{1.6}$$

If we additionally demand the condition

$$\det\hat{U} = +1 \quad , \tag{1.7}$$

i.e., $\alpha = 0 \bmod 2\pi$, only $N^2 - 1$ parameters remain. This group is called the special unitary group in N dimensions (SU(N)).

Let us now consider a group element $\hat{U}$ of U(N) as a function of N^2 parameters ϕ_μ ($\mu = 1, \dots, n$). To this end, we write (1.3) as

$$\hat{U}(\phi_1, \dots, \phi_n) = \exp\left(-\mathrm{i}\sum_\mu \phi_\mu \hat{L}_\mu\right) \quad , \tag{1.8}$$

where $\hat{L}_\mu$ are for the time being unknown operators:

$$-\mathrm{i}\hat{L}_\mu = \left.\frac{\partial \hat{U}(\phi)}{\partial \phi_\mu}\right|_{\phi=0} \tag{1.9}$$

($\phi = (\phi_1, \dots, \phi_n)$). For small $\phi_\mu(\delta\phi_\mu)$ we can expand $\hat{U}$ in a series ($\mathbb{1}$ is the unit matrix):

$$\hat{U}(\phi) \approx \mathbb{1} - \mathrm{i}\sum_{\mu=1}^{n} \delta\phi_\mu \hat{L}_\mu - \frac{1}{2}\sum_{\mu,\nu} \delta\phi_\mu \delta\phi_\nu \hat{L}_\mu \hat{L}_\mu + \dots \quad . \tag{1.10}$$

Boundary conditions (1.4) and (1.5) imply after some calculation that the operators $\hat{L}_i$ must satisfy the commutation relations

$$\left[\hat{L}_i, \hat{L}_j\right] = c_{ijk}\hat{L}_k \quad . \tag{1.11}$$

Equation (1.11) defines an algebra, the Lie–Algebra of the group U(N).

The operators $\hat{L}_i$ generate by means of (1.10) the group and are thus called *generators*. Obviously there are as many generators as the group has parameters, i.e., the group U(N) has N^2 generators and the group SU(N) $N^2 - 1$. The quantities c_{ijk} are called *structure constants* of the group. They contain all the information about the group. In the Lie algebra of the group (i.e., the $\hat{L}_k$), there is a maximal number R of commutating elements $\hat{L}_i$ ($i = 1. \dots, R$)

$$\left[\hat{L}_i, \hat{L}_j\right] = 0 \quad (i = 1, \dots, R) \quad . \tag{1.12}$$

R is called the *rank* of the group. The eigenvalues of the $\hat{L}_i$ are, as we shall see, used to classify elementary-particle spectra. We shall now discuss the concepts introduced here using the actual examples of the spin and isospin group SU(2) and the group SU(3).

SU(2). U(2) is the group of lineary independent Hermitian 2 × 2 matrices. A well-known representation of it is given by the Pauli matrices and the unit matrix

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \hat{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{1.13}$$

These span the space of Hermitian 2×2 matrices, i.e., they are linearly independent. SU(2) has only three generators; the unit matrix is not used. From (1.3) we can write a general group element of the group SU(2) as

$$\hat{U}(\phi) = \exp\left(-\mathrm{i} \sum_{i=1}^{3} \phi_i \hat{\sigma}_i\right) \tag{1.14}$$

(or, using the summation convention, $\exp(-\mathrm{i}\phi_i\hat{\sigma}_i)$). Here $\phi = (\phi_1, \phi_2, \phi_3)$ is a shorthand for the parameter of the transformation. The Pauli matrices satisfy the commutation relations

$$\left[\hat{\sigma}_i, \hat{\sigma}_j\right] = 2\mathrm{i}\varepsilon_{ijk}\hat{\sigma}_k \quad , \tag{1.15}$$

with

$$\varepsilon_{ijk} = \begin{cases} 0 & \text{for two equal indices,} \\ 1 & \text{for even permutations of the indices,} \\ -1 & \text{for odd permutations of the indices.} \end{cases}$$

Usually, instead of $\hat{\sigma}_i$, the $\hat{S}_i = \frac{1}{2}\,\hat{\sigma}_i$ are used as generators, i.e.

$$\left[\hat{S}_i, \hat{S}_j\right] = \mathrm{i}\varepsilon_{ijk}\hat{S}_k \quad .$$

According to (1.11), $\mathrm{i}\epsilon_{ijk}$ are the structure constants of SU(2). Equation (1.15) shows that no generator commutes with any other, i.e., the rank of SU(2) is 1. According to the *Racah theorem*, the rank of a group is equal to the number of Casimir operators (i.e., those operators are polynomials in the generators and commute with all generators). Thus there is one Casimir operator for SU(2), namely the square of the well-known angular momentum (spin) operator:

$$\hat{C}_{\mathrm{SU(2)}} = \sum_{i=1}^{3} \hat{S}_i^2 \quad . \tag{1.16}$$

The representation of SU(2) given in (1.13) (and generally of SU(N)) by 2×2 matrices (generally $N \times N$ matrices) is called the fundamental representation of SU(2) (SU(N)). It is the smallest nontrivial representation of SU(2) (SU(N)). It is a 2×2 representation for SU(2), a 3×3 representation for SU(3), and so on. From Schur's first lemma the Casimir operators in the fundamental representation are multiples of the unit matrix (see Exercise 1.1):

$$\hat{C}_{\mathrm{SU(2)}} = \sum_{i=1}^{3} \left(\frac{\hat{\sigma}_i}{2}\right)^2 = \frac{3}{4}\mathbb{1} \,. \tag{1.17}$$

SU(3). The special unitary group in three dimensions has $3^2 - 1 = 8$ generators. In the fundamental representation they can be expressed by the Gell-Mann matrices $\hat{\lambda}_1, \ldots, \hat{\lambda}_8$:

$$\hat{\lambda}_1=\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}\ ,\quad \hat{\lambda}_2=\begin{pmatrix}0&-\mathrm{i}&0\\\mathrm{i}&0&0\\0&0&0\end{pmatrix}\ ,\quad \hat{\lambda}_3=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix}\ ,$$
$$\hat{\lambda}_4=\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix}\ ,\quad \hat{\lambda}_5=\begin{pmatrix}0&0&-\mathrm{i}\\0&0&0\\\mathrm{i}&0&0\end{pmatrix}\ ,\quad \hat{\lambda}_6=\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix}\ , \tag{1.18}$$
$$\hat{\lambda}_7=\begin{pmatrix}0&0&0\\0&0&-\mathrm{i}\\0&\mathrm{i}&0\end{pmatrix}\ ,\quad \hat{\lambda}_8=\frac{1}{\sqrt{3}}\begin{pmatrix}1&0&0\\0&1&0\\0&0&-2\end{pmatrix}\ .$$

The Gell-Mann matrices are Hermitian:

$$\hat{\lambda}_i^\dagger=\hat{\lambda}_i \tag{1.19}$$

and their trace vanishes:

$$\mathrm{tr}\left\{\hat{\lambda}_i\right\}=0\ . \tag{1.20}$$

They define the Lie algebra of SU(3) by the commutation relations

$$\left[\hat{\lambda}_i,\hat{\lambda}_j\right]=2\mathrm{i}f_{ijk}\hat{\lambda}_k\ , \tag{1.21}$$

where the structure constants f_{ijk} are, like the ε_{ijk} in SU(2), completely antisymmetric, i.e.,

$$f_{ijk}=-f_{jik}=-f_{ikj}\quad . \tag{1.22}$$

The anticommutation relations of the $\hat{\lambda}_i$ are written as

$$\left\{\hat{\lambda}_i,\hat{\lambda}_j\right\}=\frac{4}{3}\delta_{ij}\,\mathbb{1}+2d_{ijk}\hat{\lambda}_k\quad . \tag{1.23}$$

The constants d_{ijk} are completely symmetric:

$$d_{ijk}=d_{jik}=d_{ikj}\quad . \tag{1.24}$$

The nonvanishing structure constants are given in Table 1.2.

As in SU(2), generators $\hat{F}_i=\frac{1}{2}\hat{\lambda}_i$ ("hyperspin") are used instead of the $\hat{\lambda}_i$ with the commutation relations

$$\left[\hat{F}_i,\hat{F}_j\right]=\mathrm{i}f_{ijk}\hat{F}_k\ . \tag{1.25}$$

One can easily check that among the $\hat{F}_i$ only the commutators $\left[\hat{F}_1,\hat{F}_8\right]=\left[\hat{F}_2,\hat{F}_8\right]=\left[\hat{F}_3,\hat{F}_8\right]=0$ vanish. As the $\hat{F}_i, i=1,2,3$, do not commute with each other, there are at most two commuting generators, i.e., SU(3) has rank two (in general SU(N) has rank $N-1$), and hence two Casimir operators, one of which is simply

$$\hat{C}_1=\sum_{i=1}^{8}\hat{F}_i^2=-\frac{2\mathrm{i}}{3}\sum_{i,j,k}f_{ijk}\,\hat{F}_i\,\hat{F}_j\,\hat{F}_k\quad . \tag{1.26}$$

In the fundamental representation

Table 1.2. The nonvanishing, completely antisymmetric structure constants f_{ijk} and the symmetric constants d_{ijk}

ijk	f_{ijk}
123	1
147	$1/2$
156	$-1/2$
246	$1/2$
257	$1/2$
345	$1/2$
367	$-1/2$
458	$\frac{\sqrt{3}}{2}$
678	$\frac{\sqrt{3}}{2}$

ijk	d_{ijk}
118	$\frac{1}{\sqrt{3}}$
146	$1/2$
157	$1/2$
228	$\frac{1}{\sqrt{3}}$
247	$-1/2$
256	$1/2$
338	$\frac{1}{\sqrt{3}}$
344	$1/2$
355	$1/2$
366	$-1/2$
377	$-1/2$
448	$-\frac{1}{2\sqrt{3}}$
558	$-\frac{1}{2\sqrt{3}}$
668	$-\frac{1}{2\sqrt{3}}$
778	$-\frac{1}{2\sqrt{3}}$
888	$-\frac{1}{\sqrt{3}}$

$$\left(\hat{C}_1\right)_{j\ell} = \frac{1}{4}\sum_{i=1}^{8}\sum_{k=1}^{3}\left(\hat{\lambda}_i\right)_{jk}\left(\hat{\lambda}_i\right)_{k\ell} = \frac{4}{3}\delta_{j\ell} \quad . \tag{1.27}$$

From the structure constants f_{ijk}, new matrices $\hat{U}_i$ can be constructed according to

$$\left(\hat{U}_i\right)_{jk} = -\mathrm{i} f_{ijk} \quad , \tag{1.28}$$

which also satisfy the commutation relations

$$\left[\hat{U}_i, \hat{U}_j\right] = \mathrm{i} f_{ijk}\hat{U}_k \quad . \tag{1.29}$$

This representation of the Lie algebra of SU(3) is called *adjoint* (or *regular*). In it (see Exercise 1.2)

$$\begin{aligned}(\hat{C}_l)_{kl} &= \sum_{i=1}^{8}(\hat{U}_i^2)_{kl} = \sum_{i,j}(\hat{U}_i)_{kj}(\hat{U}_i)_{jl} \\ &= -\sum_{i}\sum_{j} f_{ikj} f_{ijl} = \sum_{i,j} f_{ijk} f_{ijl} \\ &= 3\delta_{kl} \quad .\end{aligned} \tag{1.30}$$

A form of the complete SU(3) group element according to (1.3) is ($\hat{U}(0)$ designates in contrast to $\hat{U}_i$ the transformation matrix from (1.3))

$$\hat{U}(\theta) = \mathrm{e}^{-\mathrm{i}\theta\cdot\hat{F}} \quad , \tag{1.31}$$

where $\hat{\boldsymbol{F}}$ is the vector of eight generators and $\boldsymbol{\theta}$ the vector of eight parameters.

After this short digression the group structure of SU(2) and SU(3), we return to the classification of elementary particles. As indicated above, the eigenvalues of commuting generators of the group serve to classify the hadrons. For SU(3) there is only one such operator among the $\hat{T}_i$ ($i = 1, 2, 3$) that is usually chosen to be $\hat{T}_3$ (the z component). The structure of SU(2) multiplets is thus onedimensional and characterized by a number T_3. In the framework of QCD the most important application of SU(2) is as the isospin group (with generators $\hat{T}_i$) and as the angular momentum group with, for examples, the spin operator $\hat{S}_i$. The small mass difference between neutron and proton (0.14 % of the total mass) leads to the thought that both can be treated as states of a single particle, the nucleon. According to the matrix representation

$$\hat{T}_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{1}{2}\hat{\tau}_3 \quad , \tag{1.32}$$

one assigns the isospin vector $\Psi_p = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ to the proton and $\Psi_n = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ to the neutron, so that the isospin eigenvalues $T_3 = \pm\frac{1}{2}$ are assigned to the nucleons:

$$\hat{T}_3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = +\frac{1}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \tag{1.33}$$

$$\hat{T}_3 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad . \tag{1.34}$$

Analogously one introduces

$$\hat{\tau}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \hat{\tau}_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \tag{1.35}$$

such that the

$$\hat{T}_k = \frac{1}{2}\hat{\tau}_k \qquad (k = 1, 2, 3) \tag{1.36}$$

satisfy the same commutation relations as the spin operators. One can check by direct calculation that raising and lowering operators can be constructed from the $\hat{\tau}_i$:

$$\begin{aligned} \hat{\tau}_+ &= \frac{1}{2}(\hat{\tau}_1 + \mathrm{i}\hat{\tau}_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad , \\ \hat{\tau}_- &= \frac{1}{2}(\hat{\tau}_1 - \mathrm{i}\hat{\tau}_2) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad . \end{aligned} \tag{1.37}$$

They have the following well-known action on nucleon states:

$$\begin{aligned} \hat{\tau}_+\Psi_p &= 0 \quad , & \hat{\tau}_+\Psi_n &= \Psi_p \quad , \\ \hat{\tau}_-\Psi_p &= \Psi_n \quad , & \hat{\tau}_-\Psi_n &= 0 \quad , \end{aligned} \tag{1.38}$$

i.e., the operators change nucleon states into each other (they are also called *ladder operators*). From (1.3), we can give the general transformation in the abstract three-dimensional isospin space

$$\hat{U}(\phi) = \hat{U}(\phi_1, \phi_2, \phi_3) = \mathrm{e}^{-\mathrm{i}\phi_\mu \hat{T}_\mu} \quad , \tag{1.39}$$

where the ϕ_μ represent the rotation angles in isospin space. The Casimir operator of isospin SU(2) is

$$\hat{T}^2 = \hat{T}_1^2 + \hat{T}_2^2 + \hat{T}_3^2 \quad . \tag{1.40}$$

We can now describe each particle state by an abstract vector $|TT_3\rangle$ (analogously to the spin, as the isospin SU(2) is isomorphic to the spin SU(2)), where the following relations hold:

$$\hat{T}^2|TT_3\rangle = T(T+1)|TT_3\rangle \quad , \tag{1.41}$$

$$\hat{T}_3|TT_3\rangle = T_3|TT_3\rangle \quad . \tag{1.42}$$

Thus the nucleons represent an isodoublet with $T = \frac{1}{2}$ and $T_3 = \pm\frac{1}{2}$. The pions $(\pi^\pm, \pi^0)$ (masses $m(\pi^0) = 135$ MeVc^2 and $m(\pi^\pm) = 139.6$ MeV$/c^2$, i.e. a mass difference of 4.6 MeV$/c^2$) constitute an isotriplet with $T = 1$ and $T_3 = -1, 0, 1$. Obviously there is a relation between isospin and the electric charge of a particle. For the nucleons the charge operator is immediately obvious:

$$\hat{Q} = \hat{T}_3 + \frac{1}{2}\mathbb{1} \tag{1.43}$$

in units of the elementary charge e, while one finds in a similarly simple way for the pions

$$\hat{Q} = \hat{T}_3 \quad . \tag{1.44}$$

To unify both relations, one can introduce an additional quantum number Y (the so-called hypercharge) and describe any state by T_3 and Y:

$$\hat{Y}|YT_3\rangle = Y|YT_3\rangle \quad , \tag{1.45}$$

$$\hat{T}_3|YT_3\rangle = T_3|YT_3\rangle \quad . \tag{1.46}$$

In this way the nucleon is assigned $Y = 1$ and the pion $Y = 0$, so that (1.42) and (1.43) can be written as

$$\hat{Q} = \frac{1}{2}\hat{Y} + \hat{T}_3 \quad . \tag{1.47}$$

Relation (1.45) is the *Gell-Mann–Nishijima relation*. The hypercharge characterizes the center of charge. It is often customary to express Y by the strangeness S and the baryon number B using $Y = B + S$. Here $B = +1$ for all baryons, $B = -1$ for antibaryons, and $B = 0$ otherwise (in particular for mesons). Thus $Y = S$ for mesons. To classify elementary particles in the framework of SU(3), it is customary to display them in a T_3–Y diagram. The baryons with spin $\frac{1}{2}$ constitute an octet in this diagram (see Fig. 1.2).

The spectrum of antiparticles is obtained from this by reflecting the expression with respect to the Y and T_3 axes. The heavier baryons and the mesons can be

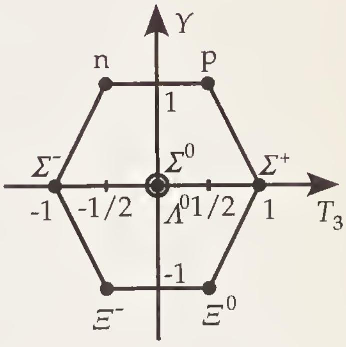


Fig. 1.2. An octet of spin-$\frac{1}{2}$ baryons

classified analogously. We introduced the hypercharge by means of the charge and have thus added another quantum number. SU(2) has rank 1, i.e., it provides only one such quantum number. SU(3), however, has rank 2 and thus two commuting generators, $\hat{F}_3$ and $\hat{F}_8$. We can therefore make the identification $\hat{T}_3 = \hat{F}_3$ and $\hat{Y} = 2/\sqrt{3}\hat{F}_8$ and interpret the multiplets as SU(3) multiplets. The SU(3)-multiplet classification was introduced by M. Gell-Mann and is initially purely schematic. There are no small nontrivial representations among these multiplets (with the exception of the singlet, interpreted as the Λ^* hyperon with mass 1405 MeV/c^2 and spin $\frac{1}{2}$). The smallest nontrivial representation of SU(3) is the triplet. This reasoning led Gell-Mann and others to the assumption that physical particles are connected to this triplet, the quarks (from James Joyce's Finnegan's Wake: "Three quarks for Muster Mark"). Today we know that there are five quarks. They are called up, down, strange, charm, and bottom quarks. A sixth quark, the top, most probably exists but it has not yet been possible to detect it directly owing to its large mass. The different kinds of quarks are called "flavors". The original SU(3) flavor symmetry is therefore only important for low energies, where c, b, and t quarks do not play a role owing to their large mass. It is, also still relevant for hadronic ground-state properties.

All particles physically observed at this time are combinations of three quarks (baryons) or a quark and an antiquark (mesons) plus, in each case, an arbitrary number of quark–antiquark pairs and gluons. This requires that quarks have

1. baryon number $\frac{1}{3}$
2. electric charges in multiples of $\pm\frac{1}{3}$.

Uneven multiples of charge $\frac{1}{3}$ have never been conclusively observed in nature, and there therefore seems to exist some principle assuring that quarks can exist in bound states in elementary particles but never free. This is the problem of quark confinement, which we shall discuss later. Up to now, we have considered the SU(3) symmetry connected with the flavor of elementary particles. Until the early 1970s it was commonly believed that this symmetry is the basis of the strong interaction. Today the true strong interaction is widely acknowledged to be connected with another quark quantum number, the *color*. The dynamics of color (chromodynamics) determines the interaction of the quarks (which is, as we shall see, flavor-blind).

In the following chapter quantum electrodynamics is reviewed. Readers familiar with it are advised to continue on page 69 with Chap. 3.

EXERCISE

1.1 The Fundamental Representation of a Lie Algebra

Problem. (a) What are the fundamental representations of the group SU(N)? (b) Show that according to Schur's lemma the Casimir operators in these fundamental representations are multiples of the unit matrix.

Solution. (a) The fundamental representations are those nontrivial representations of a group that have the lowest dimension. All higher-dimensional representations can be constructed from them. We shall demonstrate this using the special unitary groups SU(N).

SU(2). As we have learned, its representation is characterized by the angular–momentum quantum number $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$, and states are classified by $(j) \equiv |jm\rangle$, $m = -j, \ldots, -j$. The scalar representation is $j = 0$. The lowest-dimensional representation with $j \neq 0$ would then be $j = \frac{1}{2}$. From it we can construct all others by simply coupling one to another

$$\left[\frac{1}{2}\right] \times \left[\frac{1}{2}\right] = \left[\frac{1}{2}\right]^2 = [1] + [0] \quad , \tag{1a}$$

$$\left[\frac{1}{2}\right] \times \left[\frac{1}{2}\right] \times \left[\frac{1}{2}\right] = \left[\frac{1}{2}\right]^3 = \left[\frac{3}{2}\right] + \left[\frac{1}{2}\right] + \left[\frac{1}{2}\right] \; . \tag{1b}$$

"×" indicates the direct product, "+" the direct sum. The first two $j = \frac{1}{2}$ representations can be coupled to $j = 0, 1$. Adding another $j = \frac{1}{2}$, it couples with $j = 1$ to give $j = \frac{3}{2}, \frac{1}{2}$ and with $j = 0$ to give only $j = \frac{1}{2}$. In total, $\left[\frac{1}{2}\right]^3$ contains the representations $\left[\frac{3}{2}\right]$, $\left[\frac{1}{2}\right]$, $\left[\frac{1}{2}\right]$. Figure 1.3 depicts this angular momentum coupling graphically. It must be noted that a representation can appear more than once, e.g., $\left[\frac{1}{2}\right]$ appears twice in $\left[\frac{1}{2}\right]^3$ and $[1]$ thrice in $\left[\frac{1}{2}\right]^4$.

In the next example, an alternative representation according to "maximal weight" is of interest. For this, all operators in the algebra that commute with each other are considered (*Cartan subalgebra*). Their eigenvalues classify states in a representation. In the case of SU(2) there is only one operator commuting with itself. This can be chosen to be any of the j_i, usualy one takes j_3, the third component of the angular momentum vector. Its eigenvalues are $m = -j, \ldots, +j$. The "maximal weight" is $m_{\max} = j$. In direct products $\left[\frac{1}{2}\right]^n$ the maximal weight is $m_{\max} = \frac{n}{2}$, which is the "maximal weight" of the "straight coupling" (see Fig. 1.3).

SU(3). Its representations (multiplets) are classified by the eigenvalues of the Casimir operators. These give us, in the case of SU(3), two numbers $[p, q]$. These are in turn connected to the rank of the algebra, i.e., the number of commuting generators in the algebra. In general, the representations of SU(N) are characterized by $N - 1$ numbers. Another possibility would be to classify representations by their "maximal weight". As is known, each state in a representation of SU(3) (a multiplet) is labeled by the eigenvalues of the third component of isospin $\hat{T}_3$ and hypercharge $\hat{Y}$. The weight is given by the tuple (T_3, Y). A weight (T_3, Y) is higher than (T_3', Y') if

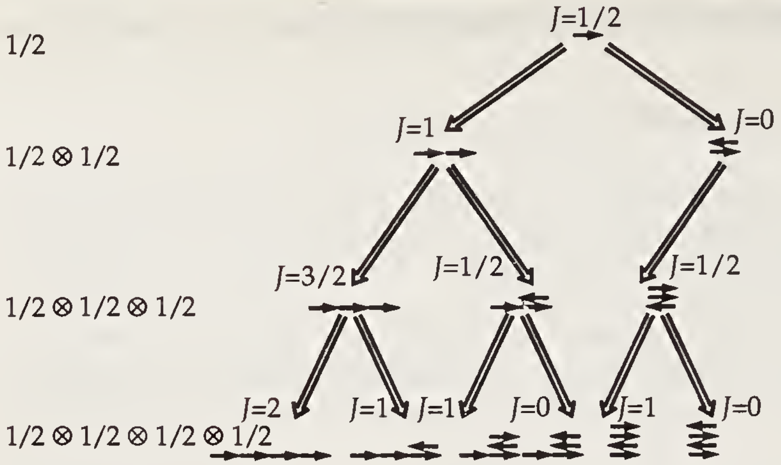


Fig. 1.3. Multiple coupling of spins $\frac{1}{2}$ to various total spins J

$$T_3 > T_3' \quad \text{or} \quad T_3 = T_3' \quad \text{and} \quad Y > Y' \quad . \tag{2}$$

The highest weight in a representation is given by the maximal value of T_3, and, if there is more than one, by the maximal value of Y. This is demonstrated by the following examples:

(1) $[p, q] = [1, 0]$.

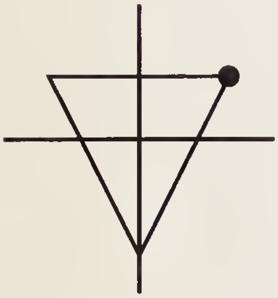


Fig. 1.4. The Quark weight diagram

This is the representation whose "weight diagram" is depicted in Fig. 1.4. The states carry the weights

$$(T_3, Y) = \left(\frac{1}{2}, \frac{1}{3}\right), \left(-\frac{1}{2}, \frac{1}{3}\right), \left(0, -\frac{2}{3}\right) \quad .$$

The tuple $\left(\frac{1}{2}, \frac{1}{3}\right)$ is the maximal weight.

(2) $[p, q] = [0, 1]$.

This is the representation of antiquarks with the following "weight diagram".

The states carry the weights

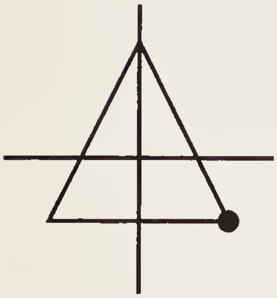


Fig. 1.5. The antiquark weight diagram

$$(T_3, Y) = \left(0, \frac{2}{3}\right), \left(\frac{1}{2}, -\frac{1}{3}\right), \left(-\frac{1}{2}, -\frac{1}{3}\right) \quad .$$

The state of maximal weight is $\left(\frac{1}{2}, -\frac{1}{3}\right)$.

In the case of SU(3), the trivial (scalar) representation is $[p, q] = [0, 0]$. The first nontrivial representations would be (1,0) and (0,1) of the same lowest dimension. By definition we thus have two fundamental representations. All others can be constructed from these two representations! To do so, we must construct the direct product of states

$$\begin{aligned}
&(1,0)^p(0,1)^q \rightarrow \\
&|T_3(1)Y(1)\rangle |T_3(2)Y(2)\rangle \cdots \\
&|T_3(p)Y(p)\rangle |\overline{T}_3(1)\overline{Y}(1)\rangle |\overline{T}_3(2)\overline{Y}(2)\rangle \cdots |\overline{T}_3(q)\overline{Y}(q)\rangle \quad .
\end{aligned} \tag{3}$$

Owing to the additivity of the isospin component $\hat{T}_3$ and the hypercharge $\hat{Y}$, it holds that

Exercise 1.1.

$$\hat{T}_3 = \sum_i \hat{T}_3(i) \quad , \tag{4a}$$

$$\hat{Y} = \sum_i \hat{Y}(i) \quad . \tag{4b}$$

Thus many-quark states have T_3 and Y eigenvalues

$$(T_3, Y) = \left(\sum_{i=1}^{p} T_3(i) + \sum_{i=1}^{q} \overline{T}_3(i), \sum_{i=1}^{p} Y(i) + \sum_{i=1}^{q} \overline{Y}(i) \right) \quad . \tag{5}$$

In these, there is one state of maximal weight, namely the one that is composed of p quarks of maximal weight $\left(\frac{1}{2}, \frac{1}{3}\right)$ and q antiquarks of maximal weight $\left(\frac{1}{2}, -\frac{1}{3}\right)$, i.e.,

$$(T_3)_{\text{max}} = \frac{p+q}{2} \quad , \quad (Y)_{\text{max}} = \frac{p-q}{3} \quad . \tag{6}$$

It characterizes a representation contained in (5). If we subtract it, there is a remainder. Within this there is another state (or several states) of maximal weight. They are analogously given tuples $[p, q]$, i.e., a multiplet. We repeat the above steps until nothing is left, i.e., the direct product is completely reduced. In this way we can construct all SU(3) decompositions.

We consider $[p_1, q_1] \times [p_2, q_2] = [1, 0] \times [0, 1]$ and first add the two weight diagrams, i.e., at each point of the one diagram, we add the other diagram (see Fig. 1.6).

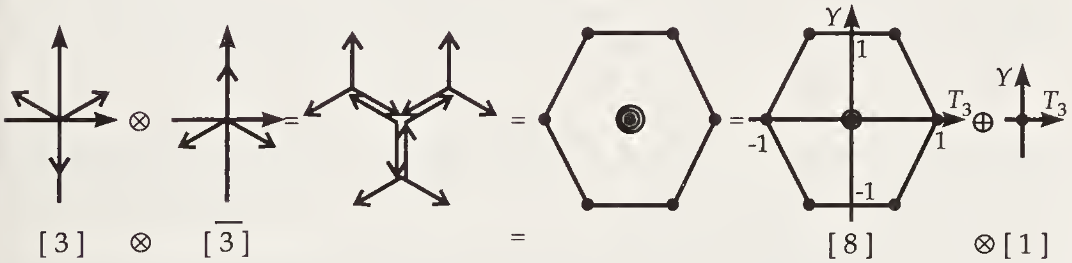


Fig. 1.6. Adding $[1, 0]$ and $[0, 1]$ weight diagrams

We are thus led to a weight diagram whose center is occupied three times! The maximal weight appearing there is

$$(T_3, Y)_{\text{max}} = (1, 0) \quad . \tag{7}$$

For $[p, q]$, it follows from (6) that

$$[p, q] = [1, 1] \quad , \tag{8}$$

corresponding to an octet with dimension 8. On subtracting the octet which is twice degenerate at the center, only the singlet remains

$$(T_3, Y)_{\text{max}} = (0, 0) \quad , \tag{9a}$$

that is,

$$[p, q] = [0, 0] \quad . \tag{9b}$$

Exercise 1.1.

We thus obtained the following result:

$$[1,0]\times[0,1]=[1,1]+[0,0] \quad . \tag{10}$$

Note: Constructing $[1,0]\times[0,1]$ with this method, we obtain

$$[1,0]\times[1,0]=[2,0]+[0,1] \quad . \tag{11}$$

On the right-hand side, $[0,1]$ appears. Mathematically, we can construct $[0,1]$ from $[1,0]$. Thus one is inclined to call only $[1,0]$ the fundamental representation. Physically, however, the right-hand side describes two-quark states and not, as does $[0,1]$, antiquark states.

SU(N). Its solutions are classified by $N-1$ numbers:

$$[h_1,\cdots,h_{N-1}] \quad . \tag{12}$$

Analogously to SU(3), there is the scalar (trivial) representation

$$[0,\cdots,0] \tag{13}$$

and $N-1$ fundamental representations

$$\begin{aligned} &[1,0,\cdots,0] \quad ,\\ &[0,1,\cdots,0] \quad ,\\ &\quad\vdots\\ &[0,\cdots,0,1] \quad . \end{aligned} \tag{14}$$

From these, all other multiplets in (12) can be constructed by direct products.

Solution. (b) Schur's lemma indicates that any operator $\hat{H}$ commuting with all operators $\hat{U}(\alpha)$ (α = group parameter), in particular with the generators $\hat{L}_i$,

$$\left[\hat{H},\hat{U}(\alpha)\right]=0 \Leftrightarrow \quad \left[\hat{H},\hat{L}_i\right]=0 \quad \Rightarrow \quad \left[\hat{H},\hat{C}(\lambda)\right]=0 \quad ,$$

has the property that every state in a multiplet of the group is an eigenvector and that all states in a multiplet are degenerate. $\hat{C}(\lambda)$ is a Casimir operator of the group in the irreducible representation λ.

Since $\hat{C}(\lambda)$ commutes with $\hat{H}$, $\hat{C}(\lambda)$ and $\hat{H}$ can be simultaneously diagonalized, i.e., $\hat{C}(\lambda)$, too, is diagonal with respect to any state of the irreducible representation (multiplet) of the group. Calling $C(\lambda)$ the eigenvalues of $\hat{C}(\lambda)$, $\hat{C}(\lambda)$ has the following form with respect the irreducible representation of the group:

$$\hat{C}(\lambda)=C(\lambda)1\!\!1(\lambda) \quad , \tag{15}$$

where $1\!\!1_\lambda$ is the unit matrix with the multiplet's dimension. As the fundamental representation is by construction irreducible, (15) holds. In matrix representation, the Casimir operator has the following form:

$$\begin{pmatrix} C(\lambda_1)1\!\!1(\lambda_1) & 0 & 0 & \cdots \\ 0 & C(\lambda_2)1\!\!1(\lambda_2) & 0 & \cdots \\ 0 & 0 & C(\lambda_3)1\!\!1(\lambda_3) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad .$$

Each diagonal submatrix appearing in it is of the form $C(\lambda)1\!\!1(\lambda)$ and characterizes a representation (multiplet) of the same dimension as this multiplet.

EXERCISE

1.2 Casimir Operators of SU(3)

Problem. The regular (adjoint) representation of SU(3) is given by the eight generators $\hat{U}_i$, $i = 1, \dots, 8$ with

$$(\hat{U}_i)_{jk} = -\mathrm{i} f_{ijk} \tag{1}$$

($\hat{U}_i$ are 8×8 matrices). Show that for $\hat{C}_1$, one of the two Casimir operators of SU(3) in the regular representation it holds that

$$\hat{C}_1 = \sum_{i=1}^{8} \hat{U}_i^2 = 3\,\mathbb{1}_{8\times 8} \tag{2}$$

Table 1.3. The eigenvalues of the Casimiroperator $\hat{C}_1$ for the regular representation

m	$\sum_{ij} f_{ijm}^2$
1	$2f_{123}^2 + 2f_{147}^2 + 2f_{156}^2 = 3$
2	$2f_{123}^2 + 2f_{246}^2 + 2f_{257}^2 = 3$
3	$2f_{123}^2 + 2f_{345}^2 + 2f_{367}^2 = 3$
4	$2f_{246}^2 + 2f_{345}^2 + 2f_{147}^2 + 2f_{458} = 3$
5	$2f_{156}^2 + 2f_{257}^2 + 2f_{345}^2 + 2f_{458} = 3$
6	$2f_{156}^2 + 2f_{246}^2 + 2f_{367}^2 + 2f_{678} = 3$
7	$2f_{147}^2 + 2f_{257}^2 + 2f_{367}^2 + 2f_{678} = 3$
8	$2f_{458}^2 + 2f_{678}^2 = 3$

Solution. Each irreducible representation of SU(3) is uniquely determined by the eigenvalues of its Casimir operators. Each state in a multiplet has the same eignvalues with respect to $\hat{C}_1$. Thus this operator must be proportional to the unit matrix. This is checked here using an example. Using (1) it follows for $\hat{C}_1$ that

$$(\hat{C}_1)_{lm} = -\sum_{i,j} f_{ilj} f_{ijm} \quad . \tag{3}$$

From the table of f_{ijk}, one recognizes that $f_{ilj} \neq 0$ and $f_{ijm} \neq 0$, which implies that $l = m$:

$$(\hat{C}_1)_{lm} = +\sum_{i,j} f_{ijm}^2 \delta_{lm} = 3\delta_{lm} \quad . \tag{4}$$

This proves (2).

2. Review of Relativistic Field Theory

2.1 Spinor Quantum Electrodynamics

As a general introduction, this chapter reviews the basics of spinor quantum electrodynamics that are referenced to in the following. Section 2.2 will give a similar review of scalar quantum electrodynamics. Readers who are familiar with this material should continue with Chap. 3.

2.1.1 The Free Dirac Equation and Its Solution

The equation of motion for the free spinor field Ψ is the free Dirac equation (we use natural units: $\hbar = c = 1$):

$$\mathrm{i}\frac{\partial}{\partial t}\Psi = (-\mathrm{i}\hat{\alpha}\cdot\nabla - \beta m_0)\Psi \quad . \tag{2.1}$$

Its solution is of the form

$$\Psi = w\mathrm{e}^{-\mathrm{i}p\cdot x} \quad , \tag{2.2}$$

where

$$p \equiv p^{\mu} = (p^0, \boldsymbol{p}) \tag{2.3}$$

is the momentum four-vector and w a four-component Dirac spinor. The spinor w is usually decomposed into the two two-component spinors φ and χ:

$$w = \begin{pmatrix} \varphi \\ \chi \end{pmatrix} \quad . \tag{2.4}$$

With this, the Dirac equation becomes a coupled system of equations for φ and χ:

$$p^0 \begin{pmatrix} \varphi \\ \chi \end{pmatrix} = \begin{pmatrix} m_0 \mathbb{1} & \hat{\sigma}\cdot\boldsymbol{p} \\ \hat{\sigma}\cdot\boldsymbol{p} & -m_0\mathbb{1} \end{pmatrix} \begin{pmatrix} \varphi \\ \chi \end{pmatrix} \quad , \tag{2.5}$$

where $\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the 2×2 unit matrix and $\hat{\sigma}$ the vector of the 2×2 Pauli matrices. Equation (2.5) possesses solutions of positive and negative energy.

Plane Waves of Positive Energy. In this case

$$p^0 \equiv E = +\sqrt{\boldsymbol{p}^2 + m_0} > 0 \quad . \tag{2.6}$$

Exploiting the covariance of the Dirac equation, we first give the solutions for a particle at rest for which

$$p^0 = m_0 \quad , \quad \boldsymbol{p} = \boldsymbol{0} \tag{2.7}$$

holds. The system of equations (2.5) then has the form

$$m_0 \begin{pmatrix} \varphi \\ \chi \end{pmatrix} = \begin{pmatrix} m_0 \mathbb{1} & 0 \\ 0 & -m_0 \mathbb{1} \end{pmatrix} \begin{pmatrix} \varphi \\ \chi \end{pmatrix} \tag{2.8}$$

and leads to

$$\chi = 0 \tag{2.9}$$

and

$$w(p^0 = m_0) = \begin{pmatrix} \varphi \\ 0 \end{pmatrix} \quad . \tag{2.10}$$

The two linearly independent solutions for the two-spinor φ are

$$\begin{aligned} \varphi^1 &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (\text{Spin } \uparrow) \quad , \\ \varphi^2 &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (\text{Spin } \downarrow) \quad , \end{aligned} \tag{2.11}$$

which clearly shows that the Dirac equation describes particles of spin $\frac{1}{2}$. For nonvanishing spatial momentum

$$\boldsymbol{p} \neq \boldsymbol{0} \quad , \quad p^0 = E = \sqrt{\boldsymbol{p}^2 + m_0^2} \quad , \tag{2.12}$$

χ can be expressed by φ using (2.5), and one obtains the spinors of positive energy in the form

$$w^s = \begin{pmatrix} \varphi^s \\ \frac{\hat{\sigma}\cdot\boldsymbol{p}}{E+m_0}\varphi^s \end{pmatrix} \quad , \quad s = 1, 2 \quad . \tag{2.13}$$

Plane Waves of Negative Energy. These are characterized by

$$p^0 \equiv -E = -\sqrt{\boldsymbol{p}^2 + m_0} \quad , \tag{2.14}$$

where E always indicates the positive square root $\sqrt{\boldsymbol{p}^2 + m_0^2}$, i.e., in this notation $E > 0$. To construct the solutions we proceed as above. For a particle at rest with $p^\mu = (p^0 = -m_0, \boldsymbol{p} = \boldsymbol{0})$ a system analogous to (2.8) leads to

$$\varphi = 0 \tag{2.15}$$

and to the four-spinor

$$w(p^0 = -m_0) = \begin{pmatrix} 0 \\ \chi \end{pmatrix} \quad , \tag{2.16}$$

respectively. For nonvanishing spatial momentum, i.e., for the four-momentum $p^\mu = (-E, \boldsymbol{p})$, φ can now be eliminated and one obtains

$$w = \begin{pmatrix} -\frac{\hat{\sigma}\cdot\boldsymbol{p}}{E+m_0}\chi \\ \chi \end{pmatrix} \quad . \tag{2.17}$$

We give the following important definition, which can be understood from hole theory. A particle (electron) is identified with a solution of positive energy and positive momentum $\boldsymbol{p}$, i.e.,

$$\Psi \sim \mathrm{e}^{-\mathrm{i}p\cdot x} \quad , \quad p = (E, \boldsymbol{p}) \quad , \tag{2.18}$$

and an antiparticle (positron) with the solution of negative energy and negative momentum, i.e.,

$$\Psi \sim \mathrm{e}^{\mathrm{i}p\cdot x} = \mathrm{e}^{-\mathrm{i}(-p\cdot x)} \equiv \mathrm{e}^{-\mathrm{i}p'\cdot x} \quad , \quad p' = (-E, -\boldsymbol{p}) \quad , \tag{2.19}$$

The particle–antiparticle solution are therefore connected by the transformation

$$p^\mu \longrightarrow -p^\mu \quad . \tag{2.20}$$

Similarly, one expects for the spin that a missing particle of spin ↑ corresponds to an antiparticle of spin ↓. In other words, electron solutions of negative energy, negative momentum, and spin ↓ correspond to positron solutions of positive energy, positive momentum, and spin ↑. For this reason one puts

$$\chi^1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \chi^2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad . \tag{2.21}$$

The four-spinors w representing particles and antiparticles are now

$$w^s = \begin{pmatrix} \frac{\hat{\sigma}\cdot\boldsymbol{p}}{E+m_0}\chi^s \\ \chi^s \end{pmatrix} \quad , \quad s = 1, 2 \quad . \tag{2.22}$$

With definitions (2.18–21) it is guaranteed that the quantities E and $\boldsymbol{p}$, as well as the basis spinors χ^1 and χ^2 that appear in the solutions (2.22), always denote energy, momentum, and spin ↑ or spin ↓ of the (physically observed) antiparticle.

2.1.2 Density and Current Density

The density ϱ and current density $\boldsymbol{j}$ of the Dirac field are, independent of the sign of the energy, given by

$$\varrho = \Psi^\dagger \Psi \quad , \tag{2.23a}$$

$$\boldsymbol{j} = \Psi^\dagger \hat{\boldsymbol{\alpha}} \Psi \quad , \tag{2.23b}$$

and satisfy the continuity equation

$$\frac{\partial}{\partial t}\varrho + \nabla \cdot \boldsymbol{j} = 0 \quad . \tag{2.24}$$

For any spinor of the form

$$\Psi = w\, u(\boldsymbol{x}, t) \tag{2.25}$$

it follows that

$$\varrho = w^\dagger w |u(\boldsymbol{x},t)|^2 \quad , \tag{2.26a}$$

$$\boldsymbol{j} = w^\dagger \boldsymbol{\alpha} w |u(\boldsymbol{x},t)|^2 \quad . \tag{2.26b}$$

Obviously, $\varrho \geq 0$ always holds, and this is independent of (2.25) designating a particle or an antiparticle solution. Similarly, $\boldsymbol{j}$ does not change its sign when moving from particle to antiparticle solutions. This is most quickly verified for the z-component:

$$\begin{aligned}
(j_z)_{\mathrm{e}^-\uparrow} &= \begin{pmatrix} \begin{pmatrix}1\\0\end{pmatrix} \\ \frac{p_z}{E+m_0}\begin{pmatrix}1\\0\end{pmatrix} \end{pmatrix}^\dagger \begin{pmatrix} 0 & \hat{\sigma}_z \\ \hat{\sigma}_z & 0 \end{pmatrix} \begin{pmatrix} \begin{pmatrix}1\\0\end{pmatrix} \\ \frac{p_z}{E+m_0}\begin{pmatrix}1\\0\end{pmatrix} \end{pmatrix} = \frac{2p_z}{E+m_0} \quad , \\
(j_z)_{\mathrm{e}^+\uparrow} &= \begin{pmatrix} -\frac{p_z}{E+m_0}\begin{pmatrix}0\\1\end{pmatrix} \\ \begin{pmatrix}0\\1\end{pmatrix} \end{pmatrix}^\dagger \begin{pmatrix} 0 & \hat{\sigma}_z \\ \hat{\sigma}_z & 0 \end{pmatrix} \begin{pmatrix} -\frac{p_z}{E+m_0}\begin{pmatrix}0\\1\end{pmatrix} \\ \begin{pmatrix}0\\1\end{pmatrix} \end{pmatrix} = \frac{2p_z}{E+m_0} \quad .
\end{aligned} \tag{2.27}$$

A sign change of charge and current density is, however, desired. To put it in by hand one inserts an extra minus sign whenever an electron (fermion) of negative energy appears in a final state. This rule is very naturally included in the definition of the Feynman propagator.

2.1.3 Covariant Notation

It is customary to introduce γ matrices, which replace $\hat{\boldsymbol{\alpha}}$ and $\hat{\beta}$ (we leave out the operator hats in the following):

$$\begin{aligned}
\gamma^0 &= \hat{\beta} \quad , \quad (\gamma^0)^2 = \mathbb{1} \quad , \\
\gamma^i &= \hat{\beta}\hat{\alpha}_i \quad , \quad (\gamma^i)^2 = -\mathbb{1} \quad , \quad i = 1,2,3 \quad , \\
\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu &= g^{\mu\nu}\mathbb{1} \quad .
\end{aligned} \tag{2.28}$$

Here $\mathbb{1}$ designates the unit matrix. The free Dirac equation (2.1) takes the form

$$\left(\mathrm{i}\gamma^\mu \frac{\partial}{\partial x^\mu} - m_0\right)\Psi = 0 \quad ,$$

or

$$(\mathrm{i}\not{\partial} - m_0)\Psi = 0 \quad , \tag{2.29}$$

using the (Feynman) dagger notation ($\not{a} \equiv \gamma^\mu a_\mu$). Density ϱ and current density $\boldsymbol{j}$ can be combined to form a four-current density:

$$j^\mu = \bar{\Psi}\gamma^\mu\Psi \quad , \tag{2.30}$$

and the continuity equation (2.24) can be written as a four-divergence:

$$\frac{\partial}{\partial x^\mu} j^\mu = 0 \quad . \tag{2.31}$$

Here

$$\bar{\Psi} = \Psi^{\dagger}\gamma^{0} \tag{2.32}$$

designates the adjoint spinor.

2.1.4 Normalization of Dirac Spinors

It is useful to consider again the normalization of spinor wave functions. Let us first consider plane waves of positive energy,

$$\Psi^{1,2} = N' w^{1,2}\, \mathrm{e}^{-\mathrm{i}p\cdot x} \quad . \tag{2.33}$$

We normalize such a a wave in a box of volume V in such a way that

$$\int d^3x\, \varrho = 2E \tag{2.34}$$

holds. This normalization differs from the usual normalization to unity of quantum mechanics but is often used in field theory. Using the explicit form of the spinors $\omega^{1,2}$ of positive energy yields the normalization factor

$$N' = \sqrt{\frac{E + m_0}{V}} \quad . \tag{2.35}$$

Usually one absorbs the factor $\sqrt{E - m_0}$ in the definition of the spinor w^s and designates the spinors for positive energy by $u(p,s)$:

$$\begin{gathered} u(p,s) = \sqrt{E + m_0}\begin{pmatrix} \varphi^1 \\ \frac{\hat{\sigma}\cdot p}{E+m_0}\varphi^2 \end{pmatrix} \qquad s = 1,2 \quad , \\ \varphi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \quad \varphi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad . \end{gathered} \tag{2.36}$$

One similarly introduces spinors $v(p,s)$ for negative energy:

$$\begin{gathered} v(p,s) = \sqrt{E + m_0}\begin{pmatrix} \frac{\hat{\sigma}\cdot p}{E+m_0}\chi^1 \\ \chi^2 \end{pmatrix} \qquad s = 1,2 \quad , \\ \chi^1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad , \quad \chi^2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad . \end{gathered} \tag{2.37}$$

We again emphasize that the spinors $v(p,s)$ are constructed such that E, $\boldsymbol{p}$, and $s = 1,2$ in (2.37) correspond to energy, momentum, and spin projection ↑ or ↓ of the positron. It is easy to check that

$$u^{+}u = v^{+}v = 2E \tag{2.38}$$

and

$$(\not{p} - m_0)u = 0 \quad , \tag{2.39a}$$

$$(\not{p} + m_0)v = 0 \tag{2.39b}$$

hold. Equations (2.39a,b) are the momentum-space Dirac equation for the (free) solutions of positive and negative energy, respectively. Correspondingly one finds the Dirac equations for the adjoint spinors $\bar{u}$ and $\bar{v}$:

$$\bar{u}(\not{p} - m_0) = 0 \quad , \tag{2.40a}$$

$$\bar{v}(\not{p} + m_0) = 0 \quad . \tag{2.40b}$$

The normalization conditions for the spinors can be summarized as

$$\bar{u}(p,s)u(p',s') = 2m_0\delta_{ss'} \quad , \tag{2.41a}$$

$$\bar{v}(p,s)v(p',s') = -2m_0\delta_{ss'} \tag{2.41b}$$

or

$$\overline{w}^r(p)w^r(p') = 2m_0\varepsilon_r\delta_{rr'} \quad , \quad \varepsilon_r = \begin{cases} 1 & \text{for} \quad r = 1,2 \\ -1 & \text{for} \quad r = 3,4 \end{cases} \quad . \tag{2.42}$$

We want to emphasize that another normalization of $u(p,s)$ and $v(p,s)$ can also quite often be found:

$$u' = \frac{1}{\sqrt{2m_0}}u \quad , \quad v' = \frac{1}{\sqrt{2m_0}}v \quad . \tag{2.43}$$

The advantage of the normalization used here is its covariance. This is understandable from (2.38): the densities u^+u and v^+v are proportional to the energy E and transform as 0-components of a four-vector. When we compute cross sections, this and the corresponding transformation properties of phase space and flow factors make Lorentz invariance obvious.

With these conventions we can write electron and positron wave functions as

$$\Psi(\mathrm{e}^-) = Nu(p,s)\mathrm{e}^{-\mathrm{i}p\cdot x} \quad , \tag{2.44a}$$

$$\Psi(\mathrm{e}^+) = Nv(p,s)\mathrm{e}^{\mathrm{i}p\cdot x} \quad , \quad N = \frac{1}{\sqrt{V}} \quad . \tag{2.44b}$$

These explicit expressions enable us to write down directly the transition currents, as we shall see below.

2.1.5 Interaction with a Four-Potential A^μ

The interaction of the field with an electromagnetic potential A^μ is introduced by the so-called "minimal" coupling to preserve gauge invariance. For an electron (of charge minus e), the minimal substitution is

$$\frac{\partial}{\partial x_\mu} \equiv \partial_\mu \to \partial_\mu - \mathrm{i}eA_\mu \quad . \tag{2.45}$$

Thus the free Dirac equation (2.1) is changed into (written in noncovariant form)

$$\mathrm{i}\frac{\partial}{\partial t}\psi = (-\mathrm{i}\hat{\alpha}\cdot\nabla + \hat{\beta}m_0 + \hat{V})\psi \quad , \tag{2.46}$$

where the interaction $\hat{V}$ is given by

$$\hat{V} = -eA^0 \mathbb{1} + e\,\hat{\alpha} \cdot \boldsymbol{A} \quad . \tag{2.47}$$

In covariant form, the Dirac equation with interaction (substitution of (2.49) into (2.29)) reads

$$\left[\mathrm{i}\gamma^\mu (\partial_\mu - \mathrm{i}eA_\mu) - m_0 \right] \Psi = 0 \tag{2.48}$$

and the interaction is written as

$$\gamma^0 \hat{V} = -e\gamma^\mu A_\mu \quad . \tag{2.49}$$

2.1.6 Transition Amplitudes

The transition amplitude (S-matrix element) of an initial electron state $\Psi_\mathrm{i}(\mathrm{e}^-;p,s)$ with four-momentum p and spin projection s into a final electron state $\Psi_\mathrm{f}(\mathrm{e}^-;p',s')$ characterized by momentum p' and spin s' is in first-order perturbation theory:

$$\begin{aligned} S_{\mathrm{fi}}^{(1)} &= -\mathrm{i} \int \mathrm{d}^4x\; \Psi_\mathrm{f}^\dagger(\mathrm{e}^-;p',s') \hat{V}\, \Psi_\mathrm{i}(\mathrm{e}^-;p,s) \\ &= \mathrm{i} \int \mathrm{d}^4x\; \Psi_\mathrm{f}^\dagger(-e\gamma^\mu A_\mu)\Psi_\mathrm{i} \\ &= \mathrm{i} \int \mathrm{d}^4x\; J^\mu(\mathrm{e}^-) A_\mu \quad , \end{aligned} \tag{2.50}$$

where

$$J^\mu(\mathrm{e}^-) = (-e)\bar{\Psi}_\mathrm{f}\, \gamma^\mu\, \Psi_\mathrm{i} \tag{2.51}$$

are the electron (fermion) transition current densities. Using the plane waves (2.44a) this becomes explicitly

$$J^\mu(\mathrm{e}^-) = \frac{(-e)}{V} \bar{u}_\mathrm{f}(p',s')\, \gamma^\mu\, u_\mathrm{i}(p,s)\, \mathrm{e}^{\mathrm{i}(p'-p)\cdot x} \quad , \quad N_\mathrm{i} = N_\mathrm{f} = \frac{1}{\sqrt{V}} \tag{2.52}$$

and now allows the calculation of scattering process in lowest order according to (2.50).

2.1.7 Discrete Symmetries

We restrict ourselves here to a summarizing and tabulating the properties of the discrete symmetry transformations parity $\hat{P}$, charge conjugation $\hat{C}$, and time reversal $\hat{T}$.

Dirac fields can be combined into the following bilinear forms (currents), distinguished by their tensor character:

$$S(x) = \bar{\Psi}(x)\Psi(x) \quad \text{scalar} \quad , \tag{2.53a}$$

$$V^\mu(x) = \bar{\Psi}(x)\gamma^\mu\Psi(x) \quad \text{vector} \quad , \tag{2.53b}$$

$$T^{\mu\nu}(x) = \bar{\Psi}(x)\sigma^{\mu\nu}\Psi(x) \quad \text{tensor} \quad , \tag{2.53c}$$

$$P(x) = \mathrm{i}\bar{\Psi}(x)\gamma_5\Psi(x) \quad \text{pseudoscalar} \quad , \tag{2.53d}$$

$$A^\mu(x) = \bar{\Psi}(x)\gamma_5\gamma^\mu\Psi(x) \quad \text{pseudovector} \quad . \tag{2.53e}$$

Table 2.1. The behaviour of the currents (2.53) under the transformations $\hat{P}$, $\hat{C}$, $\hat{T}$, and $\hat{O}=\hat{P}\hat{C}\hat{T}$

	$S(x)$	$V^{\mu}(x)$	$T^{\mu\nu}(x)$	$P(x)$	$A^{\mu}(x)$
$\hat{P}$	$S(\tilde{x})$	$V^{\mu}(\tilde{x})$	$T^{\mu\nu}(\tilde{x})$	$-P(\tilde{x})$	$-A^{\mu}(\tilde{x})$
$\hat{C}$	$S(x)$	$-V^{\mu}(x)$	$-T^{\mu\nu}(x)$	$P(x)$	$A^{\mu}(x)$
$\hat{T}$	$S(-\tilde{x})$	$V^{\mu}(-\tilde{x})$	$-T^{\mu\nu}(-\tilde{x})$	$-P(-\tilde{x})$	$A^{\mu}(-\tilde{x})$
$\hat{O}$	$S(-x)$	$-V^{\mu}(-x)$	$T^{\mu\nu}(-x)$	$P(-x)$	$-A^{\mu}(-x)$

The behavior of these currents under the transformations $\hat{P}$, $\hat{C}$, $\hat{T}$, as well as under the mixed symmetry $\hat{O}=\hat{P}\hat{C}\hat{T}$, is given in Table 2.1 where $\tilde{x}=(t,-\boldsymbol{x})$.

We also give the corresponding transformations for the electromagnetic four-potential A_{μ}:

$$\begin{aligned}\hat{P}A^{\mu}(x)\hat{P}^{+}&=A^{\mu}(x)\quad,\quad \hat{C}A^{\mu}(x)\hat{C}^{+}=-A^{\mu}(x)\quad,\\ \hat{T}A^{\mu}(x)\hat{T}^{+}&=A^{\mu}(-x)\quad,\quad \hat{O}A^{\mu}(x)\hat{O}^{+}=-A^{\mu}(-x)\quad.\end{aligned}\tag{2.54}$$

2.2 Scalar Quantum Electrodynamics

2.2.1 The Free Klein–Gordon Equation and Its Solutions

It is known that pions as spin-0 particles satisfy the Klein–Gordon equation. Here we compile the main results of pion quantum electrodynamics. Starting from the four-momentum vector and relativistic energy conservation

$$p^{\mu}=(E,\boldsymbol{p})\quad,\tag{2.55}$$

$$p_{\mu}p^{\mu}=E^2-\boldsymbol{p}^2=m_0^2\tag{2.56}$$

and the correspondence between momentum and momentum operator

$$p^{\mu}\rightarrow\hat{p}^{\mu}=\mathrm{i}\partial^{\mu}\quad,\tag{2.57}$$

the free Klein–Gordon equation follows:

$$\left(\hat{p}_{\mu}\hat{p}^{\mu}-m_o^2\right)\phi(\boldsymbol{x},t)=0\quad,\tag{2.58a}$$

which can be written as

$$\left(\Box+m_0^2\right)\phi(\boldsymbol{x},t)=0\tag{2.58b}$$

using the *d'Alembertian operator (quabla operator)*, which is defined by

$$\Box\equiv\partial_{\mu}\partial^{\mu}=\frac{\partial^2}{\partial t^2}-\nabla^2\quad.\tag{2.59}$$

Plane waves of the form

$$\phi(\boldsymbol{x},t)=N\mathrm{e}^{-\mathrm{i}p\cdot x}=N\mathrm{e}^{-\mathrm{i}(Et-\boldsymbol{p}\cdot\boldsymbol{x})}\tag{2.60}$$

are solutions of (2.58) if condition (2.56) is satisfied. Therefore we also have solutions of positive and negative energy:

$$E = \pm\sqrt{\boldsymbol{p}^2 + m_0^2} \quad . \tag{2.61}$$

The question of their interpretation is thus raised. To answer it, we shall derive expressions for the probability density ϱ and the probability current density $\boldsymbol{j}$ by multiplying (2.58) by ϕ^* and its complex conjugate-equation by ϕ and subtracting each from the other. This leads to the continuity equation ($\partial_t \equiv \frac{\partial}{\partial t}$)

$$\partial_t \varrho + \nabla \cdot \boldsymbol{j} = 0 \quad , \tag{2.62}$$

where

$$\varrho = \mathrm{i}\left[\phi^*(\partial_t \phi) - (\partial_t \phi^*)\phi\right] \tag{2.63a}$$

and

$$\boldsymbol{j} = -\mathrm{i}\left[\phi^*(\nabla\phi) - (\nabla\phi^*)\phi\right] \quad . \tag{2.63b}$$

In four-dimensional notation, this is concisely written as

$$\partial_\mu j^\mu = 0 \quad , \tag{2.64}$$

with the four-current density

$$j^\mu = (\varrho, \boldsymbol{j}) = \mathrm{i}\left[\phi^* \left(\partial^\mu \phi\right) - \left(\partial^\mu \phi^*\right) \phi\right] \quad . \tag{2.65}$$

The three-current density $\boldsymbol{j}$ in (2.63b) is formally identical with that known from the Schrödinger equation. However, the probability density ϱ contains, in contrast to the Schrödinger density, additional time derivatives. This has the consequence that ϱ is not positive definite, which can be immediately checked using plane waves (2.60), taking into account (2.61). In this way it follows for (2.63a) that

$$\varrho = 2\,|N|^2\,E \quad . \tag{2.66}$$

The probability current density (2.63b) is obtained as

$$\boldsymbol{j} = 2\,|N|^2\,\boldsymbol{p} \quad . \tag{2.67}$$

Since E can be positive or negative owing to (2.61), the above statement about ϱ is obvious. To interpret ϱ nonetheless as a probability, one must make use of the particle–antiparticle interpretation. By the Feynman–Stückelberg prescription, it holds that:

A solution of negative energy for a particle propagating backward in time corresponds to a solution of positive energy for an antiparticle propagating forward in time.

The Klein–Gordon equation describes both neutral and charged mesons. In the case of charged scalar particles we not only have to analyze their spatial propagation but also the assignment of charges as discussed next for the charged pions π^+ and π^-. To this end we consider the scattering of a particle (e.g., a π^+) off a potential in second-order perturbation theory. The space-time diagram of such a process is shown in Fig. 2.1. An incoming π^+ scatters off the potential at position x_1 and time t_1 and propagates to position x_2, where it scatters again at a later time $t_2 \gg t_1$ and then moves on freely. According to Stückelberg and Feynman there must be the possibility that particles are scattered backward in time (Fig. 2.2). Thus one must allow in relativistic quantum field theory for the processes shown in the these figures.

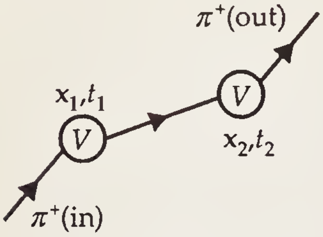


Fig. 2.1. Double scattering of a particle (π^+) off a potential

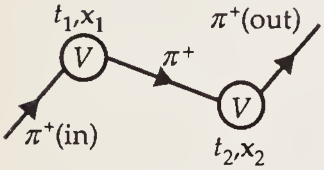


Fig. 2.2. Scattering backward in time: The particle here has, according to Stückelberg and Feynman, negative energy

We interpret this second process according to Feynman in such a way that particle solutions of positive energy propagate exclusively forward and particle solutions of negative energy exclusively backward in time. The π^+ moving backward in time between t_2 and t_1 must have negative energy. It is equivalent to a π^- (antiparticle) moving forward in time. This is obviously implied by charge conservation: only particle–antiparticle pairs can be created or annihilated. Fig. 2.3 illustrates this reinterpretation of Fig. 2.2. At t_2, a $\pi^+\pi^-$ pair is created whose π^- – which is identical to the originally incoming π^+ – is annihilated at t_1 and whose π^+ propagates on.

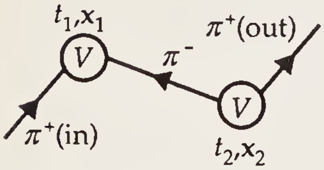


Fig. 2.3 Feynman's reinterpretation of the scattering process shown in Fig. 2.2

There is also another way to demonstrate the concept of a charged Klein–Gordon field. The charged currents for π^+ and π^- at positive energy are obtained by multiplying the charge density (2.11), calculated for waves by positive energy, by the positive and negative unit charge ($e > 0$) respectively, that is

$$\begin{aligned} j^\mu(\pi^\pm) = (\pm e) \times \text{probability current density} \\ \text{for a } \pi^\pm \text{ at positive energy.} \end{aligned} \tag{2.68}$$

Inserting the plane wave (2.60) into (2.63), we have

$$j^\mu(\pi^+) = (+e) 2\,|N|^2 \left(\sqrt{\boldsymbol{p}^2 + m_0^2}, \boldsymbol{p}\right) \tag{2.69}$$

and

$$j^\mu(\pi^-) = (-e) 2\,|N|^2 \left(-\sqrt{\boldsymbol{p}^2 + m_0^2}, \boldsymbol{p}\right) \quad . \tag{2.70}$$

Comparing (2.70) with (2.69), we see that it is obvious that (2.70) can also be written as

$$j^\mu(\pi^-) = (+e) 2\,|N|^2 \left(-\sqrt{\boldsymbol{p}^2 + m^2}, -\boldsymbol{p}\right) \quad , \tag{2.71}$$

which equals the current density of a π^+ with negative energy and negative momentum. In other words, a π^- thus corresponds to a π^+ with inverse four-momentum.

This correspondence can be expressed more precisely: if a system S emits a π^- of positive energy $E > 0$ and momentum $\boldsymbol{p}$ (see Fig. 2.4), the energy of S is reduced by E, its momentum by $\boldsymbol{p}$, and its charge by $(-e)$. But all this is equivalent to the absorption of a π^+ with negative four-momentum $(-E, -\boldsymbol{p})$, as demonstrated in Fig. 2.5. We summarize this with the following statement:

The emission (absorption) of an antiparticle with four-momentum p^μ is physically equivalent to the absorption (emission) of a particle with four-momentum $-p^\mu$.

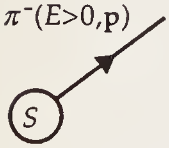


Fig. 2.4 Emission of a π^- with $(E > 0, \boldsymbol{p})$ by the system S

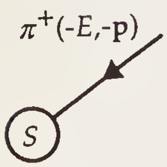


Fig. 2.5 Reinterpretation of the emission process in Fig. 2.4 as an absorption process

2.2.2 Interaction of a π^+ with a Potential A_μ

Just as in the case of the Dirac equation, the electromagnetic potential A^μ is coupled in by the minimal-coupling prescription (π^+ has the charge $+e$)

$$\partial^\mu \to \partial^\mu + \mathrm{i}eA^\mu \tag{2.72}$$

to preserve gauge invariance. If this is inserted into (2.58b), one obtains the Klein–Gordon equation with electromagnetic interaction

$$(\Box + m_0^2)\phi = -\mathrm{i}e(\partial_\mu A^\mu + A^\mu\partial_\mu)\phi + e^2A^2\phi \equiv -\hat{V}\phi \quad . \tag{2.73}$$

In contrast to the Dirac theory a coupling term quadratic in A^μ appears. However, we shall neglect it whenever scattering processes are considered in lowest order. In this approximation the coupling potential reduces to

$$\hat{V}(x) = \mathrm{i}e(\partial_\mu A^\mu(x) + A^\mu(x)\partial_\mu) \quad . \tag{2.74}$$

To calculate scattering processes, we also need the scattering amplitude. This is for scattering in first order of a potential $\hat{V}$, as before, given by

$$S_{\mathrm{fi}}^{(1)} = -\mathrm{i}\int \mathrm{d}^4x\, \phi_{\mathrm{f}}^* \hat{V} \phi_{\mathrm{i}} \tag{2.75}$$

It is displayed by the graph in Fig. 2.6.

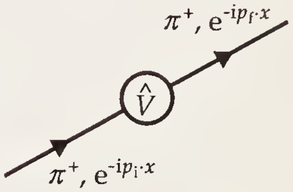

π^+, $\mathrm{e}^{-\mathrm{i}p_{\mathrm{f}}\cdot x}$
$\hat{V}$
π^+, $\mathrm{e}^{-\mathrm{i}p_{\mathrm{i}}\cdot x}$

Fig. 2.6. π^+ scattering off a potential V to lowest order. The potential is denoted by the vertex x

We shall now calculate the transition amplitude (2.75). Incoming and outgoing π^+-states are described by plane waves

$$\phi_{\mathrm{i}} = N_{\mathrm{i}}\,\mathrm{e}^{-\mathrm{i}p_{\mathrm{i}}\cdot x} \quad , \tag{2.76a}$$

$$\phi_{\mathrm{f}} = N_{\mathrm{f}}\,\mathrm{e}^{-\mathrm{i}p_{\mathrm{f}}\cdot x} \quad . \tag{2.76b}$$

Together with (2.74), (2.75) becomes

$$\begin{aligned} S_{\mathrm{fi}}^{(1)} &= -\mathrm{i}N_{\mathrm{i}}N_{\mathrm{f}}\int \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}p_{\mathrm{f}}\cdot x}(\mathrm{i}e)(\partial_\mu A^\mu + A^\mu\partial_\mu)\mathrm{e}^{-\mathrm{i}p_{\mathrm{i}}\cdot x} \\ &= -\mathrm{i}N_{\mathrm{i}}N_{\mathrm{f}}(p_{\mathrm{i}} + p_{\mathrm{f}})_\mu \int \mathrm{d}^4x\, \mathrm{e}^{-\mathrm{i}q\cdot x}A^\mu(x) \\ &= -\mathrm{i}eN_{\mathrm{i}}N_{\mathrm{f}}(p_{\mathrm{i}} + p_{\mathrm{f}})_\mu A^\mu(q) \quad , \end{aligned} \tag{2.77}$$

where the four-momentum transfer

$$q = p_{\mathrm{i}} - p_{\mathrm{f}} \tag{2.78}$$

has been introduced. Also, in calculating (2.77), a partial integration of the type

$$\int_{-\infty}^{+\infty} \mathrm{d}x^\mu f\frac{\mathrm{d}g}{\mathrm{d}x^\mu} = fg\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \mathrm{d}x^\mu \frac{\mathrm{d}f}{\mathrm{d}x^\mu}g \tag{2.79}$$

has been performed twice. Here the assumption has been made that contributions of the form gf vanish at infinity, i.e.,

$$fg\Big|_{-\infty}^{\infty} = 0 \tag{2.80}$$

This requires that either the potentials A_μ or the wave amplitudes decay fast enough at infinity. Exact plane waves do not have this property. But, strictly speaking, any particle is always represented by a wave packet. Even if it can be, for large distances from the scattering center and for large times before or after the scattering, arbitrarily delocalized, it will, however, decay asymptotically. Taking into account that using plane waves (2.76a,b) is just made to simplify calculations, the surface contributions can be neglected and the S-matrix element (2.75) can be written in a more convenient form:

$$\begin{aligned} S^{(1)}_{\mathrm{fi}} &= e\int \mathrm{d}^4x\, \phi_{\mathrm{f}}^*(\partial_\mu A^\mu + A^\mu\partial_\mu)\phi_{\mathrm{i}} \\ &= e\int \mathrm{d}^4x\, \left[-(\partial_\mu\phi_{\mathrm{f}}^*)\phi_{\mathrm{i}} + \phi_{\mathrm{f}}^*(\partial_\mu\phi_{\mathrm{i}})\right] A^\mu \\ &= -\mathrm{i}\int \mathrm{d}^4x\, j_\mu(\pi^+)A^\mu \quad . \end{aligned} \tag{2.81}$$

Here

$$j_\mu(\pi^+) = \mathrm{i}e\left[\phi_{\mathrm{f}}^*\left(\partial_\mu\phi_{\mathrm{i}}\right) - \left(\partial_\mu\phi_{\mathrm{f}}^*\right)\phi_{\mathrm{i}}\right] \tag{2.82}$$

is the transition current density for the π^+ meson. For plane waves (2.76a,b) this is particularly simple:

$$j_\mu(\pi^+) = eN_{\mathrm{i}}N_{\mathrm{f}}(p_{\mathrm{i}} + p_{\mathrm{f}})_\mu\, \mathrm{e}^{\mathrm{i}(p_{\mathrm{f}}-p_{\mathrm{i}})\cdot x} \quad , \tag{2.83}$$

which also appears in (2.77). In the following problem (2.1), the steps discussed here are illustrated once more.

EXERCISE

2.1 The Matrix Element for a Pion Scattered by a Potential

Problem. Consider the matrix element

$$M_{\mathrm{fi}} = \int \mathrm{d}^3x \int \mathrm{d}t\, \mathrm{e}^{\mathrm{i}p_{\mathrm{f}}\cdot x}\left(\partial_\mu A^\mu(x) + A^\mu(x)\partial_\mu\right)\mathrm{e}^{-\mathrm{i}p_{\mathrm{i}}\cdot x} \quad . \tag{1}$$

Assume that the four-potential fulfills the conditions

$$A^0(\boldsymbol{x},t) \to 0 \quad \text{for} \quad t \to \pm\infty \quad , \tag{2a}$$

$$|\boldsymbol{A}(\boldsymbol{x},t)| \to 0 \quad \text{for} \quad |\boldsymbol{x}| \to \infty \quad , \tag{2b}$$

and show that

$$\text{(a)} \int \mathrm{d}t\, \mathrm{e}^{\mathrm{i}p_{\mathrm{f}}\cdot x}\partial_t\left(A^0\mathrm{e}^{-\mathrm{i}p_{\mathrm{i}}\cdot x}\right) = (-\mathrm{i}(p_{\mathrm{f}})_0)\int \mathrm{d}t\, \mathrm{e}^{\mathrm{i}p_{\mathrm{f}}\cdot x}A^0\mathrm{e}^{-\mathrm{i}p_{\mathrm{i}}\cdot x} \quad , \tag{3}$$

$$\text{(b)} \int \mathrm{d}^3x\, \mathrm{e}^{\mathrm{i}p_{\mathrm{f}}\cdot x}\nabla\cdot\left(\boldsymbol{A}\mathrm{e}^{-\mathrm{i}p_{\mathrm{i}}\cdot x}\right) = \mathrm{i}\boldsymbol{p}_{\mathrm{f}}\cdot\int \mathrm{d}^3x\, \mathrm{e}^{\mathrm{i}p_{\mathrm{f}}\cdot x}\boldsymbol{A}\mathrm{e}^{-\mathrm{i}p_{\mathrm{i}}\cdot x} \tag{4}$$

and therefore also

(c) $\int d^4x\, e^{ip_f\cdot x}\left(\partial_\mu A^\mu + A^\mu\partial_\mu\right) e^{-ip_i\cdot x} = -i(p_f + p_i)_\mu \cdot \int d^4x\, e^{ip_f\cdot x} A^\mu e^{-ip_i\cdot x}$ (5)

Exercise 2.1.

hold.

Solution. (a) A partial integration of the time integral yields

$$\int\limits_{-\infty}^{\infty} dt\, e^{ip_f\cdot x}\partial_t A^0 e^{-ip_i\cdot x} = \left[A^0 e^{i(p_f-p_i)\cdot x}\right]_{t=-\infty}^{t=+\infty} - \int\limits_{-\infty}^{+\infty} dt\, A^0 e^{-ip_i\cdot x}\partial_t\, e^{ip_f\cdot x}$$

$$= -i(p_f)_0 \int\limits_{-\infty}^{+\infty} dt\, e^{ip_f\cdot x} A^0 e^{-ip_i\cdot x} \quad . \tag{6}$$

The surface term vanishes because of the boundary condition (2a).

(b) Analogously a partial integration over the spacial coordinates leads to (Gauss's theorem)

$$\int d^3x\, e^{ip_f\cdot x}\nabla\cdot\left(\boldsymbol{A}e^{-ip_i\cdot x}\right)$$

$$= \int\limits_{\text{surface, } |\boldsymbol{x}|\to\infty} d\boldsymbol{F}\cdot\boldsymbol{A}\, e^{i(p_f-p_i)\cdot x} - \int d^3x\, e^{-ip_i\cdot x}\boldsymbol{A}\cdot\nabla\, e^{ip_f\cdot x}$$

$$= i\boldsymbol{p}_f \cdot \int d^3x\, e^{ip_f\cdot x}\boldsymbol{A}\, e^{-ip_i\cdot x} \quad . \tag{7}$$

Owing to boundary condition (2b), the surface integral again vanishes.

(c) Summarizing (a) and (b) we obtain

$$\int d^4x\, e^{ip_f\cdot x}\partial_\mu\left(A^\mu e^{-ip_i\cdot x}\right) = -i(p_f)_\mu \int d^4x\, e^{ip_f\cdot x} A^\mu e^{-ip_i\cdot x} \quad . \tag{8}$$

On the other hand we have

$$\partial_\mu e^{-ip_i\cdot x} = -i(p_i)_\mu e^{-ip_i\cdot x} \quad , \tag{9}$$

i.e.,

$$M_{fi} = \int d^4x\, e^{ip_f\cdot x}\left(\partial_\mu A^\mu + A^\mu\partial_\mu\right) e^{-ip_i\cdot x}$$

$$= -i(p_f + p_i)_\mu \int d^4x\, e^{ip_f\cdot x} A^\mu\, e^{-ip_i\cdot x} \quad . \tag{10}$$

2.2.3 π^+K^+ Scattering

As a further example we now consider π^+K^+scattering and again evaluate the transition matrix element $S^{(1)}_{\pi^+K^+}$. Being a spin-0 particle, the K^+ meson obeys the same wave equation as the pion. Since π^+ and K^+ are distinguishable particles, they need not be symmetrized and exchange amplitudes do not have to be taken into accout. The scattering reaction can be described in the following manner. The electric charge of the K^+ creates a vector potential, by which the π^+ is scattered. First we have to determine this vector potential, because it enters the scattering amplitude (2.75).

A^μ obeys Maxwell's equations

$$\Box A^\mu - \partial^\mu(\partial_\nu A^\nu) = j^\mu_{\mathrm{em}} \quad . \tag{2.84}$$

Here j^μ_{em} denotes an electromagnetic current density which will be further specified later. It is well known that (2.84) can be simplified by choosing a specific gauge. One should remember that (2.84) remains invariant under gauge transformations of the form

$$A^\mu = A'^\mu - \partial^\mu \Lambda \quad , \tag{2.85}$$

with an arbitrary scalar function $\Lambda(x)$, i.e., A'^μ obeys the same equations (2.84) as A^μ. One can therefore always choose the gauge $\Lambda(x)$ in such a way that

$$\partial_\mu A^\mu = 0 \tag{2.86}$$

holds. Equation (2.86) is referred to as the *Lorentz condition*. By requiring condition (2.86) we have fixed a certain gauge and are now able to determine A^μ. In this Lorentz gauge the Maxwell equations reduce to

$$\Box A^\mu = j^\mu_{\mathrm{em}} \quad . \tag{2.87}$$

In order to derive the vector potential A^μ of K^+ mesons, the K^+ transition current has to be specified and inserted into the right hand side of (2.87). As already mentioned the K^+ is a Klein–Gordon particle just like the π^+ and we can therefore construct $j^\mu(K^+)$ in complete analogy to the pion current (2.82) or (2.83):

$$\begin{aligned} j^\mu(K^+) &= \mathrm{i}e\left[\varphi_4^*(\partial^\mu\varphi_2) - (\partial^\mu\varphi_4^*)\varphi_2\right] \\ &= eN_2N_4(p_2+p_4)^\mu \mathrm{e}^{\mathrm{i}(p_4-p_2)\cdot x} \quad . \end{aligned} \tag{2.88}$$

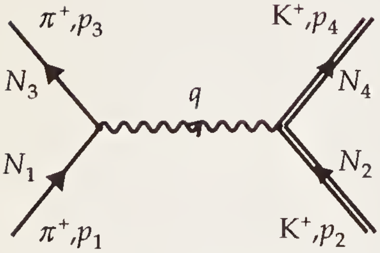


Fig. 2.7. A Feynman diagram for π^+K^+ scattering to lowest order (one-photon exchange). The normalization factors are also shown

The notation is explained in Fig. 2.7, which represents π^+K^+ scattering to lowest order. The formal solutions of (2.87) corresponding to the K^+ transition current (2.88) are

$$A^\mu = \Box^{-1} j^\mu(K^+) \quad , \tag{2.89}$$

where the inverse quabla operator is defined by

$$\Box^{-1}\Box = \mathbb{1} \quad . \tag{2.90a}$$

$\Box^{-1}$ can be identified by its action on a plane wave:

$$\Box^{-1}\left(\Box \mathrm{e}^{-\mathrm{i}q\cdot x}\right) = \Box^{-1}\left(-q^2 \mathrm{e}^{-\mathrm{i}q\cdot x}\right) = \mathrm{e}^{-\mathrm{i}q\cdot x} \quad , \tag{2.90b}$$

$$\Box^{-1}\mathrm{e}^{-\mathrm{i}q\cdot x} = -\frac{1}{q^2}\mathrm{e}^{-\mathrm{i}q\cdot x} \quad . \tag{2.90c}$$

Now the four-potential (2.89), which is created by the transition current (2.88), is readily obtained:

$$A^\mu(\mathrm{K}^+) = -\frac{1}{q^2} j^\mu(\mathrm{K}^+) = -\frac{1}{q^2} e N_2 N_4 (p_2 + p_4)^\mu \mathrm{e}^{\mathrm{i}q\cdot x} \quad . \tag{2.91}$$

Here the transferred four-momentum is

$$q^\mu = (p_4 - p_2)^\mu = (p_1 - p_3)^\mu \quad . \tag{2.92}$$

Inserting this result into the scattering amplitude of (2.81) leads to

$$\begin{aligned} S^{(1)}_{\mathrm{fi}}(\pi^+\mathrm{K}^+) &= -\mathrm{i}\int \mathrm{d}^4x\, j_\mu(\pi^+) A^\mu(\mathrm{K}^+) \\ &= \mathrm{i}\int \mathrm{d}^4x\, j_\mu(\pi^+) \frac{1}{q^2} j^\mu(\mathrm{K}^+) \\ &= \mathrm{i}e^2 N_1 N_2 N_3 N_4 (p_1 + p_3)_\mu \frac{1}{q^2}(p_2 + p_4)^\mu \int \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}(p_3 - p_1)\cdot x}\mathrm{e}^{\mathrm{i}(p_4 - p_2)\cdot x} \\ &= -N_1 N_2 N_3 N_4 (2\pi)^4 \delta^4(p_3 + p_4 - p_1 - p_2) \\ &\quad \times e(p_1 + p_3)_\mu \left(-\frac{\mathrm{i}g^{\mu\nu}}{q^2}\right) e(p_2 + p_4)_\nu \\ &= -\mathrm{i}N_1 N_2 N_3 N_4 (2\pi)^4 \delta^4(p_3 + p_4 - p_1 - p_2) F_{\mathrm{fi}} \quad . \end{aligned} \tag{2.93}$$

The last step includes the definition of the *reduced scattering amplitude* F_{fi}, which is mainly given by the current–current coupling connected with the photon propagator. Now we can interpret the factors occurring in this result (2.93):

1. Every external line in a Feynman graph yields a normalization factor N_i.
2. The mesons interact via exchange of a virtual photon, which is represented in the graph by a wavy line. This line corresponds to the photon propagator

   $$D_{\mu\nu}(q) = g_{\mu\nu} D(q^2) = -\mathrm{i}g_{\mu\nu}\frac{1}{q^2} \tag{2.94}$$

   in (2.93). The last step of (2.94) shows how the photon propagator is represented in graphs (diagrams). The square of the momentum transfer q^2 is often referred to as the squared mass of the virtual photon. This is completely analogous to $q^2 = m_0^2$, which holds for every four-momentum of a particle with rest mass m_0. A free photon obeys the homogeneous Maxwell equations

   $$\Box A^\mu = 0 \quad . \tag{2.95}$$

   This equation is only solved by a plane wave of the form $\exp(-iqx)$ if

   $$q^2 = 0 \tag{2.96}$$

   holds. But this condition shows that real photons are massless ($m_0 = 0$). Virtual photons, however, which are exchanged by electromagnetically interacting particles, are characterized by $q^2 \neq 0$, and referred to as *off mass shell*. We should emphasize that the form (2.94) of the photon propagator is only valid within Lorentz gauge. It is defined by

$$\Box D_{\mu\nu}(x-x') = -\mathrm{i} g_{\mu\nu}\delta^4(x-x') \tag{2.97}$$

according to the Maxwell equations in Lorentz gauge. We shall later discuss the form of the photon propagator in different gauges.

3. There are two vertices in the graph for π^+K^+ scattering. Since the virtual photon propagator $D_{\mu\nu}$ is a tensor with respect to Lorentz indices, there must be four-vectors on the left- and the right-hand sides, in order to produce a scalar. In the case of spinless mesons, however, there is only one characterizing four-vector, which is the four-momentum. This fact and the symmetry of the initial and final lines at the vertex explain the factors $e(p_1+p_3)$ and $e(p_2+p_4)$ in (2.93). The tensor character of the photon propagator is due to the photon being a spin-1 particle.
4. The transition currents in momentum space (vertices) as well as the photon propagator have been defined with factors ±i in a way that yields the correct sign also at higher orders. The main advantage of this convention is that scattering amplitudes for arbitrary graphs can immediately be constructed.
5. The four-momenta of the incoming and outgoing particles (the external lines in the graph) are subject to four-momentum conservation, which is taken into account by the factor $(2\pi)^4\,\delta^4(p_1+p_2-p_3-p_4)$.

2.2.4 The Cross Section

We have already mentioned that plane waves (2.60) with probability density (2.63a) are not normalized as usual to one particle per volume V but rather to $2E_i$ particles per volume V, i.e.,

$$\varrho_i = |N_i|^2\,2E_i \quad , \tag{2.98}$$

where we have made use of (2.66) once again. This makes sense because both ϱ and E are the zero components of four-vectors. We have already become acquainted with the covariant normalization of Dirac spinors and its advantages in describing meson–meson scattering processes. Of course, one has to choose flux and phase-space factors correspondingly, since the cross section must not depend on the normalization scheme. Here we employ the normalization

$$\int_V \mathrm{d}^3x\;\varrho_i = 2E_i \quad , \tag{2.99}$$

which leads to $N_i = 1/\sqrt{V}$. The transition probability per unit volume and unit time is given by

$$P_{\mathrm{fi}} = |S_{\mathrm{fi}}|^2/VT \quad . \tag{2.100}$$

Making use of the final result (2.93) and taking into account that the square of momentum functions , δ^4, can be expressed as

$$\begin{aligned}\left[(2\pi)^4\delta^4(p_3+p_4-p_1-p_2)\right]^2 &= (2\pi)^4\delta^4(p_3+p_4-p_1-p_2)(2\pi)^4\delta^4(0)\\ &= (2\pi)^4\delta^4(p_3+p_4-p_1-p_2)VT \quad ,\end{aligned} \tag{2.101}$$

we obtain

$$P_{\mathrm{fi}}^{(1)} = (2\pi)^4\,\delta^4(p_3+p_4-p_1-p_2)(N_1N_2N_3N_4)^2\,|F_{\mathrm{fi}}|^2 \quad . \tag{2.102}$$

This transition rate is still proportional to the flux of incoming particles and the density of target particles. In order to eliminate this dependence we have to divide by these two quantities. The beam particle flux (projectile quantities carry index 1, here, for example, π^+-particles) is defined as the number of incoming particles per unit area that can reach the target (K$^+$ particles, index 2) per unit time. The velocity of such beam particles is $v = v_1 - v_2$. The densities are normalized to $2E/V$ particles per unit volume (see (2.99)) and therefore the flux factor is

$$|j_1| = |v|\,\frac{2E_1}{V} \quad . \tag{2.103}$$

For the density of target particles we obtain

$$\varrho_2 = \frac{2E_2}{V} \quad . \tag{2.104}$$

If the target particles are at rest, (2.104) assumes the value $2m_2/V$. All together (2.102) has to be multiplied by the factor

$$\frac{1}{|v|}\frac{V}{2E_1}\frac{V}{2E_2} \quad . \tag{2.105}$$

In order to derive the cross section, one has to sum over all two-particle final states. If the volume V contains a particle, this yields an integration over the two-particle phase space with the voulume element

$$\frac{V}{(2\pi)^3}\mathrm{d}^3p_3\frac{V}{(2\pi)^3}\mathrm{d}^3p_4 \quad . \tag{2.106}$$

According to the normalization employed above, the phase space factor is

$$\frac{V}{(2\pi)^3}\frac{\mathrm{d}^3p_3}{2E_3}\frac{V}{(2\pi)^3}\frac{\mathrm{d}^3p_4}{2E_4} \quad , \tag{2.107}$$

which then yields the following cross section:

$$\begin{aligned}\mathrm{d}\sigma &= P_{\mathrm{fi}}\frac{V^2}{2E_12E_2\,|v|}\frac{V}{(2\pi)^3}\frac{\mathrm{d}^3p_3}{2E_3}\frac{V}{(2\pi)^3}\frac{\mathrm{d}^3p_4}{2E_4}\\ &= \frac{|F|^2}{2E_12E_2\,|v|}(2\pi)^4\,\delta^4(p_3+p_4-p_1-p_2)\frac{\mathrm{d}^3p_3}{(2\pi)^32E_3}\frac{\mathrm{d}^3p_4}{(2\pi)^32E_4} \quad .\end{aligned} \tag{2.108}$$

Since $N_i = 1/\sqrt{V}$, all factors V cancel and the cross section does not depend on the normalization volume. The Lorentz invariant form of the flux factor (see Exercise 2.2) is

$$E_1E_2\,|v| = \sqrt{(p_1\cdot p_2)^2 - m_1^2m_2^2} \quad . \tag{2.109}$$

This identity holds only for collinear collisions. For non collinear collisions only the relativistically invariant expression on the right-hand side of (2.109) remains valid. We therefore introduce the Lorentz-invariant phase-space factor

$$\mathrm{d\,Lips}(s;p_3,p_4) = (2\pi)^4\,\delta^4(p_3+p_4-p_1-p_2)\frac{1}{(2\pi)^3}\frac{d^3p_3}{2E_3}\frac{1}{(2\pi)^3}\frac{d^3p_4}{2E_4} \quad ,(2.110)$$

where

$$s = (p_1+p_2)^2 \tag{2.111}$$

denotes one of the so-called Mandelstam variables. Equation (2.108) then assumes the form

$$\mathrm{d}\sigma = \frac{|\,F\,|^2}{4\sqrt{(p_1\cdot p_2)^2 - m_1^2m_2^2}}\mathrm{d\,Lips}(s;p_3,p_4) \quad . \tag{2.112}$$

EXERCISE

2.2 The Flux Factor

Problem. Verify that in the center-of-mass system as well as in the laboratory system the flux factor $4E_1E_2|\boldsymbol{v}|$ is given by the invariant expression

$$4E_1E_2|\boldsymbol{v}| = 4\sqrt{(p_1p_2)^2 - m_1^2m_2^2} \quad .$$

Solution. In the center-of-mass system the momenta of projectile and target point in opposite directions, i.e.,

$$\begin{aligned} E_1 &= \sqrt{m_1^2+\boldsymbol{p}^2} \quad , \quad \boldsymbol{p}_1 = +\boldsymbol{p} \quad , \\ E_2 &= \sqrt{m_2^2+\boldsymbol{p}^2} \quad , \quad \boldsymbol{p}_2 = -\boldsymbol{p} \quad . \end{aligned} \tag{1}$$

Therefore $p_1p_2 = E_1E_2 + \boldsymbol{p}^2$, and we obtain

$$\begin{aligned} (p_1p_2)^2 - m_1^2m_2^2 &= (E_1E_2+\boldsymbol{p}^2)^2 - m_1^2m_2^2 \\ &= E_1^2(E_2^2-m_2^2) + 2E_1E_2\boldsymbol{p}^2 + \boldsymbol{p}^4 + (E_1^2-m_1^2)m_2^2 \\ &= \boldsymbol{p}^2(E_1^2 + 2E_1E_2 + E_2^2) \\ &= \boldsymbol{p}^2(E_1+E_2)^2 \\ &= (E_1E_2)^2\left|\frac{\boldsymbol{p}_1}{E_1} + \frac{\boldsymbol{p}_2}{E_2}\right|^2 \quad , \end{aligned} \tag{2}$$

which immediately leads to

$$\begin{aligned} 4\sqrt{(p_1p_2)^2 - m_1^2m_2^2} &= 4E_1E_2\left|\frac{\boldsymbol{p}_1}{E_1} - \frac{\boldsymbol{p}_2}{E_2}\right| \\ &= 4E_1E_2\,|\boldsymbol{v}_1 - \boldsymbol{v}_2| \quad . \end{aligned} \tag{3}$$

In the laboratory system the target particle is at rest and we have

Exercise 2.2.

$$E_1 = \frac{m_1}{\sqrt{1-v^2}} \quad , \quad \boldsymbol{p}_1 = \frac{m_1 \boldsymbol{v}}{\sqrt{1-v^2}}$$
$$E_2 = m_2 \quad , \quad \boldsymbol{p}_2 = \boldsymbol{0} \quad . \tag{4}$$

Then the scalar product $p_1 \cdot p_2$ is simply equal to $m_1 m_2 / \sqrt{1-v^2}$ and we obtain

$$\begin{aligned} 4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2} &= 4 m_1 m_2 \sqrt{\frac{1}{1-v^2} - 1} \\ &= 4 \frac{m_1 m_2}{\sqrt{1-v^2}} |\boldsymbol{v}| \\ &= 4 E_1 E_2 |\boldsymbol{v}| \quad . \end{aligned} \tag{5}$$

EXERCISE

2.3 The Mandelstam Variable s

Problem. Introduce the Mandelstam variable $s = (p_1 + p_2)^2$ and show that

$$4\left[(p_1 \cdot p_2)^2 - m_1^2 m_2^2\right] = \left[s - (m_1 + m_2)^2\right]\left[s - (m_1 - m_2)^2\right]$$

holds.

Solution. The Mandelstam variable s provides an invariant measure for the energy of the particles participating in the reaction. In the center-of-mass system, where one has $\boldsymbol{p}_1 + \boldsymbol{p}_2 = \boldsymbol{0}$, $\sqrt{s}$ is equal to the sum of all particle energies:

$$\begin{aligned} s = (p_1 + p_2)^2 &= p_1^2 + 2 p_1 \cdot p_2 + p_2^2 \\ &= m_1^2 + 2 p_1 \cdot p_2 + m_2^2 \quad , \end{aligned}$$

i.e.,

$$2 p_1 \cdot p_2 = s - m_1^2 - m_2^2 \quad . \tag{1}$$

The square of the flux factor then assumes the form

$$(2 p_1 \cdot p_2 - 2 m_1 m_2)(2 p_1 \cdot p_2 + 2 m_1 m_2) = \left[s - (m_1 + m_2)^2\right]\left[s - (m_1 - m_2)^2\right] \quad .$$

EXERCISE

2.4 The Lorentz-Invariant Phase-Space Factor

Problem. Show that the phase-space factor

$$\mathrm{d}\,\mathrm{Lips}(s;p_3,p_4) = (2\pi)^4\,\delta^4(p_3+p_4-p_1-p_2)\frac{1}{(2\pi)^3}\frac{d^3p_3}{2E_3}\frac{1}{(2\pi)^3}\frac{d^3p_4}{2E_4} \tag{1}$$

is Lorentz invariant.

Solution. The phase-space factor is apparently invariant under spacial rotations. We must therefore investigate its behavior under proper Lorentz transformations, which are induced by the matrix $\Lambda(v_b)$ with a *boost velocity* v_b. Owing to rotational invariance one can, without loss of generality, put v_b parallel to the z axis. This considerably simplifies the dependence of the particle coordinates in the system at rest p_n $(n = 1, 2, 3, 4)$ on the new coordinates p_n' in the moving reference system. In general this dependence is

$$p_n' = \Lambda(v_\mathrm{b})p_n \quad , \tag{2}$$

with the inversion

$$p_n = \Lambda^{-1}(v_\mathrm{b})p_n' = \Lambda(-v_\mathrm{b})p_n' \quad . \tag{3}$$

For the differentials $\mathrm{d}p^x_{n'}$, $\mathrm{d}p^y_{n'}$, $\mathrm{d}p^z_{n'}$ we obtain ($\gamma = 1/\sqrt{1-v_\mathrm{b}^2}$):

$$\begin{aligned} \mathrm{d}p_n^x &= \mathrm{d}p_n^{x\prime} \quad , \\ \mathrm{d}p_n^y &= \mathrm{d}p_n^{y\prime} \quad , \\ \mathrm{d}p_n^z &= \gamma\left(\mathrm{d}p_n^{z\prime} + |v_\mathrm{b}\mathrm{d}E_n'\right) \\ &= \gamma\mathrm{d}p_n^{z\prime}\left(1 + |v_\mathrm{b}|\frac{p_n^{z\prime}}{E_n'}\right) \\ &= \mathrm{d}p_n^{z\prime}\frac{E_n}{E_n'} \quad . \end{aligned} \tag{4}$$

Here we have employed the relations

$$E_n \equiv p_n^0 = (p_n^2 + m_n^2)^{\frac{1}{2}} = \gamma(E_n' + |v_\mathrm{b}|p_n^{z\prime}) \tag{5}$$

and

$$\frac{\mathrm{d}E_n'}{\mathrm{d}p_n^{z\prime}} = \frac{\mathrm{d}}{\mathrm{d}p_n^{z\prime}}\left[(p_n^{z\prime})^2 + (p_n^{y\prime})^2 + (p_n^{x\prime})^2\right]^{\frac{1}{2}} = \frac{p_n^{z\prime}}{E_n'} \quad . \tag{6}$$

For the volume element $\mathrm{d}^3p_n = \mathrm{d}p_n^x\mathrm{d}p_n^y\mathrm{d}p_n^z$ we therefore have

$$\frac{\mathrm{d}^3p_n}{E_n} = \frac{\mathrm{d}^3p_n'}{E_n'} \quad . \tag{7}$$

Exercise 2.4.

It remains to prove that the four-delta function is a Lorentz scalar. By definition we have

$$\int \mathrm{d}^4 p \, \delta^4(p) = 1 \tag{8}$$

in any reference frame. Owing to the properties of the matrix $\Lambda(v)$ the volume element is a Lorentz scalar too:

$$\mathrm{d}^4 p = \left| \frac{\partial p^\mu}{\partial p'^\nu} \right| \mathrm{d}^4 p' = |\det(\lambda(-v_\mathrm{b}))| \mathrm{d}^4 p' = \mathrm{d}^4 p' \quad . \tag{9}$$

EXAMPLE

2.5 $\pi^+\pi^+$ and $\pi^+\pi^-$ Scattering

As an example of the scattering of identical particles we study $\pi^+\pi^+$ scattering. The main modification compared to the $\pi^+\mathrm{K}^+$ scattering discussed above is the symmetrization of initial and final states. The total scattering amplitude has to be symmetric under exchange of the incoming or outgoing identical bosons. The direct graph of $\pi^+\pi^+$ scattering is depicted in Fig. 2.8.

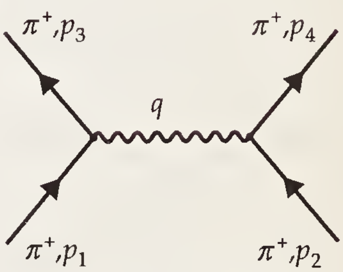


Fig. 2.8. A Feynman diagram for the direct amplitude of $\pi^+\pi^+$ scattering in the one-photon exchange approximation

Substituting the final state p_3 for p_4 and vice versa leads to the corresponding exchange graph (Fig. 2.9). The complete scattering amplitude, is then

$$S_{\mathrm{fi}}^{(1)}(\pi^+\pi^+) = S_{\mathrm{fi}}^{(1)}(\text{direct}) + S_{\mathrm{fi}}^{(1)}(\text{exchange}) \quad .$$

Employing (2.93) $S_{\mathrm{fi}}^{(1)}$ assumes the form

$$\begin{aligned} S_{\mathrm{fi}}^{(1)}(\pi^+\pi^+) &= -\mathrm{i}(2\pi)^4 \delta^4(p_3 + p_4 - p_1 - p_2) N_1 N_2 N_3 N_4 \\ &\quad \times \left[\frac{-e^2 (p_1 + p_3)_\mu (p_2 + p_4)^\mu}{(p_2 - p_4)^2} + \frac{-e^2 (p_1 + p_4)_\mu (p_2 + p_3)^\mu}{(p_2 - p_3)^2} \right] \\ &\equiv -\mathrm{i}(2\pi)^4 \delta^4(p_3 + p_4 - p_1 - p_2) \\ &\quad \times N_1 N_2 N_3 N_4 F_{\pi^+\pi^+}(p_1 p_2; p_3 p_4) \quad . \end{aligned} \tag{1}$$

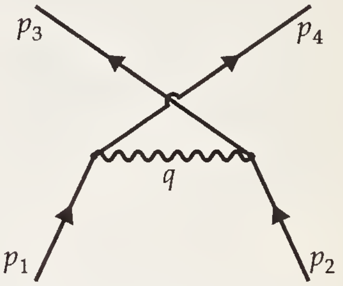


Fig. 2.9. The exchange amplitude of $\pi^+\pi^+$ scattering

Again $F_{\pi^+\pi^+}$ denotes the invariant scattering amplitude. All further steps can be performed in complete analogy to $\pi^+\mathrm{K}^+$ scattering.

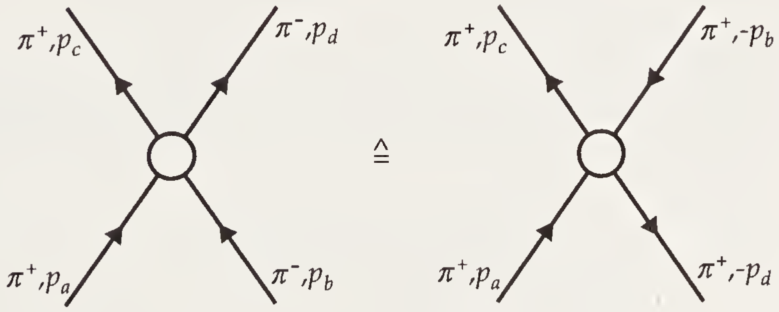


Fig. 2.10. The interrelation between $\pi^+\pi^+$ and $\pi^+\pi^-$ scattering

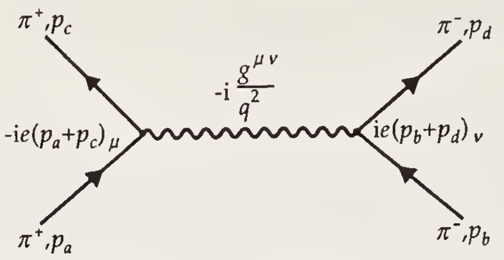


Fig. 2.11. The one-photon exchange amplitude for $\pi^+\pi^-$ scattering in detail

A similar argument holds for $\pi^+\pi^-$ scattering, if we take the antiparticle interpretation into account (cf. the discussion connected with (2.14–17)). We can therefore make the identification shown in Fig. 2.10. Owing to the antiparticle concept disussed earlier we can interpret an incoming π^- with four-momentum p_b as an outgoing π^+ with four-momentum $-p_\mathrm{b}$ and an outgoing π^- with four-momentum p_d as an incoming π^+ with four-momentum $-p_\mathrm{d}$. Hence we can immediately write down the invariant scattering amplitude for $\pi^+\pi^-$ scattering:

$$F_{\pi^+\pi^-}(p_\mathrm{a}p_\mathrm{b};p_\mathrm{c}p_\mathrm{d}) = F_{\pi^+\pi^+}(p_\mathrm{a}-p_\mathrm{d};p_\mathrm{c}-p_\mathrm{b}) \quad , \tag{2}$$

which explicitly is

$$\begin{aligned} F_{\pi^+\pi^-}(p_\mathrm{a}p_\mathrm{b};p_\mathrm{c}p_\mathrm{d}) = \Bigg[& \frac{-e^2(p_\mathrm{a}+p_\mathrm{c})_\mu(-p_\mathrm{d}-p_\mathrm{b})^\mu}{(-p_\mathrm{d}+p_\mathrm{b})^2} \\ & + \frac{-e^2(p_\mathrm{a}-p_\mathrm{b})_\mu(+p_\mathrm{c}-p_\mathrm{d})^\mu}{(p_\mathrm{a}+p_\mathrm{b})^2} \Bigg] \quad . \end{aligned} \tag{3}$$

A graphical representation of the two contributions more detailed than that given above is given in Fig. 2.11. The difference between the direct parts of the amplitudes for $\pi^+\pi^-$ scattering (or $\pi^+\mathrm{K}^+$ scattering) and $\pi^+\pi^+$ scattering is only that of sign if the corresponding four-momenta are considered equal (because of the different masses of K^+ and π^- they are in general not equal). The different signs of these two processes correspond to attractive and repulsive interaction, respectively.

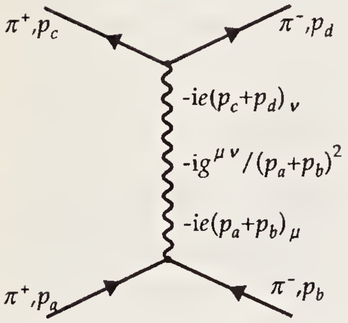


Fig. 2.12. The detailed exchange graph for $\pi^+\pi^-$ scattering. Obviously this amplitude can also be understood as one-photon $\pi^+\pi^-$ annihilation connected with $\pi^+\pi^-$ pair creation

EXERCISE

2.6 The Cross Section for Pion–Kaon Scattering

Problem. Derive the explicit form of the differential cross section for electromagnetic $\pi^+\mathrm{K}^+$ scattering in the center-of-momentum system (cms).

Solution. According to (2.112) the cross section for $\pi^+\mathrm{K}^+$ scattering is

$$\mathrm{d}\sigma = \frac{|F|^2}{4\sqrt{(p_1p_2)^2 - m_1^2m_2^2}}\,\mathrm{d\,Lips}(s;p_3p_4) \quad , \tag{1}$$

where

Exercise 2.6.

$$F = -\frac{e^2}{q^2}(p_1 + p_3)\cdot(p_2 + p_4) \quad ,$$
$$q^\mu = (p_3 - p_1)^\mu = -(p_4 - p_2)^\mu \tag{2}$$

and

$$\mathrm{d\,Lips}(s;p_3p_4) = (2\pi)^4\,\delta^4(p_3 + p_4 - p_1 - p_2)\frac{1}{(2\pi)^3}\frac{\mathrm{d}^3\mathrm{p}_3}{2\mathrm{E}_3}\frac{1}{(2\pi)^3}\frac{\mathrm{d}^3\mathrm{p}_4}{2\mathrm{E}_4} \tag{3}$$

denote the invariant scattering amplitude and the Lorentz-invariant phase-space factor, respectively. The cm system is defined by

$$\boldsymbol{p}_1 + \boldsymbol{p}_2 = \boldsymbol{p}_3 + \boldsymbol{p}_4 = \boldsymbol{0} \quad . \tag{4}$$

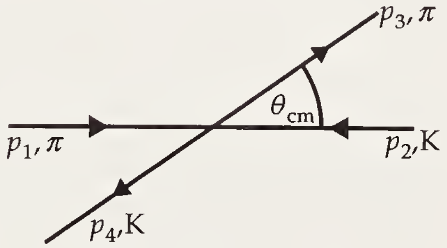


Fig. 2.13. $\pi^+\mathrm{K}^+$ scattering in the cm system

In this system the scattering process is described by the scattering angle θ_{cms} (see Fig. 2.13). Now we transform the four-momenta to the cm system,

$$\begin{aligned} p_1^\mu &= (E_1, \boldsymbol{p}) \quad , \quad p_2^\mu = (E_2, -\boldsymbol{p}) \quad , \\ p_3^\mu &= (E_3, \boldsymbol{p}') \quad , \quad p_4^\mu = (E_4, -\boldsymbol{p}') \quad , \end{aligned} \tag{5}$$

which leads to the total energy

$$\begin{aligned} E_{\mathrm{cms}} &= E_1 + E_2 = E_3 + E_4 \\ &= \sqrt{\boldsymbol{p}^2 + m_1^2} + \sqrt{\boldsymbol{p}^2 + m_2^2} \\ &= \sqrt{\boldsymbol{p}'^2 + m_1^2} + \sqrt{\boldsymbol{p}'^2 + m_2^2} \\ &= \sqrt{p^2 + m_1^2} + \sqrt{p^2 + m_2^2} \quad . \end{aligned} \tag{6}$$

Here and in the following we denote the absolute value of the spatial momentum by

$$|\boldsymbol{p}| = |\boldsymbol{p}'| \equiv p \quad . \tag{7}$$

By integrating the invariant phase-space factor $\mathrm{d\,Lips}(s;p_3p_4)$ over the spatial momenta d^3p_4 we obtain

$$\int \frac{\mathrm{d}^3p_4}{E_4}\,\delta^4(p_3 + p_4 - p_1 - p_2) = \frac{1}{E_4}\delta(E_3 + E_4 - E_1 - E_2) \quad . \tag{8}$$

The right-hand side of (8) contains E_4 as well as $|\boldsymbol{p}_4|$. These two variables, however, are not independent of each other: they are connected by

Exercise 2.6.

$$\boldsymbol{p}_4 = \boldsymbol{p}_1 + \boldsymbol{p}_2 - \boldsymbol{p}_3 \quad , \quad E_4 = \sqrt{\boldsymbol{p}_4 + m_2^2} \quad . \tag{9}$$

Next we transform $\mathrm{d}^3\mathrm{p}_3$ into spherical coordinates

$$\mathrm{d}^3 p_3 = p_3^2 \mathrm{d}p_3 \mathrm{d}\Omega \quad , \tag{10}$$

where $p_3 = |\boldsymbol{p}_3|$ and $\mathrm{d}\Omega$ denotes the spherical angle into which the π^+ is scattered. Because of

$$E_3^2 = p_3^2 + m_1^2 \tag{11}$$

we have

$$E_3 \mathrm{d}E_3 = p_3 \mathrm{d}p_3 \tag{12}$$

and

$$\mathrm{d}\,\mathrm{Lips}(s;p_3p_4) = \frac{1}{(4\pi)^2}\delta(E_3 + E_4 - E_1 - E_2)\frac{p_3 \mathrm{d}E_3}{E_4}\mathrm{d}\Omega \quad . \tag{13}$$

This formula is valid in any Lorentz system. Now we go into the cm system by making use of the relations

$$E_3^2 = p_2 + m_1^2 \quad , \quad E_4^2 = p^2 + m_2^2 \tag{14}$$

and

$$E_3 \mathrm{d}E_3 = E_4 \mathrm{d}E_4 = p\,\mathrm{d}p \quad . \tag{15}$$

Introducing the free variable

$$E' = E_3 + E_4 \quad , \tag{16}$$

we also have, according to (15)

$$\mathrm{d}E' = \frac{p}{E_3}\mathrm{d}p + \frac{p}{E_4}\mathrm{d}p = \frac{E'}{E_3E_4}p\,\mathrm{d}p = \frac{E'}{E_4}\mathrm{d}E_3 \quad . \tag{17}$$

Now (13) assumes the form

$$\mathrm{d}\,\mathrm{Lips}(s;p_3p_4)\Big|_{\mathrm{cms}} = \frac{1}{(4\pi)^2}\delta(E_{\mathrm{cms}} - E')\frac{p}{E'}\mathrm{d}E'\mathrm{d}\Omega \quad . \tag{18}$$

An integration over E' then yields

$$\mathrm{d}\,\mathrm{Lips}(s;p_3p_4)\Big|_{\mathrm{cms}} = \frac{1}{(4\pi)^2}\frac{p}{E_{\mathrm{cms}}}\mathrm{d}\Omega \tag{19}$$

for the two-particle phase-space factor in the cm system. Finally the flux factor has to be rewritten in terms of cm variables. With the help of (5) and (6) we immediately find that

$$\begin{aligned}\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2} &= \sqrt{(E_1E_2 + p^2)^2 - (E_1^2 - p^2)(E_2^2 - p^2)} \\ &= p\,E_{\mathrm{cms}} \quad .\end{aligned} \tag{20}$$

Exercise 2.6.

The result for the differential cross section is then

$$\left.\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right|_{\mathrm{cms}} = \frac{1}{(8\pi E_{\mathrm{cms}})^2}|F|^2 \quad . \tag{21}$$

By introducing the second Mandelstam variable t the result (21) can be brought into an invariant form. t is simply defined as the square of the four-momentum transfer:

$$t = q^2 = (p_1 - p_3)^2 = (p_2 - p_4)^2 \quad . \tag{22}$$

In the cm system we consequently have

$$\begin{aligned} t &= (0, \boldsymbol{p} - \boldsymbol{p}')^2 = -(\boldsymbol{p} - \boldsymbol{p}')^2 \\ &= -(\boldsymbol{p}^2 - 2\,\boldsymbol{p} \cdot \boldsymbol{p}' + \boldsymbol{p}'^2) \\ &= -2p^2\,(1 - \cos\theta_{\mathrm{cms}}) \quad , \end{aligned} \tag{23}$$

which leads to

$$\mathrm{d}t = 2p^2 \mathrm{d}\cos\left(\theta_{\mathrm{cms}}\right) \quad . \tag{24}$$

Since the cross section for spinless particles is cylindrically symmetric around the beam axis, i.e.

$$\mathrm{d}\Omega_{\mathrm{cms}} = 2\pi \mathrm{d}\cos\left(\theta_{\mathrm{cms}}\right) \quad , \tag{25}$$

we obtain the relation

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\pi}{p^2}\frac{\mathrm{d}}{\mathrm{d}\Omega_{\mathrm{cms}}} \quad . \tag{26}$$

Therefore the two-particle cross section is, in invariant form

$$\begin{aligned} \frac{\mathrm{d}\sigma}{\mathrm{d}t} &= \frac{1}{64\pi}\frac{1}{(pE_{\mathrm{cms}})^2}|F|^2 \\ &= \frac{1}{64\pi}\frac{|F|^2}{(p_1 \cdot p_2)^2 - m_1^2 m_2^2} \quad , \end{aligned} \tag{27}$$

where we have inserted (20). Finally we introduce the Mandelstam variable s (see Exercise 2.3) and take $p_1^2 = m_1^2$ and $p_2^2 = m_2^2$ into account. Equation (27) then assumes the form

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{1}{16\pi}\frac{|F|^2}{\left[s - (m_1 + m_2)^2\right]\left[s - (m_1 - m_2)^2\right]} \quad . \tag{28}$$

As already mentioned these expressions are valid for any scattering reaction with two unpolarized particles in both initial and final states. Now we want to express $|F|^2$ in the case of $\pi^+\mathrm{K}^+$ scattering by invariant Mandelstam variables. From the definitions

$$\begin{aligned} s &= (p_1 + p_2)^2 = (p_3 + p_4)^2 \quad , \\ u &= (p_1 - p_4)^2 = (p_2 - p_3)^2 \end{aligned} \tag{29}$$

we derive the relations

Exercise 2.6.

$$\begin{aligned} 2p_1\cdot p_2 &= 2p_3\cdot p_4 = s - m_1^2 - m_2^2 \quad , \\ 2p_1\cdot p_4 &= m_1^2 + m_2^2 - u \quad . \end{aligned} \tag{30}$$

With the help of these relations the invariant scattering amplitude (2) assumes the form

$$|F|^2 = \left(\frac{4\pi\alpha}{t}\right)(s-u)^2 \quad , \tag{31}$$

where we have transformed the fine-structure constant

$$\alpha = \frac{e^2}{4\pi} \simeq \frac{1}{137} \tag{32}$$

to so-called Heaviside–Lorentz units. In these units Gauss's law reads $\nabla\cdot\boldsymbol{E} = p$. It is particularly simple to transform (31) into the cm system, since we have, according to (29) and (5),

$$\begin{aligned} s &= (E_1 + E_2, \mathbf{0})^2 = E_{\mathrm{cms}}^2 \\ u &= (E_1 - E_4, \boldsymbol{p} + \boldsymbol{p}')^2 = (E_1 - E_4)^2 - (\boldsymbol{p} + \boldsymbol{p}')^2 \\ &= E_1^2 + E_4^2 - 2E_1E_4 - (E_1^2 - m_1^2) - (E_4^2 - m_2^2) - 2\boldsymbol{p}\cdot\boldsymbol{p}' \\ &= m_1^2 + m_2^2 - 2p^2\cos(\theta_{\mathrm{cms}}) - 2E_1E_4 \quad . \end{aligned} \tag{33}$$

The variable t has already been written down in (23).

2.2.5 Spin-1 Particles and Their Polarization

In the preceding sections on the basic elements of scalar QED we discussed the scattering of charged spin-0 mesons. Their mutual interaction is mediated by the exchange of virtual photons (massless spin-1 vector bosons). The kind of virtual quanta exchanged is of course specific for each interaction. For example in pion Compton scattering, virtual pions are exchanged (see Example 2.8), and the weak interaction in lepton scattering is mediated by vector bosons ($Z^0, W^{\pm}$). One difference between such particles is the number of internal degrees of freedom, which depends on their spin (or polarization).

2.2.5.1 Massive Spin-1 Particles. Massive spin-1 bosons are described in the framework of the Proca theory. From the Lagrangian density of the classical four-vector field $\phi^\mu(x)$

$$L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}M^2\phi_\mu\phi^\mu \quad , \tag{2.113a}$$

$$F^{\mu\nu} = \partial^\mu\phi^\nu - \partial^\nu\phi^\mu \quad , \tag{2.113b}$$

the wave equation follows:

$$\partial_\alpha F^{\alpha\mu} + M^2\phi^\mu = 0 \quad . \tag{2.114}$$

Taking the four-divergence of this equation, we find that

$$M^2 \partial_\mu \phi^\mu = 0 \quad . \tag{2.115}$$

As it is assumed that $M^2 \neq 0$ here, the divergence of ϕ^μ vanishes, and (2.114) is reduced to the Proca equation

$$\left(\Box + M^2\right) \phi^\mu = 0 \quad , \quad \partial_\mu \phi^\mu = 0 \quad . \tag{2.116}$$

We first consider the polarization vectors of these massive vector bosons. In the system of rest of such particles there are three possible positions for spin 1, i.e., three spin vectors, which can generally be chosen as

$$\begin{aligned} \varepsilon^{(1)} &= (1,0,0) \quad , \\ \varepsilon^{(2)} &= (0,1,0) \quad , \\ \varepsilon^{(3)} &= (0,0,1) \quad . \end{aligned} \tag{2.117a}$$

These obviously satisfy the orthogonality relations

$$\varepsilon^{(i)} \cdot \varepsilon^{(j)} = \delta^{ij} \quad . \tag{2.117b}$$

It is more useful to use the spherical representation $\varepsilon(\lambda)$ with

$$\begin{aligned} \varepsilon(\lambda = 1) &= -\frac{1}{\sqrt{2}}(1,\mathrm{i},0) \quad , \\ \varepsilon(\lambda = 0) &= (0,0,1) \quad , \\ \varepsilon(\lambda = -1) &= \frac{1}{\sqrt{2}}(1,-\mathrm{i},0) \quad , \end{aligned} \tag{2.118a}$$

and

$$\varepsilon^*(\lambda) \cdot \varepsilon(\lambda') = \delta_{\lambda\lambda'} \tag{2.118b}$$

instead of the Cartesian representation (2.117a).

In the massless case, i.e., for photons, this change of basis vectors corresponds to the transition from linearly to circularly polarized light. As is known from nonrelativistic quantum mechanics, $\lambda = \pm 1, 0$ is the projection of the particle's spin on, for example, the z axis.

We shall now formulate (2.118) in a manifestly covariant way. For a spin-1 particle in motion, four-vectors $\varepsilon^\mu(\lambda)$ must be found that transform into (2.118) in the rest frame. Setting in the rest frame

$$\varepsilon^0(\lambda) = 0 \quad , \qquad \text{for} \quad \lambda = 0, \pm 1 \tag{2.119}$$

and thus defining (in the rest frame)

$$\varepsilon^\mu(\lambda) = \left(\varepsilon^0(\lambda), \varepsilon(\lambda)\right) \quad , \tag{2.120}$$

the polarization vector in any other inertial system can be found by the Lorentz transformation. Since the four-momentum in the rest frame is given by

$$p^\mu = (M, \mathbf{0}) \quad , \tag{2.121}$$

it follows that

$$p \cdot \varepsilon(\lambda) = 0 \quad . \tag{2.122}$$

Equation (2.122) is basically a direct consequence of the condition $\partial_\mu \phi^\mu = 0$, since it reduces the number of relevant degrees of freedom to 3. Since a general free solution of the wave equation (2.116) can always be written as a superposition of linearly independent solutions in the form

$$\phi^\mu(x) = \sum_{\lambda=0,\pm1} \varepsilon^\mu(p;\lambda)\mathrm{e}^{-\mathrm{i}p\cdot x} \quad , \quad p^2 = M^2 \quad ,$$

(2.122) becomes evident. The normalization of polarization vectors is given by

$$\varepsilon^{*\mu}(p;\lambda)\varepsilon_\mu(p;\lambda') = -\delta_{\lambda\lambda'} \quad . \tag{2.123}$$

Let us consider, for example, a system in which the particle is moving along the z axis with momentum $\boldsymbol{p}$. Hence its four-momentum is

$$p^\mu = (E,0,0,p) \quad , \quad |\boldsymbol{p}| \equiv p \quad , \tag{2.124}$$

and we recognize immediately from (2.122) that the tranverse polarization vectors in this system are the same as those in the rest system,

$$\varepsilon^\mu(p;\lambda = \pm 1) \quad , \tag{2.125}$$

but that the longitudinal polarization vector must be given by

$$\varepsilon^\mu(p;\lambda = 0) = \frac{1}{M}(p,0,0,E) \quad , \tag{2.126}$$

in order to satisfy (2.122) and the normalization (2.123). It turns out that the p and E dependence of the longitudinal polarization vector have very interesting consequences for massive spin-1 particles such as the Z and W$^\pm$ bosons mediating the weak interaction. The polarization vectors $\varepsilon^\mu(p;\lambda)$ satisfy the completeness relation

$$\sum_\lambda \varepsilon^{\mu^*}(p;\lambda)\varepsilon^\nu(p;\lambda) = -g^{\mu\nu} + \frac{p^\mu p^\nu}{M^2} \quad , \tag{2.127}$$

which we shall prove in Exercise 2.7. The factor on the right-hand side projects out the physical states and appears, as we shall see, in the propagator of virtual spin-1 particles.

EXERCISE

2.7 Polarization States of a Massive Spin-1 Particle

Problem. The polarization vectors of a massive spin-1 particle with four-momentum p^μ and helicity λ are denoted by $\varepsilon^\mu(p;\lambda)$. It holds that

$$\varepsilon^{\mu^*}(p;\lambda)\varepsilon_\mu(p;\lambda') = -\delta_{\lambda\lambda'} \quad . \tag{1}$$

The minus sign occurs because these vectors are spacelike. Owing to Lorentz covariance, the sum over the polarization states

$$\sum_\lambda \varepsilon^*_\mu(p;\lambda)\varepsilon_\nu(p;\lambda) \equiv \eta_{\mu\nu}(p) \tag{2}$$

has to be of the form

$$\eta_{\mu\nu}(p) = Ag_{\mu\nu} + Bp_\mu p_\nu \quad . \tag{3}$$

Find arguments for this fact and determine the constants A and B. Make use of scalar multiplications by p^μ, p^ν, and $g^{\mu\nu}$. Note that $g^{\mu\nu}g_{\mu\nu} = 4$.

Solution. The condition $\partial_\mu\phi^\mu = 0$ for the wave function of a spin-1 particle leads to $p^\mu\varepsilon_\mu(p;\lambda) = 0$. p^μ and $\varepsilon^\mu(p;\lambda)$ are a system of four linearly independent, orthogonal vectors; i.e., any four-vector a^μ can be represented as a linear combination

$$a^\mu = a_p p^\mu + \sum_\lambda a_\lambda \varepsilon^\mu(p;\lambda) \quad . \tag{4}$$

Now we evaluate

$$\begin{aligned} a^\mu\eta_{\mu\nu}(p) &= \sum_\lambda \left[a_p p^\mu \varepsilon^*_\mu(p;\lambda) + \sum_{\lambda'} a_{\lambda'}\varepsilon^{*\mu}(p;\lambda)\varepsilon_\mu(p;\lambda')\right]\varepsilon_\nu(p;\lambda) \\ &= \sum_\lambda \left[a_p \cdot 0 + \sum_{\lambda'}(-\delta_{\lambda\lambda'})a_{\lambda'}\right]\varepsilon_\nu(p;\lambda) \\ &= -\sum_\lambda a_\lambda \varepsilon_\nu(p;\lambda) \quad . \end{aligned} \tag{5}$$

This result does not depend on the specific choice of the $\varepsilon_\mu(p;\lambda)$ and shows that $-\eta_{\mu\nu}(p)$ eliminates the part of a given four-vector that is proportional to p^μ. Therefore $\eta_{\mu\nu}(p)$ can only depend on p^μ, i.e., $a^\perp_\nu(p) = -a^\mu\eta_{\mu\nu}(p)$. Since the $\varepsilon_\mu(p;\lambda)$ are four-vectors, the polarization sum $\eta_{\mu\nu}(p)$ is transformed like a second rank tensor. Any symmetric tensor of second rank that is built by a four-vector p^μ is of the general form

$$\eta_{\mu\nu}(p) = A(p^2)g_{\mu\nu} + B(p^2)p_\mu p_\nu \quad . \tag{6}$$

Exercise 2.7.

There are no other possibilities, because the only Lorentz-covariant quantities available are $g_{\mu\nu}$, p_μ, and p^2. Here we have $p^2 = M^2 =$ const., and therefore A and B must be constants. Multiplying the polarization sum by p^μ yields

$$\begin{aligned} p^\mu \eta_{\mu\nu} &= \sum_\lambda p \cdot \varepsilon^*(p;\lambda)\varepsilon_\nu(p;\lambda) = 0 \\ &\rightarrow p^\mu(A g_{\mu\nu} + B p_\mu p_\nu) = (A + Bp^2)p_\nu = 0 \\ &\rightarrow B = -\frac{A}{M^2} \quad . \end{aligned} \tag{7}$$

Hence the polarization sum assumes the form $A(g_{\mu\nu} - p_\mu p_\nu / M^2)$. A contraction with $g^{\mu\nu}$ leads to

$$\begin{aligned} g^{\mu\nu}\eta_{\mu\nu}(p) = \eta^\mu{}_\mu(p) &= \sum_\lambda \varepsilon^{\mu^*}(p;\lambda)\varepsilon_\mu(p;\lambda) \\ &= \sum_\lambda (-\delta_{\lambda\lambda}) = -3 \end{aligned} \tag{8}$$

$$\begin{aligned} \rightarrow \eta^\mu{}_\mu(p) = A g^\mu{}_\mu + Bp^2 &= A\left(g^\mu{}_\mu - \frac{p^2}{M^2}\right) \\ &= A(4-1) = 3A \\ \rightarrow A = -1 \quad . \end{aligned} \tag{9}$$

The final result is then

$$\sum_\lambda \varepsilon^*_\mu(p;\lambda)\varepsilon_\nu(p;\lambda) = -\left(g_{\mu\nu} + \frac{p_\mu p_\nu}{M^2}\right) \quad . \tag{10}$$

2.2.5.2 Massless Spin-1 Particles: Photons. Photons do not possess a rest system as massive vector bosons do, since from $p^2 = 0 \leftrightarrow |E| > 0$ we immediately get $\boldsymbol{p} \neq 0$ in any inertial frame. It is therefore impossible to proceed as for massive spin-1 particles, formulating the polarization vectors in the rest frame and then obtaining them in any frame by a Lorentz transformation, which was the method just discussed. To find the number of relevant internal degrees of freedom of the photon field we will use gauge invariance, which holds for massless, but not for massive, vector bosons. It will turn out that real photons have only two transverse polarization degrees of freedom.

We first summarize. For a free photon field, the wave equation and the Lorentz condition are

$$\Box A^\mu = 0 \quad , \tag{2.128a}$$

$$\partial_\mu A^\mu = 0 \quad . \tag{2.128b}$$

This auxiliary condition can be always satisfied in a special gauge – the Lorentz gauge – and only in this gauge does the wave equation (2.128) have this simple form. For $p^2 = 0$ (real photons) its solutions are plane waves

$$A^\mu = N\varepsilon^\mu \mathrm{e}^{-\mathrm{i}p\cdot x} \quad , \tag{2.129}$$

where N is a normalization factor and ε^μ the polarization vector of the photon. The Lorentz condition (2.128) leads immediately to the condition

$$p \cdot \varepsilon = 0 \tag{2.130}$$

for the polarization vector. Equations (2.128) and (2.130) are analogous to (2.116) and (2.122), derived for massive vector bosons. There $\partial_\mu \phi^\mu$ followed directly from the field equations (2.114) and $p \cdot \varepsilon = 0$ was first derived in the rest frame and then recognized to be covariant in general. Now (2.128) and (2.130) follow from an arbitrary choice of gauge for the photon field. The Lorentz condition (2.128) reduces the number of internal degrees of freedom to three. But as we shall see in the following, the gauge condition reduces this number to two.

In the Lorentz gauge, one can still perform the symmetry transformation

$$A^\mu \longrightarrow A^\mu - \partial^\mu \Lambda = A'^\mu \quad , \tag{2.131}$$

provided Λ satisfies the Klein–Gordon equation for the massless scalar field,

$$\Box \Lambda = 0 \quad . \tag{2.132}$$

Such regauging obviously does not change the Lorentz condition (2.128). For the plane waves (2.129), the regauging amounts to changing the polarization vector ε^μ by a multiple of p^μ:

$$\begin{aligned} A^\mu - \partial^\mu \Lambda &= N \varepsilon^\mu \mathrm{e}^{-\mathrm{i}p\cdot x} - \alpha \partial^\mu \mathrm{e}^{-\mathrm{i}p\cdot x} \\ &= N\left(\varepsilon^\mu + \beta p^\mu\right) \mathrm{e}^{-\mathrm{i}p\cdot x} \quad , \end{aligned} \tag{2.133}$$

that is,

$$\varepsilon^\mu \longrightarrow \varepsilon'^\mu = \varepsilon^\mu + \beta p^\mu \quad . \tag{2.134}$$

Moreover, the condition

$$\varepsilon' \cdot p = 0 \tag{2.135}$$

still holds since $\partial_\mu A'^\mu = 0$ holds as well. The freedom expressed by (2.134) has profound consequences. To illustrate this, we consider a photon with the four-momentum

$$p^\mu = \left(p^0, \boldsymbol{p}\right) \tag{2.136}$$

and the polarization vector

$$\varepsilon^\mu = \left(\varepsilon^0, \boldsymbol{\varepsilon}\right) \quad , \tag{2.137}$$

which satisfy the Lorentz condition $\varepsilon \cdot p = 0$. Gauge invariance now allows us to add according to (2.134) any multiple of p^μ to ε^μ, still obtaining admittable polarization vectors. We can therefore always choose the gauge such that the time component of ε^μ vanishes in (2.137) and the four-dimensional equation (2.135) is reduced to the three-dimensional equation

$$\boldsymbol{\varepsilon} \cdot \boldsymbol{p} = 0 \quad . \tag{2.138}$$

This means that there are only two linearly independent polarization vectors for photons (massless bosons). For a plane photon wave propagating in the z direction, we can choose $\varepsilon^{(1)}$ and $\varepsilon^{(2)}$ from (2.117a) for linearly polarized photons and $\varepsilon(\lambda = +1)$ and $\varepsilon(\lambda = -1)$ for circularly polarized photons.

This is the result – known from classical electrodynamics – that the electric and magnetic field strengths

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

are purely transverse. Although photons are described by a vector field A^μ and thus must have spin 1, there are only two independent spin projections and not three, as one might naively expect. As we have seen, this result is rooted in the photons having no mass. If the wave equation (2.128) had a mass term, as in the Proca equation (2.116), the theory would no longer be gauge invariant and gauge freedom would be lost.

We thus describe incoming photons with four-momentum p and polarization state λ by the wave function

$$A^\mu = N\varepsilon^\mu(\lambda)\mathrm{e}^{-\mathrm{i}p\cdot x} \quad , \quad \lambda = \pm 1 \quad , \tag{2.139}$$

and outgoing photons by

$$A^\mu = N\varepsilon^{*\mu}(\lambda)\mathrm{e}^{\mathrm{i}p\cdot x} \quad , \quad \lambda = \pm 1 \quad . \tag{2.140}$$

As in the (2.123) for massive spin-1 particles, the orthogonality

$$\varepsilon^*(\lambda)\cdot\varepsilon(\lambda') = -\delta_{\lambda\lambda'} \tag{2.141}$$

holds. Normalizing in the same way as with the pion wave function, we obtain

$$N = \frac{1}{\sqrt{V}} \tag{2.142}$$

as the normalization factor.

In a certain sense we can associate the gauge system where (2.138) holds, i.e., where photons are tranverse, to the rest system (2.118, 119) for particles with a mass. In this rest system $\varepsilon(\lambda)\cdot \boldsymbol{p} = 0$ also holds, since the momentum $\boldsymbol{p}$ does vanish there.

2.2.6 The Propagator for Virtual Photons

In view of the different fermion–boson scattering processes that will be derived in the next section, we now proceed to derive the pion propagator. This derivation is completely analogous to that for the photon propagator. The equation corresponding to (2.87) is

$$\left(\Box + m_0^2\right)\phi(x) \equiv -J(x) \quad . \tag{2.143}$$

The Green function for this inhomogeneous Klein–Gordon equation is

$$\left(\Box + m_0^2\right)G(x-x') = -\mathrm{i}\delta^4(x-x') \quad . \tag{2.144}$$

By Fourier transformation, we obtain

$$G(p^2) = \frac{\mathrm{i}}{p^2 - m_0^2 + \mathrm{i}\varepsilon} \qquad (2.145)$$

where $p^2 \neq m_0^2$ has been assumed for virtual pions. We refer to standard QED textbooks for the detailed proof that the propagator is given by (2.145) and that the Feynman interpretation of waves with positive and negative energy corresponds to the $+\mathrm{i}\varepsilon$ prescription for treating the poles in (2.145).

EXAMPLE

2.8 Compton Scattering by Pions

Elastic photon scattering by a charged particle is called Compton scattering. In the case of a pion this process is written as

$$\gamma + \pi^+ \rightarrow \gamma + \pi^+ \quad .$$

First we consider the direct graph (symmetrization is necessary!), which is depicted in Fig. 2.14a. The lower vertex, which is separately shown in Fig. 2.14b, corresponds to the absorption of a photon with four-momentum k_1 and polarization ε_1 and to the transition of the pion state from p_1 to q. The corresponding scattering amplitude is proportional to (cf. (2.81))

$$\sim \mathrm{i}e \int \mathrm{d}^4x\, \phi_\mathrm{f}^*(\pi^+, q)(\partial_\mu A^\mu + A^\mu \partial_\mu)\phi_\mathrm{i}(\pi^+, p_1) \quad . \qquad (1)$$

Inserting the plane waves

$$\begin{aligned} \phi_\mathrm{i} &\approx \mathrm{e}^{-\mathrm{i}p_1\cdot x} \quad , \\ \phi_\mathrm{f}^* &\approx \mathrm{e}^{\mathrm{i}q\cdot x} \quad , \\ A^\mu &\approx \varepsilon_1^\mu \mathrm{e}^{-\mathrm{i}k_1\cdot x} \end{aligned} \qquad (2)$$

into (1) leads to a vertex amplitude proportional to

$$F_1 \approx e(p_1 + q)_\mu \varepsilon_1^\mu \quad , \qquad (3)$$

where

$$q = p_1 + k_1 \qquad (4)$$

denotes the four-momentum of the virtual photon. Energy conservation is ensured by the delta function

$$\delta^4(q - p_1 - k_1) \quad . \qquad (5)$$

An analogous procedure for the upper vertex in Fig. 2.14a (cf. Fig. 2.14c) leads to the vertex amplitude

$$F_2 \approx e(q + p_2)_\nu \varepsilon_2^{\nu *} \qquad (6)$$

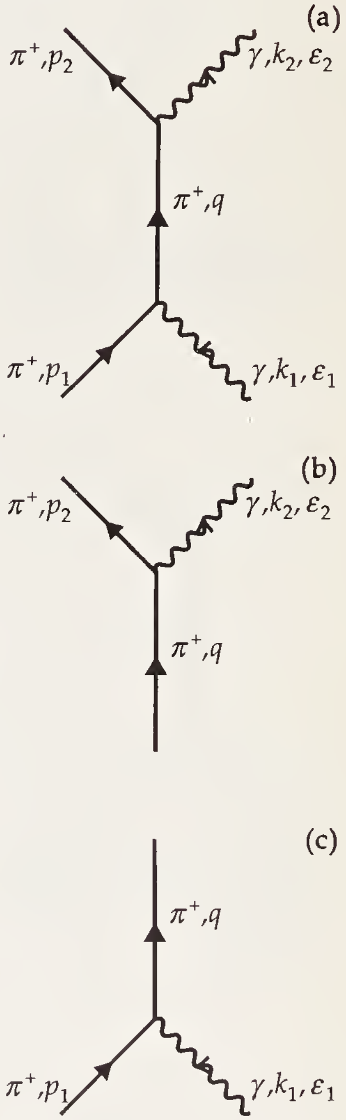

(a)
π^+, p_2
$\gamma, k_2, \varepsilon_2$
π^+, q
π^+, p_1
$\gamma, k_1, \varepsilon_1$
(b)
π^+, p_2
$\gamma, k_2, \varepsilon_2$
π^+, q
(c)
π^+, q
π^+, p_1
$\gamma, k_1, \varepsilon_1$

Fig. 2.14a–c. The direct graph of $\pi^+\gamma$ Compton scattering. The vertices of $\pi^+\gamma$ Compton scattering

Example 2.8.

with four-momentum conservation,

$$q = p_2 + k_2 \quad , \quad \delta^4(p_2 + k_2 - q) \quad . \tag{7}$$

Putting graphs 2.14b and 2.14c together we obtain the complete direct graph 2.14a. Formally this connection is achieved by inserting the virtual photon propagator

$$G(q^2) = \frac{\mathrm{i}}{q^2 - m_0^2} \tag{8}$$

between the scattering amplitudes (3) and (6). The invariant scattering amplitude for the direct graph of Compton scattering by a pion is then

$$F_{\gamma\pi}^{(\mathrm{d})} = e^2 \varepsilon_1 \cdot (p_1 + q) \frac{1}{q^2 - m_0^2} \varepsilon_2^* \cdot (q + p_2) \quad , \tag{9}$$

where energy conservation is ensured by $\delta^4(p_2 + k_2 - p_1 - k_1)$. Taking into account the Lorentz-gauge condition

$$\varepsilon_1 \cdot k_1 = 0 \quad , \qquad \varepsilon_2 \cdot k_2 = 0 \tag{10}$$

and introducing the Mandelstam variable

$$s = q^2 = (p_1 + k_1)^2 = (p_2 + k_2)^2 \quad , \tag{11}$$

* we can easily transform (9) into

$$F_{\pi\gamma}^{(\mathrm{d})} = \frac{4e^2(\varepsilon_1 \cdot p_1)(\varepsilon_2^* \cdot p_2)}{s - m_0^2} \quad . \tag{12}$$

Now we have to symmetrize with respect to the two photons (or the two pions). The resulting exchange graph is shown in Fig. 2.15. First the outgoing photon k_2, ε_2 is emitted and later the incoming photon k_1, ε_1 is absorbed. In analogy to (9) the scattering amplitude for this process is

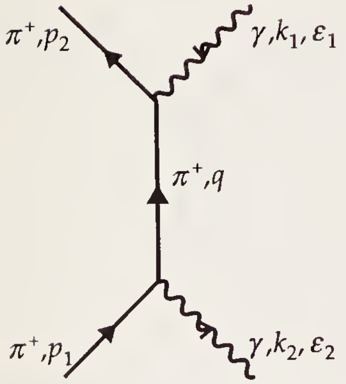


Fig. 2.15. The exchange graph for $\pi^+\gamma$ Compton scattering

$$F_{\gamma\pi}^{(\mathrm{e})} = e^2 \varepsilon_1 \cdot (q' + p_2) \frac{1}{q'^2 - m_0^2} \varepsilon_2^* \cdot (q' + p_1) \quad , \tag{13}$$

with

$$q' = p_1 - k_2 = p_2 - k_1 \quad . \tag{14}$$

If we employ the Mandelstam variable

$$u = (p_1 - k_2)^2 = (p_2 - k_1)^2 = q'^2 \quad , \tag{15}$$

which represents the squared mass of the virtual pion in the exchange graph, and take (10) into account, the amplitude (13) assumes the form

$$F_{\gamma\pi}^{(\mathrm{e})} = \frac{4e^2(\varepsilon_1 \cdot p_2)(\varepsilon_2^* \cdot p_1)}{u - m_0^2} \quad . \tag{16}$$

This exchange amplitude is sometimes called the u-channel contribution (amplitude).

Now the question arises whether there are contributions to Compton scattering that are of order e^2 caused by the squared interaction term

$$-e^2 A_\mu A^\mu \quad . \tag{17}$$

This interaction will directly lead to graphs of the form shown in Fig. 2.16, i.e., to vertices with two photon lines. Such contributions are called contact terms. They can be interpreted using gauge invariance arguments. To this end we first consider the process shown in Fig. 2.17, where the initial state consists of one photon and one particle and the final state of two particles. One photon is absorbed and two particles are emitted in the final channel. Of course, one of the outgoing particles can again be a photon, i.e., the Compton scattering treated above is included in such a process. The corresponding amplitude (see Fig. 2.17) must be linear in the polarization ε^μ and can therefore be factorized into

$$A = \varepsilon^\mu T_\mu \quad , \tag{18}$$

where T_μ contains all the details of the process. In Lorentz gauge, the condition

$$\varepsilon \cdot k = 0$$

must hold. As we have already discussed in Sect. 2.2.5.2 an additional gauge transformation

$$\varepsilon'^\mu = \varepsilon^\mu + \beta k^\mu$$

may be performed without changing any physical results. This transformation corresponds to an additional gauge transformation within the Lorentz gauge. Since the amplitude (18) must be gauge invariant, we are lead to the condition

$$k_\mu T^\mu = 0 \quad . \tag{19}$$

The total scattering amplitude for $\gamma\pi^+$ Compton scattering derived above can be written as

$$\begin{aligned} F_{\gamma\pi} &= F^{(\mathrm{d})}_{\gamma\pi} + F^{(\mathrm{e})}_{\gamma\pi} \\ &= 4e^2 \varepsilon_1^\mu \varepsilon_2^{\nu *} \left(\frac{p_{1\mu} p_{2\nu}}{s - m_0^2} + \frac{p_{2\mu} p_{1\nu}}{u - m_0^2} \right) \\ &\equiv \varepsilon_1^\mu \varepsilon_2^{\nu *} T_{\mu\nu} \quad . \end{aligned} \tag{20}$$

According to the transition from (18) to (19) we have to replace ε_1 by k_1 and evaluate

$$k_1^\mu \varepsilon_2^{\nu *} T_{\mu\nu} \quad .$$

The scalar products thar occur,

$$2p_1 \cdot k_1 = s - m_0^2 \ , \quad 2p_2 \cdot k_1 = -(u - m_0^2) \quad , \tag{21}$$

then give

$$\begin{aligned} k_1^\mu \varepsilon_2^{\nu *} T_{\mu\nu} &= 2e^2 \varepsilon_2^{\nu *} (p_2 - p_1)_\nu \\ &= 2e^2 \varepsilon_2^* \cdot k_1 \neq 0 \quad . \end{aligned} \tag{22}$$

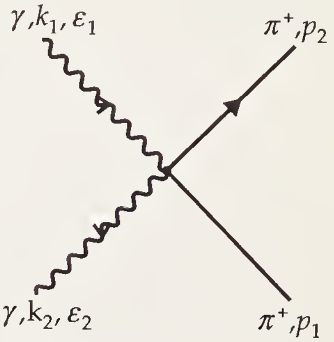


Fig. 2.16. The four-point contact graph for $\gamma\pi^+ \to \gamma\pi^+$

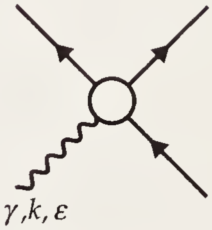


Fig. 2.17. The graph of a general one-photon process

Example 2.8.

This result is surprising, because the total scattering amplitude (20) derived so far explicitly violates condition (19). If we also replace ε_2 by k_2 in (22), we obtain

$$k_1^\mu k_2^\nu T_{\mu\nu} = 2e^2 k_1 \cdot k_2 \neq 0 \quad . \tag{23}$$

Again this result is not equal to zero. The only possible explanation for this observation is that the amplitude (20) is not gauge invariant. But where did we make a mistake? We wanted to evaluate a process of second order in e, but so far we have not taken into account the contact graphs according to the interaction (17), which are of the same order. In order to restore gauge invariance and fulfill conditions (19) T_μ (or $T_{\mu\nu}$ in (20), respectively) must contain all the interactions of a given order. If the coupling constant e is interpreted as a variable quantity (which then assumes some fixed value), gauge invariance must separately be fulfilled in every order in e. We therefore expect an additional term $F_{\gamma\pi}^{(c)}$ for the complete Compton scattering amplitude, i.e.,

$$F_{\gamma\pi} = F_{\gamma\pi}^{(e)} + F_{\gamma\pi}^{(c)} \quad , \tag{24}$$

where the superscript (c) denotes the contact term. Also $F_{\gamma\pi}^{(c)}$ must be linear in ε_1 and ε_2^* and by means of the replacements

$$\varepsilon_1 \to k_1 \quad , \quad \varepsilon_2^* \to k_2 \tag{25}$$

it must yield (22) or (23) with the opposite sign. Only in this way can the gauge invariance of the scattering amplitude (24) be ensured. Apparently

$$F_{\gamma\pi}^{(c)} = -2e^2 \varepsilon_1 \cdot \varepsilon_2^* \tag{26}$$

must hold. This scattering amplitude is caused by the quadratic interaction. One must understand that if we had in general ignored the interaction $-e^2A^2$, gauge invariance in second order would have demanded its existence. This is a first example of the power of gauge symmetry. A further comment on (26): this term is the only one that is linear in ε_1 and ε_2^* and up to a sign equal to (22) and (23). There are no other terms fulfilling these conditions!

Taking into account (20), (24), and (26), we get for the total invariant scattering amplitude for Compton scattering by a pion

$$F_{\gamma\pi} = e^2 \varepsilon_1^\mu \varepsilon_2^{\nu *} \left(\frac{4p_{1\mu}p_{2\nu}}{s - m_0^2} + \frac{4p_{2\mu}p_{1\nu}}{u - m_0^2} - 2g_{\mu\nu} \right) \quad . \tag{27}$$

The factor 2 in front of the $g_{\mu\nu}$ term is plausible, because each factor A_μ in A^2 can represent one absorption and one emission process. Multiplying (27) by the four-momentum conservation $(2\pi)^4\,\delta^4(p_2 + k_2 - p_1 - k_1)$ and normalization factors then yields the complete Compton S-matrix element.

2.3 Fermion–Boson and Fermion–Fermion Scattering

Example 2.8.

In this section we discuss a number of problems and examples to review and deepen our knowledge of QED. We shall encounter well-known subjects in new forms and also gain new insights. Our notation will get closer to that employed in high-energy physics.

EXERCISE

2.9 Elastic $e^-\pi^+$ Scattering (I)

Problem. Determine the scattering amplitude and explain the formal steps necessary to evaluate the cross section.

Solution. The graph for $e^-\pi^+$ scattering is of the following form (see Fig. 2.18). Most of the above notation is readily understood. Only the factor $+\mathrm{i}$ in the transition current at the electron vertex, which has been denoted by $(\mathrm{i}e\gamma_\mu)$, needs additional explanation. Reviewing our knowledge of QED, we start with the scattering amplitude (2.50), i.e., with

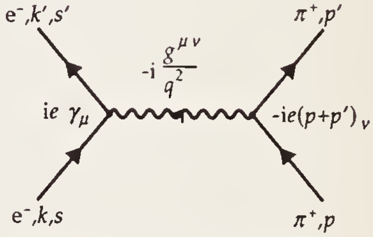


Fig. 2.18. The Feynman diagram for $e^-\pi^+$ scattering in the one-photon exchange approximation

$$S^{(1)}_{\mathrm{fi}} = -\mathrm{i}\int \mathrm{d}^4x\, j_\mu(\mathrm{e}^-)A^\mu \quad , \tag{1}$$

where according to (2.52)

$$j_\mu(\mathrm{e}^-) = (-e)NN'\,\bar{u}(k',s')\,\gamma_\mu\, u(k,s)\,\mathrm{e}^{\mathrm{i}(k'-k)\cdot x} \tag{2}$$

denotes the electron transition current density. The electromagnetic four-potential A^μ in (1) is created by the pion transition current density

$$j^\mu(\pi^+) = (+e)\bar{N}\bar{N}'(p+p')^\mu\,\mathrm{e}^{\mathrm{i}(p'-p)\cdot x} \quad . \tag{3}$$

According to (2.91) we have

$$A^\mu = -\frac{1}{q^2}j^\mu(\pi^+) \quad , \tag{4}$$

with the four-momentum transfer

$$q = p' - p = k - k' \quad . \tag{5}$$

The scattering amplitude (1) in detail is then

$$\begin{aligned} S^{(1)}_{\mathrm{fi}} &= -\mathrm{i}\int \mathrm{d}^4x\, j_\mu(e^-)\left(-\frac{1}{q^2}\right)j^\mu(\pi^+) \\ &= \mathrm{i}NN'\bar{N}\bar{N}'\int \mathrm{d}^4x\left[\bar{u}(k',s')\gamma_\mu u(k,s)\,\mathrm{e}^{\mathrm{i}(k'-k)\cdot x}\right]\left(-\frac{e^2}{q^2}\right)\left[(p+p')^\mu\,\mathrm{e}^{\mathrm{i}(p-p')\cdot x}\right] \\ &= -\mathrm{i}NN'\bar{N}\bar{N}'(2\pi)^4\,\delta^4(k'+p'-k-p)(-e)\bar{u}(k',s')\gamma_\mu u(k,s) \\ &\quad\times\left(-\frac{g^{\mu\nu}}{q^2}\right)(+e)(p+p')_\nu \\ &= -\mathrm{i}NN'\bar{N}\bar{N}'(2\pi)^4\,\delta^4(p'+k'-p-k)F_{ss'}(kp;k'p') \quad . \end{aligned} \tag{6}$$

Example 2.9.

In the last step we have introduced the invariant scattering amplitude

$$F_{ss'}(kp;k'p') = (-e)\left[\bar{u}(k',s')\gamma_\mu u(k,s)\right]\left(-\frac{g^{\mu\nu}}{q^2}\right)(+e)\left[(p+p')_\nu\right] \quad . \tag{7}$$

Only the spin variables occur in addition. Now we see from (6) that the above Feynman rules yield the correct total sign for the scattering amplitude if a factor +i is assigned to the vertex of the leptonic transition current. Figure 2.18 already contains this factor. Note that the spinor combinations $(\bar{u}\gamma_\mu u)$ are the components of a four-vector. Contracting this vector with $(p+p')_\nu$ yields a Lorentz scalar and therefore a Lorentz-invariant scattering amplitude. The derivation of the cross section consists in the same steps, which have been discussed in detail for $\pi^+\mathrm{K}^+$ scattering. Employing the four-vectors

$$\begin{aligned} k^\mu &= (\omega,\boldsymbol{k}) \quad , \quad k'^\mu = (\omega',\boldsymbol{k}') \quad , \\ p^\mu &= (E,\boldsymbol{p}) \quad , \quad p'^\mu = (E',\boldsymbol{p}') \quad , \end{aligned} \tag{8}$$

we find the differential cross section to be

$$\mathrm{d}\sigma_{ss'} = (2\pi)^4\delta(k'+p'-k-p)\frac{|F_{ss'}|^2}{2E\,2\omega|\boldsymbol{v}|}\frac{1}{(2\pi)^3}\frac{\mathrm{d}^3k'}{2\omega'}\frac{1}{(2\pi)^3}\frac{\mathrm{d}^3p'}{2E'} \quad . \tag{9}$$

The scattering amplitude $F_{ss'}$ can be easily evaluated if we insert the spinors (see (2.36))

$$u(k,s) = \sqrt{w+m_0}\begin{pmatrix} \phi^s \\ \frac{\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{k}}{\omega+m_0}\phi^s \end{pmatrix} \tag{10}$$

into (7). This procedure is quite tedious and, more importantly, does not yield the quantity observed in most of the actual experiments. The interesting quantity is the so-called nonpolarized cross section, which is obtained from (9) by averaging over the initial spins and summing over the final spins, i.e.,

$$\begin{aligned} \mathrm{d}\bar{\sigma} &= \frac{1}{2}(d\sigma_{\uparrow\uparrow} + \mathrm{d}\sigma_{\uparrow\downarrow} + \mathrm{d}\sigma_{\uparrow\downarrow} + \mathrm{d}\sigma_{\downarrow\downarrow}) \\ &= \frac{1}{2}\sum_{ss'}\mathrm{d}\sigma_{ss} \quad . \end{aligned} \tag{11}$$

These lengthy summations need not be performed explicitly. Instead, employing Feynman's trace techniques enables us to drastically simplify the spin summations (11).

2.3.1 Traces and Spin Summations

Let us briefly review the basic facts about trace techniques. The trace of a matrix is the sum of its diagonal elements, i.e.,

$$\mathrm{tr}\{A\} = \sum_i A_{ii} \quad . \tag{2.146}$$

Cyclic permutability holds under the trace

$$\begin{aligned} \mathrm{tr}\{AB\} &= \mathrm{tr}\{BA\} \quad , \\ \mathrm{tr}\{ABC\} &= \mathrm{tr}\{CAB\} = \mathrm{tr}\{BCA\} \quad . \end{aligned} \tag{2.147}$$

The most important relations for traces over products of γ matrices and Feynman "daggers" needed in this context are as follows. By using the anticommutation relation

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}$$

the following traces can be easily evaluated:

$$\begin{aligned} \mathrm{tr}\{\mathbb{1}\} &= 4 \quad , \\ \mathrm{tr}\{\gamma_\mu\gamma_\nu\} &= 4\, g_{\mu\nu} \quad , \\ \mathrm{tr}\{\not{a}\not{b}\} &= 4\, a \cdot b \quad , \\ \mathrm{tr}\{\not{a}\not{b}\not{c}\not{d}\} &= 4\big[(a \cdot b)(c \cdot d) + (a \cdot d)(b \cdot c) - (a \cdot c)(b \cdot d)\big] \quad , \end{aligned} \tag{2.148}$$

where use has been made of the fact that

$$\not{a}\not{b} = -\not{b}\not{a} + 2\, a \cdot b \tag{2.149}$$

holds (see Exercise 2.10). An expression for the scattering cross section in lepton–pion scattering was derived in the Exercise 2.9. There the square of the scattering cross section appeared:

$$\begin{aligned} |F_{ss'}|^2 = \left(\frac{e^2}{q^2}\right)^2 & \left\{\left[\bar{u}(k', s')\gamma_\mu u(k, s)\right][p + p']^\mu\right\} \\ & \times \left\{\left[\bar{u}(k', s')\gamma_\nu u(k, s)\right][p + p']^\nu\right\}^* \quad . \end{aligned} \tag{2.150}$$

Since $F_{ss'}$ is a complex number (all matrix indices are summed over), we can replace the conjugate term $\{\dots\}^*$ by the Hermitian conjugate; we find that

$$\begin{aligned} \left\{\left[\bar{u}(k', s')\gamma_\nu u(k, s)\right][p + p']^\nu\right\}^\dagger &= \left\{\left[u^\dagger(k, s)\gamma_\nu^\dagger\gamma_0 u(k', s')\right][p + p']^\nu\right\} \\ &= \left\{\left[\bar{u}(k, s)\gamma_\nu u(k', s')\right][p + p']^\nu\right\} \quad , \end{aligned} \tag{2.151}$$

since

$$\gamma_0\gamma_\nu^\dagger\gamma_0 = \gamma_\nu, \quad \gamma_0^\dagger = \gamma_0 \tag{2.152}$$

holds. The four-vector $(p + p')^\nu$ has real components and is therefore not changed by the operations $(\dots)^*$ and $(\dots)^\dagger$. In performing the spin summation according to (11) in Exercise 2.9 we must calculate the following expression:

$$\begin{aligned} \frac{1}{2}\sum_{s,s'}|F_{ss'}|^2 &= \frac{1}{2}\left(\frac{e^2}{q^2}\right)^2 \sum_{s,s'}\left[\bar{u}(k', s')\gamma_\mu u(k, s)\right] \\ &\quad \times \left[\bar{u}(k, s)\gamma_\nu u(k', s')\right](p + p')^\mu(p + p')^\nu \\ &\equiv \left(\frac{e^2}{q^2}\right)^2 L_{\mu\nu}T^{\mu\nu} \quad , \end{aligned} \tag{2.153}$$

where we have introduced the so-called lepton tensor

$$L_{\mu\nu} = \frac{1}{2}\sum_{s,s'} \bar{u}(k',s')\gamma_\mu u(k,s)\bar{u}(k,s)\gamma_\nu u(k',s') \tag{2.154}$$

and the so-called hadron tensor

$$T^{\mu\nu} = (p+p')^\mu (p+p')^\nu \quad . \tag{2.155}$$

The factorization of the scattering cross section in a leptonic and a hadronic part stems from the one-photon approximation. To higher order, the situation is more complicated. The middle term (the direct product of two spinors)

$$\sum_s u(k,s) \otimes \bar{u}(k,s) = \sum_s u_\beta(k,s)\bar{u}_\gamma(k,s) \quad , \tag{2.156}$$

which no longer depends on s (s is summed over), is a 4×4 matrix. Making use of the explicit form $u(k,s)$ of the spinor and the fact that for two-component spinors ϕ^s

$$\begin{aligned} \sum_s \phi^s \phi^{s\dagger} &= \phi^1\phi^{1\dagger} + \phi^2\phi^{2\dagger} \\ &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} (1,0) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} (0,1) \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1} \end{aligned} \tag{2.157}$$

holds, we can deduce that

$$\begin{aligned} \sum_s u(k,s)\bar{u}(k,s) &= u(k,\uparrow)\bar{u}(k,\downarrow) + u(k,\downarrow)\bar{u}(k,\downarrow) \\ &= (\omega + m_0)\begin{pmatrix} \mathbb{1} & -\frac{\hat{\sigma}\cdot k}{\omega+m_0} \\ \frac{\hat{\sigma}\cdot k}{\omega+m_0} & -\frac{\omega-m_0}{\omega+m_0}\mathbb{1} \end{pmatrix} \\ &= (\not{k} + m_0) \quad . \end{aligned} \tag{2.158}$$

In Exercise 2.10, we shall perform this calculation in detail. Using this result, (2.156) can be expressed by (2.158) and inserted into (2.154), which leads to

$$L_{\mu\nu} = \frac{1}{2}\sum_{s'}\sum_{\alpha,\beta,\gamma,\delta} \bar{u}_\alpha(k',s')(\gamma_\mu)_{\alpha\beta}(\not{k} + m_0)_{\beta\gamma}(\gamma_\nu)_{\gamma\delta}u_\delta(k',s') \tag{2.159}$$

written in expanded matrix notation. Each single factor in this term represents a c number, and the factors can therefore be reordered arbitrarily. In particular, the sum

$$\sum_{s'} u_\delta(k',s')\,\bar{u}_\alpha(k',s') = (\not{k}' + m_0)_{\delta\alpha} \tag{2.160}$$

can be performed using (2.158), and the lepton tensor is thus reduced to the form

$$\begin{aligned} L_{\mu\nu} &= \frac{1}{2} \sum_{\alpha,\beta,\gamma,\delta} (\not{k}' + m_0)_{\delta\alpha}(\gamma_\mu)_{\alpha\beta}(\not{k} + m_0)_{\beta\gamma}(\gamma_\nu)_{\gamma\delta} \\ &= \frac{1}{2}\mathrm{tr}\{(\not{k}' + m_0)\gamma_\mu(\not{k} + m_0)\gamma_\nu\} \quad . \end{aligned} \tag{2.161}$$

Using the trace formulas from (2.148) we obtain

$$\begin{aligned} \mathrm{tr}\{(\not{k}' + m_0)\gamma_\mu(\not{k} + m_0)\gamma_\nu\} &= \mathrm{tr}\{\not{k}'\gamma_\mu\not{k}\gamma_\nu\} + m_0^2\mathrm{tr}\{\gamma_\mu\gamma_\nu\} \\ &= 4(k'_\mu k_\nu + k'_\nu k_\mu - k\cdot k' g_{\mu\nu}) + 4m_0^2 g_{\mu\nu} \ . \end{aligned} \tag{2.162}$$

If we take into account that the mass squared is

$$q^2 = (k - k')^2 = k^2 + k'^2 - 2k\cdot k' = 2(m_0^2 - k\cdot k') \quad , \tag{2.163}$$

this can be inserted into (2.152), and the lepton tensor can finally be written as

$$L_{\mu\nu} = 2\left(k'_\mu k_\nu + k'_\nu k_\mu + \frac{1}{2}q^2 g_{\mu\nu}\right) \quad . \tag{2.164}$$

This tensor not only is important for elastic scattering, but also plays a major role in most quark–parton calculations. We shall perform the contraction of $L_{\mu\nu}$ with the hadron tensor $T^{\mu\nu}$ in the pion rest system ($p^\mu = (M,\boldsymbol{0})$) according to (2.153) in Exercise 2.11 to calculate the cross section. Neglecting the electron mass (the ultrarelativistic limit), the result is

$$\begin{aligned} \frac{\mathrm{d}\bar{\sigma}}{\mathrm{d}\Omega} &= \frac{\alpha^2}{4k^2\sin^4(\theta/2)}\frac{k'}{k}\cos^2\left(\frac{\theta}{2}\right) \\ &= \frac{\alpha^2}{4\omega^2\sin^4(\theta/2)}\frac{\omega'}{\omega}\cos^2\left(\frac{\theta}{2}\right) \quad , \end{aligned} \tag{2.165}$$

where

$$\begin{aligned} k &= |\boldsymbol{k}| = \omega \quad , \\ k' &= |\boldsymbol{k}'| = \omega' \quad , \end{aligned} \tag{2.166}$$

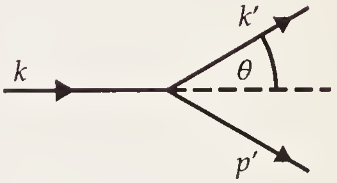


Fig. 2.19. The definition of the e$^-$ scattering angle in e$^-\pi^+$ scattering

and

$$q^2 = -4\,kk'\sin^2\left(\frac{\theta}{2}\right) \quad . \tag{2.167}$$

θ is the scattering angle of the electron (see Fig. 2.19).

The preceding considerations that culminate in the scattering cross section (2.165) are based on treating pions and electrons as pointlike. We thus label the cross section "n.s." (meaning "no structure"):

$$\left(\frac{\mathrm{d}\bar{\sigma}}{\mathrm{d}\Omega}\right)_{\text{n.s.}} \tag{2.168}$$

Today it is accepted that pions have an internal structure, being composed of quarks, antiquarks, and gluons. Leptons, on the other hand, are still considered to be elementary, i.e., without internal structure. This conclusion is drawn mainly from lepton–lepton scattering.

EXERCISE

2.10 Features of Dirac Matrices

Problem. Start with the anticommutator

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\,\mathbb{1} \tag{1}$$

and show that the following relations hold:

(a) $\not{a}\not{b} = -\not{b}\not{a} + 2a \cdot b \quad .$ (2)

(b) $$\begin{aligned}(\not{k} - m_0)(\not{k} + m_0) &= (\not{k} + m_0)(\not{k} - m_0)\\ &= 0 \quad , \quad \text{for} \quad k^2 = m_0^2 \quad .\end{aligned} \tag{3}$$

(c) $\hat{\Lambda}_+(k) = (\not{k} + m_0) \quad ,$ (4)

which eliminates the negative energy parts of an arbitrary spinor, and

$$\hat{\Lambda}_-(k) = (\not{k} - m_0) \quad , \tag{5}$$

which eliminates the positive energy parts of an arbitrary spinor.
(d) Employ the explicit forms

$$\begin{aligned}u(k,s) &= \sqrt{\omega + m_0}\begin{pmatrix}\phi^s\\ \frac{\hat{\sigma}\cdot k}{\omega + m_0}\phi^s\end{pmatrix} \quad ,\\ \phi^1 &= \begin{pmatrix}1\\0\end{pmatrix} \quad , \quad \phi^2 = \begin{pmatrix}0\\1\end{pmatrix} \quad ,\end{aligned} \tag{6}$$

$$\begin{aligned}v(k,s) &= \sqrt{\omega + m_0}\begin{pmatrix}\frac{\hat{\sigma}\cdot k}{\omega + m_0}\chi^s\\ \chi^s\end{pmatrix} \quad ,\\ \chi^1 &= \begin{pmatrix}0\\1\end{pmatrix} \quad , \quad \chi^2 = \begin{pmatrix}1\\0\end{pmatrix}\end{aligned} \tag{7}$$

and show how the 4×4 matrices $\sum_s u(k,s)\bar{u}(k,s)$ and $\sum_s v(k,s)\bar{v}(k,s)$ depend on $\hat{\Lambda}_+(k)$ and $\hat{\Lambda}_-(k)$, respectively.

Solution.

(a) $$\begin{aligned}\not{a}\not{b} + \not{b}\not{a} &= a_\mu\gamma^\mu b_\nu\gamma^\nu + b_\nu\gamma^\nu a_\mu\gamma^\mu = a_\mu b_\nu\,\{\gamma^\mu, \gamma^\nu\}\\ &= 2a_\mu b_\nu g^{\mu\nu}\,\mathbb{1} = 2a \cdot b\,\mathbb{1} \quad ;\end{aligned} \tag{8}$$

from this equation follows in particular that $\not{a}^2 = a^2\,\mathbb{1}$.

(b) $$\begin{aligned}\left(\not{k} - m_0\,\mathbb{1}\right)\left(\not{k} + m_0\,\mathbb{1}\right) &= \left(\not{k}^2 - m_0^2\right)\mathbb{1}\\ &= \left(k^2 - m_0^2\right)\mathbb{1} = 0 \quad .\end{aligned} \tag{9}$$

(c) The wave function of a particle with positive energy and four-momentum k_μ can be written as

$$\Psi^{(+)}(x) = \sum_s b_s u(k,s)\mathrm{e}^{-\mathrm{i}k\cdot x} \quad . \tag{10}$$

Exercise 2.10.

Correspondingly a solution with negative energy and four-momentum $k_{-\mu}$ is

$$\Psi^{(-)}(x) = \sum_s d_s v(k,s) \mathrm{e}^{\mathrm{i}k\cdot x} \quad . \tag{11}$$

Both wave functions must obey the Dirac equation, i.e.,

$$\begin{aligned}\left(\mathrm{i}\not{\nabla} - m_0\right)\Psi^{(+)}(x) &= \left(\not{k} - m_0\right)\Psi^{(+)}(x)\\ &= \hat{\Lambda}_{-(k)}\Psi^{(+)}(x) = 0\end{aligned} \tag{12}$$

and

$$\begin{aligned}\left(\mathrm{i}\not{\nabla} - m_0\right)\Psi^{(-)}(x) &= -\left(\not{k} + m_0\right)\Psi^{(-)}(x)\\ &= -\hat{\Lambda}_{+(k)}\Psi^{(-)}(x) = 0 \quad .\end{aligned} \tag{13}$$

Equations (12) and (13) can also be put into the form

$$\begin{aligned}\not{k}\Psi^{(+)}(x) &= m_0\Psi^{(+)}(x) \quad ,\\ \not{k}\Psi^{(-)}(x) &= -m_0\Psi^{(-)}(x) \quad .\end{aligned} \tag{14}$$

From (14) we immediately obtain

$$\begin{aligned}\hat{\Lambda}_+\Psi &= 2m_0\Psi^{(+)} \quad ,\\ \hat{\Lambda}_-\Psi &= -2m_0\Psi^{(-)} \quad .\end{aligned} \tag{15}$$

(d) $$\begin{aligned}\sum_s u(k,s)\bar{u}(k,s) &= \sum_s u(k,s)u^\dagger(k,s)\gamma^0\\ &= (\omega+m_0)\sum_s \begin{pmatrix}\phi^s\\ \frac{\hat{\sigma}\cdot\boldsymbol{k}}{\omega+m_0}\phi^s\end{pmatrix}\left(\phi^{s\dagger}, \phi^{s\dagger}\frac{\hat{\sigma}^\dagger\cdot\boldsymbol{k}}{\omega+m_0}\right)\begin{pmatrix}1\!\!1 & 0\\ 0 & -1\!\!1\end{pmatrix}\\ &= (\omega+m_0)\sum_s \begin{pmatrix}\phi^s\phi^{s\dagger} & -\phi^s\phi^{s\dagger}\frac{\hat{\sigma}^\dagger\cdot\boldsymbol{k}}{\omega+m_0}\\ \frac{\hat{\sigma}\cdot\boldsymbol{k}}{\omega+m_0}\phi^s\phi^{s\dagger} & -\frac{\hat{\sigma}\cdot\boldsymbol{k}}{\omega+m_0}\phi^s\phi^{s\dagger}\frac{\hat{\sigma}^\dagger\cdot\boldsymbol{k}}{\omega+m_0}\end{pmatrix} \quad .\end{aligned} \tag{16}$$

Now we employ the Hermiticity of the Pauli matrices, $\hat{\sigma}^\dagger = \hat{\sigma}$, and make use of the identity $\sum_s \phi^s\phi^{s\dagger} = 1\!\!1$ (cf. (2.157)). Equation (16) then assumes the form

$$(\omega+m_0)\begin{pmatrix}1\!\!1 & -\frac{\hat{\sigma}\cdot\boldsymbol{k}}{\omega+m_0}\\ \frac{\hat{\sigma}\cdot\boldsymbol{k}}{\omega+m_0} & -\frac{(\hat{\sigma}\cdot\boldsymbol{k})^2}{(\omega+m_0)^2}\end{pmatrix} \quad . \tag{17}$$

But since we have

$$\begin{aligned}(\hat{\sigma}\cdot\boldsymbol{k})^2 &= \frac{1}{2}k_i k_j\left(\hat{\sigma}_i\hat{\sigma}_j + \hat{\sigma}_j\hat{\sigma}_i\right) = k_i k_j \sigma_{ij}\\ &= \boldsymbol{k}^2 = \omega^2 - m_0^2 \quad ,\end{aligned}$$

the result becomes

$$\begin{pmatrix}(\omega+m_0)1\!\!1 & -\hat{\sigma}\cdot\boldsymbol{k}\\ \hat{\sigma}\cdot\boldsymbol{k} & (-\omega+m_0)1\!\!1\end{pmatrix} \quad . \tag{18}$$

Employing the γ-matrix representation

Exercise 2.10.

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \quad , \quad \boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix} \quad , \tag{19}$$

we obtain the result

$$\begin{aligned} \sum_s u(k,s)\bar{u}(k,s) &= \omega\gamma^0 - \boldsymbol{k}\cdot\boldsymbol{\gamma} + m_0 \\ &= (\not{k} + m_0) = \hat{\Lambda}_+(k) \quad . \end{aligned}$$

An analogous calculation for negative solutions yields

$$\begin{aligned} &\sum_s v(k,s)\bar{v}(k,s) \\ &= (\omega + m_0) \sum_s \begin{pmatrix} \frac{\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{k}}{\omega+m_0}\chi^s \\ \chi^s \end{pmatrix} \left(\chi^{s\dagger} \frac{\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{k}}{\omega + m_0}, -\chi^s \right) \\ &= (\omega + m_0) \begin{pmatrix} \frac{\omega^2 - m_0^2}{(\omega+m_0)^2} & -\frac{\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{k}}{\omega+m_0} \\ \frac{\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{k}}{\omega+m_0} & -\mathbb{1} \end{pmatrix} \\ &= \begin{pmatrix} (\omega - m_0)\mathbb{1} & -\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{k} \\ \hat{\boldsymbol{\sigma}}\cdot\boldsymbol{k} & (-\omega - m_0)\mathbb{1} \end{pmatrix} \\ &= \omega\gamma^0 - \boldsymbol{k}\cdot\boldsymbol{\gamma} - m_0 \\ &= (\not{k} - m_0) = -\hat{\Lambda}_-(k) \quad . \end{aligned} \tag{20}$$

EXERCISE

2.11 Electron–Pion Scattering (II)

Problem. Evaluate in detail the nonpolarized $\pi^+ e^-$ cross section. Start with the previously discussed formula

$$\mathrm{d}\bar{\sigma} = \frac{1}{4E\omega|\boldsymbol{v}|}\frac{1}{2}\sum_{s,s'} |F_{ss'}|^2 \mathrm{d}\,\mathrm{Lips}(s;k'p') \quad .$$

Determine $\mathrm{d}\bar{\sigma}/\mathrm{d}\Omega$ in the rest system of the pion ($p^\mu = (M, \boldsymbol{0})$).

Solution. We denote the four-momenta of the pion before and after the collision by $p^\mu = (M, \boldsymbol{0})$ and $p'^\mu = (E', \boldsymbol{p}')$, respectively. $k^\mu = (\omega, \boldsymbol{k})$ and $k'^\mu = (\omega', \boldsymbol{k}')$ are the corresponding four-momenta of the incoming and outgoing electrons. The scattering angle θ is the angle between the directions $\boldsymbol{k}$ and $\boldsymbol{k}'$ and $q = k' - k$ denotes the momentum transfer. In the following we only consider high electron energies, i.e., since $\omega, \omega' \gg m$, the electron rest mass can be neglected. Therefore we have $\omega = |\boldsymbol{k}|$ and $\omega' = |\boldsymbol{k}'|$ and the invariant flux factor is simply

$$4\sqrt{(k\cdot p)^2 - m^2M^2} \simeq 4M\omega \quad . \tag{1}$$

According to the Feynman rules the spin average of the squared scattering amplitude describing the exchange of one photon is given by the contraction of lepton and hadron tensor:

Exercise 2.11.

$$\frac{1}{2}\sum_{s,s'}|F_{ss'}|^2 = \left(\frac{4\pi\alpha}{q^2}\right)^2 L_{\mu\nu}T^{\mu\nu} \quad . \tag{2}$$

Employing (2.155) and (2.164) we obtain

$$L_{\mu\nu}T^{\mu\nu} = 8\left(2(k\cdot p)(k'\cdot p) + \frac{q^2}{2M^2}\right) \quad . \tag{3}$$

In the ultrarelativistic limit, q^2 becomes

$$\begin{aligned} q^2 &= (k-k')^2 = (\omega-\omega')^2 - (\boldsymbol{k}-\boldsymbol{k}')^2 \\ &\approx -2\omega\omega'(1-\cos\theta) = -4\omega\omega'\sin^2\left(\frac{\theta}{2}\right) \quad . \end{aligned}$$

Hence

$$\left(\frac{4\pi\alpha}{q^2}\right)^2 = \left(\frac{\pi\alpha}{\omega\omega'}\right)^2 \frac{1}{\sin^4\left(\frac{\theta}{2}\right)} \tag{4}$$

and

$$L_{\mu\nu}T^{\mu\nu} = 16M^2\omega\omega'\cos^2\left(\frac{\theta}{2}\right) \quad . \tag{5}$$

Now we have to evaluate the invariant phase-space factor:

$$\begin{aligned} \mathrm{d}\,\mathrm{Lips}(s;k'p') &= (2\pi)^4\delta^4(k'+p'-k-p)\frac{1}{(2\pi)^3}\frac{\mathrm{d}^3p'}{2E'}\frac{1}{(2\pi)^3}\frac{\mathrm{d}^3k'}{2\omega'} \\ &= \frac{1}{(4\pi)^2}\delta^3(\boldsymbol{k}'+\boldsymbol{p}'-\boldsymbol{k})\delta(E'-M+\omega'-\omega)\frac{\mathrm{d}^3p'}{E'}\frac{\mathrm{d}^3k'}{\omega'} \quad . \end{aligned} \tag{6}$$

In order to evaluate the cross section for the electron to be scattered into a given final state, we have to integrate over all final states of the pion. By means of the δ^3 function in (6) this is readily achieved (owing to momentum conservation only one final pion state is possible for a given final state of the electron). In the remaining integrand $\boldsymbol{p}'$ must then be replaced by $\boldsymbol{k}-\boldsymbol{k}' = -\boldsymbol{q}$. Because of the identity

$$\begin{aligned} E' &= \sqrt{M^2+\boldsymbol{p}'^2} = \sqrt{M^2+\boldsymbol{q}^2} \\ &\approx \sqrt{M^2+\omega^2+\omega'^2-2\omega\omega'\cos\theta} \quad . \end{aligned} \tag{7}$$

E' is not an independent quantity either. Except for this factor there are no further dependences of the integrand on $\boldsymbol{p}'$, including the scattering amplitude and flux factor. Finally $\omega' = |\boldsymbol{k}'|$ leads to

$$\frac{\mathrm{d}^3k'}{\omega'} = \omega'\,\mathrm{d}\omega'\,\mathrm{d}\Omega \quad , \tag{8}$$

where $\mathrm{d}\Omega$ denotes the spherical angle into which the electron is scattered. The ω' integration is performed by using the remaining energy δ function, i.e., for a given scattering angle the absolute value of the electron momentum is fixed by kinematic arguments. Note that in the argument of the δ function E' depends on ω' as well (cf. (7)). Therefore we have to employ the relation

Exercise 2.11.

$$\delta\left(f(x)\right) = \sum_i \frac{\delta(x - x_i)}{f'(x)|_{x=x_i}} \quad , \tag{9}$$

where the x_i denote the zeros of the function $f(x)$. According to (7) we have

$$\frac{\mathrm{d}(E' - M + \omega' - \omega)}{\mathrm{d}\omega'} = \frac{\omega' - \omega\cos\theta + E'}{E'} \quad . \tag{10}$$

Inserting now the identity $E' = M + \omega - \omega'$, which has been derived by integrating over the δ function, into (7) and (10), we obtain

$$M(\omega - \omega') = \omega\omega'(1 - \cos\theta) \quad , \tag{11}$$

$$\frac{\mathrm{d}}{\mathrm{d}\omega'}(E' - M + \omega - \omega') = \frac{\omega(1 - \cos\theta) + M}{E'} \quad , \tag{12}$$

or

$$\frac{\mathrm{d}}{\mathrm{d}\omega'}(E' + \omega') = \frac{M\omega}{E'\omega'} \quad . \tag{13}$$

The (partially integrated) Lorentz-invariant phase-space factor now assumes the form

$$\mathrm{dLips} = \frac{1}{(4\pi)^2}\frac{\omega'^2}{M\omega}\mathrm{d}\Omega \quad , \tag{14}$$

where ω' is fixed by (11). Summarizing equations (1), (4), (5), and (14) finally yields the non-polarized cross section

$$\left(\frac{\mathrm{d}\bar{\sigma}}{\mathrm{d}\Omega}\right)_{\text{n.s.}} = \frac{\alpha^2}{4\omega^2\sin^4\left(\frac{\theta}{2}\right)}\frac{\omega'}{\omega}\cos^2\left(\frac{\theta}{2}\right) \quad . \tag{15}$$

EXERCISE

2.12 Positron–Pion Scattering

Problem. Show that the cross section for $e^+\pi^+$ scattering is, in the one-photon-exchange approximation, equal to that for $e^-\pi^+$ scattering.

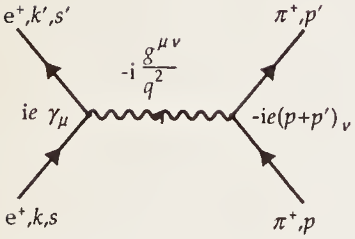


Fig. 2.20. $e^+\pi^+$ scattering in the one-photon-exchange approximation

Solution. The graph for $e^+\pi^+$ scattering is of the following form. The incoming positron with four-momentum k and spin s is described as an outgoing electron with four-momentum $-k$ and spin $-s$. Correspondingly the outgoing positron with k', s' can be interpreted as an incoming electron with $-k'$, $-s'$. Therefore we have for the positron transition current

$$j_\mu(e^+) = (-e)NN'\bar{v}(k,s)\gamma_\mu v(k',s')e^{-i(-k'+k)\cdot x} \quad , \tag{1}$$

where $v\left(k',s'\right)$ and $\overline{v}\left(k',s'\right)$ represent an incoming electron wave $-k'$, $-s'$ and an outgoing electron wave $-k$, $-s$, respectively. If we construct the scattering matrix element with $j_\mu(e^+)$, an additional minus sign is inserted according to the

Exercise 2.12.

previously introduced rules. Everything else remains the same as in $e^-\pi^+$ scattering discussed in Exercises 2.9 and 2.11. The evaluation of the cross section, too, is analogous to that of $e^-\pi^+$ scattering except for the spin average, where the sum

$$\sum_s u(k,s)\bar{u}(k,s) = (\not{k} + m_0) \tag{2}$$

in the lepton tensor must be replaced by (cf. (2.154))

$$\sum_s v(k,s)\bar{v}(k,s) = (\not{k} - m_0) \quad . \tag{3}$$

This calculation has was made in Exercise 2.10 and leads to the trace

$$\mathrm{tr}\,\{(\not{k}' - m_0)\gamma_\mu(\not{k} - m_0)\gamma_\nu\} \tag{4}$$

instead of

$$\mathrm{tr}\,\{(\not{k}' + m_0)\gamma_\mu(\not{k} + m_0)\gamma_\nu\} \quad . \tag{5}$$

Because of rules (2.148) we immediately recognize that these two traces are equal. Therefore the $e^+\pi^+$ cross section is to lowest order exactly equal to the $e^-\pi^+$ cross section. This result does not surprise us at all, since only the lepton charge changed its sign and the lowest-order cross section contains only the square of this charge.

2.3.2 The Structure of the Form Factors from Invariance Considerations

We now assume that the pion has an internal structure that we do not know exactly but that can be parametrized in some rather general way, as we shall demonstrate shortly. The photon–pion vertex with an internal pion structure is drawn in Fig. 2.21. In place of some simple vertex, a circle is drawn, symbolizing the internal strcuture of the pion, while a simple junction indicates the vertex of a pointlike pion (without internal structure).

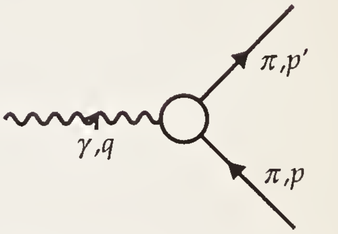
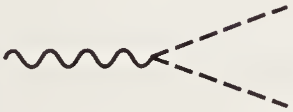
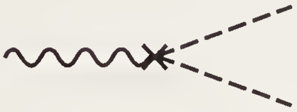


Fig. 2.21. The general photon–pion vertex. The circle indicates the internal structure of the pion





To exemplify which kind of processes can be contained in the circle, we mention the two following graphs, drawn in the conventional framework without imagining a quark structure of the pion.

Owing to the strong interaction (large coupling constant) such virtual processes can contribute considerably to the total scattering amplitude. As we shall see, even without detailed knowledge of the internal structure of the pion, general statements about the form of the required modifications of the vertex can be obtained from the Lorentz and gauge invariance of the theory. These two general requirements fix the form of the scattering amplitude for any internal structure imaginable.

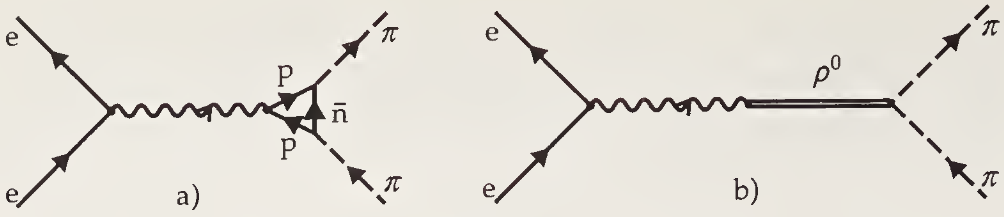


Fig. 2.22a,b. Complex virtual processes founded on strong interactions and contributing to the structure of the $\gamma\pi^+$ vertex: (**a**) a virtual nucleon loop; (**b**) a virtual ρ^0 meson

For a pointlike pion, the transition amplitude is

$$j^\mu(\pi^+) = eNN'(p+p')^\mu \, e^{i(p'-p)\cdot x} \quad . \tag{2.169}$$

We now consider the current (2.169) as a matrix element of an electromagnetic current operator (Heisenberg operator)

$$\hat{j}^\mu_{\mathrm{em}}(x) \tag{2.170}$$

and write

$$j^\mu(\pi^+) = \langle \pi^+ p' | \hat{j}^\mu_{\mathrm{em}}(x) | \pi^+ p \rangle \quad . \tag{2.171}$$

If (2.171) describes the special transition current of pointlike pions, we identify

$$\langle \pi^+ p' | \hat{j}^\mu_{\mathrm{em}}(0) | \pi^+ p \rangle = eNN'(p+p')^\mu \quad . \tag{2.172}$$

Since the matrix element (2.171) is taken with plane pion waves, its x dependence is given by the exponential factor in (2.169). This will also hold unmodified for pions with internal structure, but we expect that the strong interaction modifies the right-hand side of (2.172), i.e., the four-momentum dependence:

$$\langle \pi^+ p' | \hat{j}^\mu_{\mathrm{em}}(0) | \pi^+ p \rangle \equiv NN' \Gamma^\mu(p, p', q) \quad , \tag{2.173}$$

where the momentum transfer or four-momentum of the virtual photon q can also enter.

Taking a rather pragmatic point of view, we simply try to parametrize the four-current or Γ^μ. As Γ^μ must remain a four-vector, we shall first discuss which general four-vector Γ^μ can be constructed from the available four-vectors p, p', and q. Owing to four-momentum conservation at the vertex

$$p' = p + q \quad , \tag{2.174}$$

only two independent four-vectors are available, which can be chosen to be

$$(p'+p)^\mu \quad \text{and} \quad (p'-p)^\mu = q^\mu \quad . \tag{2.175}$$

Both can be utilized in the construction of Γ^μ. Additionally, both four-vectors (2.175) can be multiplied with an unknown scalar function. From the relations

$$\begin{aligned} &p^2 = p'^2 = M^2 \quad , \\ &p'^2 = p^2 + 2p\cdot q + q^2 \Leftrightarrow 2p\cdot q = -q^2 \quad , \end{aligned} \tag{2.176}$$

and

$$q^2 = 2M^2 - 2p\cdot p' \tag{2.177}$$

it can be seen that only one nontrivial scalar can be composed of the four-vectors (2.175), namely $p \cdot p'$ or, equivalently, q^2, the square of the momentum transfer at the vertex. Using this, we can write the vertex function most generally using Lorentz invariance as

$$\Gamma^\mu(p,p',q) = e\left[F(q^2)(p+p')^\mu + G(q^2)q^\mu\right] \quad . \tag{2.178}$$

The scalar functions $F(q^2)$ and $G(q^2)$ are called form factors.

To discover what statements can be made about the form factors from gauge invariance, we remember that the Maxwell equations are left invariant by the gauge transformation $A^\mu \to A'^\mu = A^\mu - \partial^\mu \Lambda$. The Maxwell equations are in the Lorentz gauge

$$\Box A^\mu = \Box A'^\mu = j^\mu_{\text{em}} \quad . \tag{2.179}$$

Since, owing to current conservation, only those Λ satisfying $\Box\Lambda = 0$ can be considered, this gauge condition implies current conservation:

$$\partial_\mu j^\mu_{\text{em}} = 0 \quad . \tag{2.180}$$

Charge conservation must obviously hold both for the transition current (2.169) of a pointlike pion,

$$-\mathrm{i}\partial_\mu j^\mu(\pi^+) = q_\mu j^\mu(\pi^+) = 0 \quad , \quad q\cdot(p+p') = 0 \quad , \tag{2.181}$$

and for the current

$$q_\mu\langle\pi^+p'|\hat{j}^\mu(0)|\pi^+p\rangle = 0 \quad . \tag{2.182}$$

Both conditions thus demand in general that

$$q_\mu\Gamma^\mu = q_\mu e\left[F(q^2)(p+p')^\mu + G(q^2)q^\mu\right] = 0 \quad . \tag{2.183}$$

The first term always vanishes since $q\cdot(p+p') = 0$ owing to (2.175). Only the second term will not always vanish – if $q^2 \neq 0$ – so that (2.183) can only be satisfied if

$$G(q^2) = 0 \quad . \tag{2.184}$$

In other words gauge invariance of the theory leads automatically to the statement that all structural effects of the pion (mainly caused by the strong interaction) can be described by a form factor as a function of the photon mass q^2. We thus find that

$$(\Gamma^\mu)_{\text{n.s.}} = e(p'+p)^\mu \quad , \quad (\Gamma^\mu)_{\text{w.s.}} = eF(q^2)(p'+p)^\mu \quad , \tag{2.185}$$

where the subscript "w.s." indicates "with structure". Since the charge e appears as a prefactor, (2.185) also contains the definition of charge in the sense that the form factor at vanishing four-momentum transfer ($q^2 = 0$) must be unity:

$$F(0) = 1 \quad . \tag{2.186}$$

The form factor is measured in scattering processes for all values of q^2, both for $q^2 \leq 0$ (spacelike four-momenta measured in $\mathrm{e}^-\phi^+$ scattering) and $q^2 \geq 4M^2$

(timelike four-momenta in the "crossing reaction" $e^+e^- \to \pi^+\pi^-$). Naturally these measurements ask new questions of the theory in order to explain the form factor $F(q^2)$. For physical reasons, we are led to expect that $F(q^2)$ diminishes when $|q^2|$ is increased, since it becomes increasingly difficult to transfer momentum to the various constituents of the pion in order that it stays intact (elastic scattering as opposed to inelastic scattering).

EXAMPLE

2.13 Electron–Muon Scattering

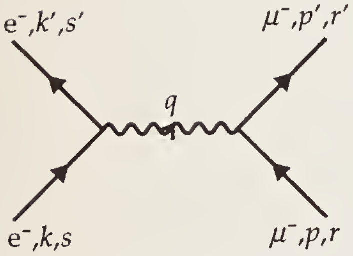


Fig. 2.23 The one-photon-exchange amplitude for $e^-\mu^-$ scattering

As an example of lepton–lepton scattering we briefly discuss $e^-\mu^-$ scattering, which is represented in the one-photon-exchange approximation by the following graph. Using the experiences of previous problems we can immediately write down the invariant scattering amplitude, which now depends on four spin indices:

$$F_{sr;s'r'} = (-e)^2\bar{u}(k',s')\gamma_\mu u(k,s)\left(-\frac{g^{\mu\nu}}{q^2}\right)\bar{u}(p',r')\gamma_\mu u(p,r) \quad . \tag{1}$$

Again the nonpolarized cross section is proportional to the square of this amplitude averaged over initial spins and summed over final spins, i.e.,

$$d\bar{\sigma} = \frac{1}{4}\sum_{r,s,r',s'} |F_{sr;s'r'}|^2 \quad . \tag{2}$$

We can perform the same steps as for $e^-\pi^+$ scattering (cf. (2.153–164)) for each transition current separately. This simplification is caused by the factorization of the currents in the one-photon-exchange approximation. We obtain

$$\begin{aligned}\frac{1}{4}\sum_{r,s,r',s'} &= \left(\frac{e^2}{q^2}\right)^2\left[\frac{1}{2}\mathrm{tr}\,\{(\not{k}' + m_0)\gamma_\mu(\not{k} + m_0)\gamma_\nu\}\right]\\ &\quad\times\left[\frac{1}{2}\mathrm{tr}\,\{(\not{p}' + M_0)\gamma^\mu(\not{p} + M_0)\gamma^\nu\}\right]\\ &\equiv \left(\frac{e^2}{q^2}\right)^2 L_{\mu\nu}M^{\mu\nu} \quad .\end{aligned} \tag{3}$$

Employing our previous results (2.164) we can immediately write down the electron tensor

$$L_{\mu\nu} = 2\left[k'_\mu k_\nu + k'_\nu k_\mu + \frac{q^2}{2}g_{\mu\nu}\right] \tag{4}$$

and the muon tensor

$$M^{\mu\nu} = 2\left[p'^\mu p^\nu + p'^\nu p^\mu + \frac{q^2}{2}g^{\mu\nu}\right] \quad . \tag{5}$$

In order to obtain the cross section we have to evaluate the contraction of these two tensors $L_{\mu\nu}M^{\mu\nu}$. The direct evaluation is straightforward but quite lengthy. We

Example 2.13.

prefer to employ the following trick, which follows from the current conservation. Because of $q^\mu = k'^\mu - k^\mu$ the electron current conservation

$$\partial^\mu j_\mu(\mathrm{e}^-) = 0 \quad ,$$
$$q^\mu \left[\overline{u}(k',s')\gamma_\mu u(k,s)\right] = 0 \tag{6}$$

can be written as

$$\overline{u}(k',s')\left(\not{k}' - \not{k}\right) u(k,s) = 0 \quad . \tag{7}$$

Equation (7) can be explicitly obtained from the corresponding Dirac equations for $\overline{u}\left(k',s'\right)$ and $u\,(k,s)$, respectively, and is valid for all possible spin projections. But since $L_{\mu\nu}$ is a product of two transition currents, we immediately get

$$q^\mu L_{\mu\nu} = q^\nu L_{\mu\nu} = 0 \quad . \tag{8}$$

This result is very useful because in evaluating the contraction $L_{\mu\nu}M^{\mu\nu}$ we can omit all terms proportional to q. Therefore we are able to simplify the quantities $p' = p + q$ and to consider the so-called *effective muon tensor*

$$M^{\mu\nu}_{\text{eff}} = 2\left[2p^\mu p^\nu + \frac{q^2}{2}g^{\mu\nu}\right] \quad , \tag{9}$$

which yields the same result for the contraction to be calculated, i.e.,

$$L_{\mu\nu}M^{\mu\nu} = L_{\mu\nu}M^{\mu\nu}_{\text{eff}} \quad . \tag{10}$$

A straightforward but cumbersome calculation yields the following result for the nonpolarized cross section in the rest frame of the muon ($p^\mu = (M,0,0,0)$):

$$\frac{\mathrm{d}\overline{\sigma}}{\mathrm{d}\Omega} = \left(\frac{\mathrm{d}\overline{\sigma}}{\mathrm{d}\Omega}\right)_{\text{n.s.}}\left(1 - \frac{q^2\tan^2\left(\frac{\theta}{2}\right)}{2M^2}\right) \quad , \tag{11}$$

where θ denotes the angle between $\boldsymbol{k}$ and $\boldsymbol{k}'$. The following remarks should be noted:

1. $\left(\mathrm{d}\overline{\sigma}/\mathrm{d}\Omega\right)_{\text{n.s.}}$ is the *no-structure* cross section known from $\mathrm{e}^-\pi^+$ scattering (cf. (2.165) and (2.168)). It is modified by an additional term proportional to $\tan^2(\theta/2)$. This effect is caused by the spin-$\frac{1}{2}$ nature of the muon. The muon not only has a charge but also a magnetic moment. The latter is automatically taken into account by the Dirac equation. In other words, compared to $\mathrm{e}^-\pi^+$ scattering we observe an additional scattering by the normal magnetic moment in the case of $\mathrm{e}^-\mu^-$ scattering.
2. The electron rest mass was neglected in the kinematics of (11), i.e., we considered only the ultrarelativistic limit.
3. We wrote down the $\mathrm{e}^-\pi^+$ as well as the $\mathrm{e}^-\mu^-$ cross sections in the rest system of the π^+ and μ^-, respectively, which can hardly be realized in experimental setups. Later this kind of cross section for structureless particles will be useful in the discussion of the quark–parton model. One has to understand these cross sections in order to acknowledge the physical content of parton dynamics.

Example 2.13.

4. The *crossed reaction* $e^+e^- \to \mu^+\mu^-$ is frequently investigated in electron–positron collisions in the context of so-called *colliding-beam experiments*. It is also important for testing the quark–parton model, if compared with the reaction

$$e^+e^- \to \text{hadrons} \quad .$$

An analogous calculation leads to the cross section

$$\frac{d\bar{\sigma}}{d\Omega} = \frac{\alpha^2}{4q^2}\left(1+\cos^2\theta\right) \quad , \tag{12}$$

where all variables are defined in the center-of-mass system of the e^+e^- pair and all masses are neglected (ultrarelativistic limit). θ denotes the angle between the axis of the incoming and the axis of the outgoing particles.

3. Scattering Reactions and the Internal Structure of Baryons

3.1 Simple Quark Models Compared

In the first chapter we showed that the baryon spectrum by itself already suggests that baryons are composed of quarks. However, the interaction between quarks cannot be easily deduced from the energies of the states, since different models yield nearly identical and well–fitting descriptions of the mass spectrum. Such models include the flavor–SU(6) model, the MIT bag model, the Skyrmion bag model, and potential models with nonrelativistic quarks. Luckily lattice calculations are now good enough to demonstrate that the *correct* model, namely QCD, gives equally good results.

The flavor–SU(6) model postulates that the up, down, and strange quark species are eigenvalues of an internal symmetry group, namely SU(3). To include spin and thus the splitting between the spin–$\frac{1}{2}$ and spin–$\frac{3}{2}$ multiplets, this group is extended to $\mathrm{SU}(6) \supset \mathrm{SU}(2) \otimes \mathrm{SU}(3)$. This symmetry is then broken in such a manner that the mass terms that appear depend only on operators that can be diagonalized simultaneously. In this way we obtain mass formulas that describe mass differences in a multiplet. The simplest expression is the Gursey–Radicati mass formula:

$$M = a + bY + c\left[T(T+1) - \frac{1}{4}Y^2\right] + dS(S+1) \quad ; \tag{3.1}$$

Y, T, and S are the hypercharge, isospin, and spin of the baryon. Using the four parameters a to d we can fit the lowest baryon resonances very well. With

$$a = 1065.5\,\mathrm{MeV},\ b = -193\,\mathrm{MeV},\ c = 32.5\,\mathrm{MeV},\ \text{and}\ d = 67.5\,\mathrm{MeV}\ , \tag{3.2}$$

for example, the numbers given in Table 3.1 are obtained. To describe the other baryons as well an internal angular momentum must be introduced, i.e., we suppose that quarks inside baryons can fill states with any angular momentum. As the rotation group is O(3), we are thus led to the $\mathrm{SU}(6) \otimes \mathrm{O}(3)$ symmetry group.

Indeed we can describe the full baryon spectrum starting from the $\mathrm{SU}(6) \otimes \mathrm{O}(3)$ mass formulas. Owing to the size of the symmetry group, many different contributions appear, making the procedure rather tedious.[1] Also the discovery of every new quark, such as the charm and bottom quarks, demand an extension of the flavor–SU(6) model, leading to yet more complicated and unsatisfactory models.

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[1] For a discussion, see M. Jones, R. H. Dalitz, and R. R. Hougan: Nucl. Phys. **B129**, 45 (1977).

Table 3.1. The Gursey–Radicati mass formula

Particles	Mass from (3.1)	Experimental data
$N1/2^+$	939 MeV	939 MeV
$\Lambda 1/2^+$	1116 MeV	1116 MeV
$\Sigma 1/2^+$	1181 MeV	1189 MeV
$\Xi 1/2^+$	1325 MeV	1318 MeV
$\Delta 3/2^+$	1239 MeV	1230–1234 MeV
$\Sigma 3/2^+$	1384 MeV	1385 MeV
$\Xi 3/2^+$	1528 MeV	1533 MeV
$\Omega 3/2^+$	1672 MeV	1672 MeV

The extremely large masses of the heavy quarks signal that flavor symmetry is heavily broken, rendering such models much less attractive.

In the MIT bag model, presented in Sect. 3.3, quarks can occupy all states satisfying the specific boundary conditions. Such states exist for any angular momentum, i.e., the single-particle spectrum of the quarks contains all states known from atomic physics: $s_{1/2}$, $p_{1/2}$, $p_{3/2}$, $d_{3/2}$, $d_{5/2}$, … . In principle, many-particle states with definite spin and parity could be constructed from this, thus deriving the corresponding masses from the bag boundary conditions. However, this procedure gives disastrously bad results. To improve these, additional residual interactions (like the one-gluon exchange) and other corrections can be taken into account, but satisfactory baryon spectra are obtained only after introducing a sufficiently large number of parameters.[2]

Also the Skyrmion bag model, which is based on totally different assumptions, yields similar results. There the baryon number is regarded as a topological quantum number. We shall not investigate this model further but shall illustrate the basic idea with a simple example. As we remarked in Sect. 1.2, spin and isospin are isomorphic. In particular, the regular representation of isospin, e.g., pions, with its three isospin unit vectors $|\pi^0\rangle$, $\frac{1}{\sqrt{2}}(|\pi^+\rangle + |\pi^-\rangle)$ and $\frac{1}{\sqrt{2}\,\mathrm{i}}(|\pi^+\rangle - |\pi^-\rangle)$ is isomorphic to angular momentum. As angular momenta can be represented by vectors in three-dimensional space, pions also can be interpreted as vectors in three-dimensional isospin space. A very interesting construction is now obtained by coupling the direction of this isospin vector to the position vector, for example by demanding that the isospin vector $\boldsymbol{t}$ at position $\boldsymbol{x}$ points in the direction $\boldsymbol{t} = \boldsymbol{x}/|\boldsymbol{x}|$ in isospin space. In this way we obtain a pion field consisting purely of π^0s along the z axis and of a mixture of π^0, π^+, and π^- at other positions (see Fig. 3.1). This construction is termed the *hedgehog* solution (because the isospin vectors point outwards like the spikes of a hedgehog). To reverse this orientation of the pion

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[2] See, for example, T. A. De Grand and R. L. Jaffe: Ann. Phys. **100**, 425 (1976) and T. A. De Grand: Ann. Phys. **101**, 496 (1976).

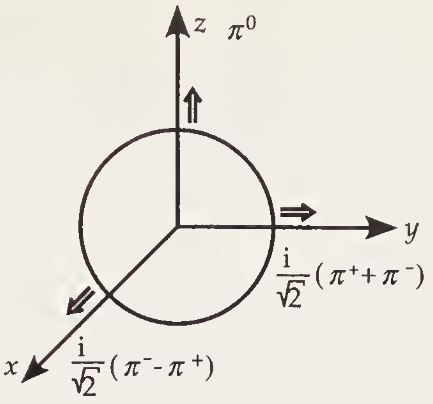
z
π^0
y
$\frac{i}{\sqrt{2}}(\pi^+ + \pi^-)$
x
$\frac{i}{\sqrt{2}}(\pi^- - \pi^+)$

Fig. 3.1. The schematic form of the *hedgehog* solution

field, one would have to change $\pi(x)$ in an infinite spatial domain (at $|x| \to \infty$), which would require infinite energy. Thus a single hedgehog is stable, and the number of hedgehogs can be identified with the baryon number. More precisely, a *topological quantum number* is defined that specifies how often $\pi(|x|)$ covers all isospin values for $|x| \to \infty$. The different states with topological quantum number 1 are then identified with the different baryons.

Potential models simply solve the Schrödinger equation for nonrelativistic quarks including a spin–spin and spin–tensor interaction. The basic Hamiltonian is

$$
\begin{aligned}
H = {} & \sum_i \frac{p_i^2}{2m_i} + \frac{1}{2}K \sum_{i>j} V_{ij}^2 \\
& + \sum_{i>j} \frac{2\alpha_s}{3m_i m_j} \left\{ \frac{8\pi}{3} s_i \cdot s_j \, \delta^3(r) + \frac{1}{r^3}\left[3(s_i \cdot r)(s_j \cdot r) - s_i \cdot s_j\right] \right\} \\
& + U(r_{ij}) \quad . \qquad (3.3)
\end{aligned}
$$

The coupling constants K, α_s, the masses m_i, and the (weak) residual interaction are fitted to the baryon ground states,[3] and excited states are then predicted. As Table 3.2 shows, the predictions obtained in this way coincide rather well with experimental values.[4]

In conclusion, completely different models describe the mass spectrum equally well, which implies that nothing can be learned about the underlying interaction from baryon masses alone. Also other parameters, such as magnetic moments, do not give more information. However, there are experimental results which are really sensitive. These are the so-called *structure functions* deduced from scattering reactions. Their definition, measurement, and meaning will be discussed in detail in this chapter. Structure functions are sensitive to the details of the interaction to such an extent that, contrary to the situation with the mass formulas, no current model yields a really satisfactory description. Only a complete solution of quantum chromodynamics could achieve this.

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[3] See, for example, N. Isgur and G. Karl: Phys. Rev. **D19** 2653 (1979a).

[4] See A. J. G. Hey and R. L. Kelly: Phys. Rep. **96**, 72 (1983).

Table 3.2. A test of the potential model

Particles	Potential model	Experiment
$N1/2^-$	1490 MeV	1520–1555 MeV
$N1/2^-$	1655 MeV	1640–1680 MeV
$N3/2^-$	1535 MeV	1515–1530 MeV
$N3/2^-$	1745 MeV	1650–1750 MeV
$\Lambda 1/2^-$	1490 MeV	1407 ± 4 MeV
$\Lambda 1/2^-$	1650 MeV	1660–1680 MeV
$\Lambda 1/2^-$	1800 MeV	1720–1850 MeV
$\Lambda 3/2^-$	1490 MeV	1519.5 ± 1 MeV
$\Lambda 3/2^-$	1690 MeV	1685–1695 MeV
$\Lambda 3/2^-$	1880 MeV	?

3.2 The Description of Scattering Reactions

To learn about the internal structure of nucleons, we must consider the scattering of particles as point like as possible, such as the scattering of high-energy electrons, muons, or neutrinos off nucleons:

$$e^-(E \gg 1\text{GeV}) + \mathrm{N} \to e^- \dots \quad , \tag{3.4}$$

$$\nu_e(E \gg 1\text{GeV}) + \mathrm{N} \to e^- \dots \quad , \tag{3.5}$$

Since highly energetic leptons have a very small wavelength, namely $\lambda \approx 1/E < 0.2\,\text{fm}$, and do not possess a resolvable internal structure, the cross sections of these reactions depend solely on the internal structure of the nucleon. As electron scattering takes place mainly by photon exchange, it senses the electromagnetic charge distribution, whereas reaction (3.5) occurs through the weak interaction and gives information about the corresponding distribution of "weak charge". By comparing the results of different scattering reactions, we thus obtain a nearly complete description of the internal structure of the nucleon. The internal structure of baryon resonances and heavy mesons cannot, of course, be determined in this way because of the small lifetime of these particles. Although some information can be obtained from their decay properties, only the structure functions of the proton, neutron, and pion are known.

We shall now discuss the scattering of an electron off a nucleon. This is often discussed in textbooks on *Quantum Electrodynamics*, leading to the Rosenbluth formula. We shall shortly repeat this discussion and introduce a new, more practical notation for the process of Fig. 3.2. Since QED is Lorentz-covariant, the vertex function Γ_μ (or, more precisely, the matrix element $\bar{u}(P',S')\,\lambda_\mu\,u(P,S)$) must be a Lorentz vector. The most general structure of Γ_μ is thus

$$\Gamma_\mu = A\gamma_\mu + BP'_\mu + CP_\mu + \mathrm{i}DP'^\nu\sigma_{\mu\nu} + \mathrm{i}EP^\nu\sigma_{\mu\nu} \quad , \tag{3.6}$$

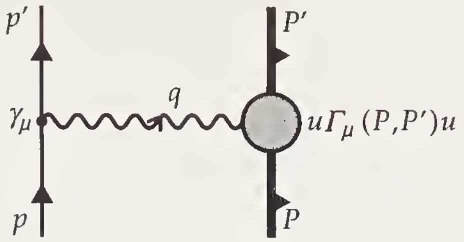

Fig. 3.2. Elastic electron–nucleon scattering

where the quantities $A, B, \ldots, E$ depend only on Lorentz invariant quantities. Since all these invariants can be expressed in terms of M_{N}^2 and q^2,

$$\begin{aligned} P\cdot P &= P'\cdot P' = M_{\mathrm{N}}^2 \quad ,\\ P\cdot P' &= -\frac{1}{2}(P-P')^2 + M_{\mathrm{N}}^2 = -\frac{1}{2}q^2 + M_{\mathrm{N}}^2 \quad ,\\ P\cdot q &= P\cdot P' - P^2 = -\frac{1}{2}q^2 \quad ,\\ P'\cdot q &= P'^2 - P'\cdot P = \frac{1}{2}q^2 \quad , \end{aligned} \tag{3.7}$$

$A = A(q^2)$, $B = B(q^2)$, etc. holds. From the demand for gauge invariance, it follows, on the other hand, that

$$q^\mu\,\bar{u}(P')\,\Gamma_\mu\,u(P) = 0 \quad . \tag{3.8}$$

Substituting from (3.6) yields $D = -E$ and $C = B$ and thus

$$\begin{aligned} \overline{u}(P')\,\Gamma_\mu(P',P)\,u(P) = \overline{u}(P')\Big[&A(q^2)\gamma_\mu + B(q^2)(P'+P)_\mu \\ &+ \mathrm{i}D(q^2)(P'-P)^\nu\sigma_{\mu\nu}\Big]u(P) \quad . \end{aligned} \tag{3.9}$$

On physical grounds we demand that the transition current must be Hermitian. For (3.9) to be invariant under the transformation $(\ldots)^+|P_\mu \to P'_\mu$, A, B, and D must be real. Using the Gordon decomposition, the second term on the right-hand side can be expressed by the first and the third terms, and we get

$$\overline{u}(P')\,\Gamma_\mu(P',P)\,u(P) = \overline{u}(P')\Big[A(q^2)\gamma_\mu + \mathrm{i}B(q^2)q^\nu\sigma_{\mu\nu}\Big]u(P) \quad . \tag{3.10}$$

The absolute square of this expression enters in the cross section

$$\begin{aligned} W_{\mu\nu} &= \sum_{\mathrm{spin}} \left[\overline{u}(P')\,\Gamma_\mu\,u(P)\right]^* \left[\overline{u}(P')\,\Gamma_\nu\,u(P)\right] \\ &= \mathrm{tr}\Big\{(A\gamma_\mu - \mathrm{i}Bq^\lambda\sigma_{\mu\lambda})(\not{P}' + M_{\mathrm{N}})(A\gamma_\nu + \mathrm{i}Bq^\rho\sigma_{\nu\rho})(\not{P} + M_{\mathrm{N}})\Big\} \quad . \end{aligned} \tag{3.11}$$

After some lengthy calculation we find that

$$\begin{aligned} W_{\mu\nu} = {} & 4(A+2M_{\mathrm{N}}B)^2\left[P_\mu P'_\nu + P'_\mu P_\nu - \left(P\cdot P' - M_{\mathrm{N}}^2\right)g_{\mu\nu}\right] \\ & + \left[-4(A+2M_{\mathrm{N}}B)2M_{\mathrm{N}}B + 4M_{\mathrm{N}}^2B^2\left(\frac{P\cdot P'}{M_{\mathrm{N}}^2}+1\right)\right] \\ & \times (P_\mu + P'_\mu)(P_\nu + P'_\nu) \quad . \end{aligned} \tag{3.12}$$

Replacing P'_μ by $q_\mu + P_\mu$ yields

$$\begin{aligned} W_{\mu\nu} = {} & 4(A+2M_{\mathrm{N}}B)^2\left[P_\mu q_\nu + P_\nu q_\mu + 2P_\mu P_\nu + \frac{q^2}{2}g_{\mu\nu}\right] \\ & + \left[-4(A+2M_{\mathrm{N}}B)2M_{\mathrm{N}}B + 4M_{\mathrm{N}}^2B^2\left(-\frac{q^2}{2M_N^2}+2\right)\right] \\ & \times (q_\mu + 2P_\mu)(q_\nu + 2P_\nu) \quad , \end{aligned} \tag{3.13}$$

and using

$$P_\mu q_\nu + P_\nu q_\mu + 2P_\mu P_\nu = \frac{1}{2}(q_\mu + 2P_\mu)(q_\nu + 2P_\nu) - \frac{1}{2}q_\mu q_\nu \tag{3.14}$$

we get for (3.13)

$$\begin{aligned} W_{\mu\nu} = {} & -2(A+2M_{\mathrm{N}}B)^2q^2\left(\frac{q_\mu q_\nu}{q^2} - g_{\mu\nu}\right) \\ & + \left(2A^2 - 2B^2q^2\right)4\left(P_\mu - q_\mu\frac{P\cdot q}{q^2}\right)\left(P_\nu - q_\nu\frac{P\cdot q}{q^2}\right) \quad . \end{aligned} \tag{3.15}$$

In the last step, we have used the fact that, owing to (3.7), $P\cdot q/q^2 = \frac{1}{2}$. We now introduce two new functions W_1, W_2 and the variable $Q^2 = -q^2$ and write the elastic scattering tensor as

$$\begin{aligned} W_{\mu\nu}^{\mathrm{el}} = {} & \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right)W_1(Q^2) \\ & + \left(P_\mu - q_\mu\frac{P\cdot q}{q^2}\right)\left(P_\nu - q_\nu\frac{P\cdot q}{q^2}\right)\frac{W_2(Q^2)}{M_{\mathrm{N}}^2} \quad . \end{aligned} \tag{3.16}$$

This structure is immediately evident if one considers that, owing to gauge invariance,

$$q^\mu W_{\mu\nu}^{\mathrm{el}} = q^\nu W_{\mu\nu}^{\mathrm{el}} = 0 \quad . \tag{3.17}$$

If we consider instead of elastic scattering special inelastic processes like $\mathrm{e}+\mathrm{N}\to\mathrm{e}+\mathrm{N}+\pi$, more momentum vectors can be combined and the general structure of $W_{\mu\nu}$ becomes more complicated. A simple expression can again be obtained if we sum over all possible processes or, more precisely, over all possible final hadron states, since this sum can again only depend on P and q. The only change is that the Lorentz invariants q^2 and $q\cdot P$ are now independent. One thus obtains the following general form for the inclusive inelastic scattering tensor:

$$\begin{aligned} W_{\mu\nu}^{\mathrm{incl.}} = {} & \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right)W_1(Q^2,\nu) \\ & + \left(P_\mu - q_\mu\frac{P\cdot q}{q^2}\right)\left(P_\nu - q_\nu\frac{P\cdot q}{q^2}\right)\frac{W_2(Q^2,\nu)}{M_{\mathrm{N}}^2} \end{aligned} \tag{3.18}$$

with the *inelasticity variable*

$$\nu = \frac{P \cdot q}{M_{\mathrm{N}}} \quad . \tag{3.19}$$

To obtain the differential cross section, we must multiply $W_{\mu\nu}$ with the corresponding tensor for the electrons

$$\begin{aligned} L_{\mu\nu} &= \frac{1}{2}\mathrm{tr}\,\{(\not{p} + m)\gamma_\mu(\not{p}' + m)\gamma_\nu\} \\ &= 2\left[p_\mu p'_\nu + p_\nu p'_\mu - g_{\mu\nu}\, p \cdot p' + g_{\mu\nu} m^2\right] \quad . \end{aligned} \tag{3.20}$$

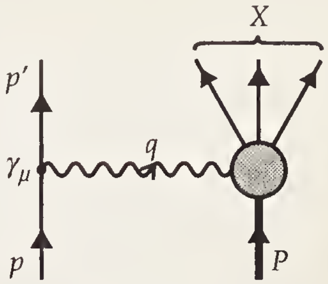

Fig. 3.3. Inclusive inelastic electron–nucleon scattering

Equation (3.20) can be obtained by setting $A = 1$, $B = 0$ in (3.12). (The factor $\frac{1}{2}$ comes from averaging over the spin directions of the incoming electrons.) The other factors appearing are the coupling constant and the photon propagator

$$\left(\frac{e^2}{q^2}\right)^2 = \frac{e^4}{Q^4} \quad . \tag{3.21}$$

Finally, this must be multiplied with normalization and phase-space factors (cf. Example 3.1). The final expression for unpolarized electron–nucleon scattering in the laboratory system is

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}E'\mathrm{d}\Omega} = \frac{E'\alpha^2}{EQ^4} L^{\mu\nu} W_{\mu\nu} \quad . \tag{3.22}$$

Here E and E' are defined by

$$\begin{aligned} p_\mu &= (E, \boldsymbol{p}) \quad , \\ p'_\mu &= (E', \boldsymbol{p}') \quad , \end{aligned} \tag{3.23}$$

and the electron mass has been neglected (the extremely relativistic approximation).

Substituting (3.18) and (3.20) into (3.22), this yields

$$\begin{aligned} \frac{\mathrm{d}^2\sigma}{\mathrm{d}E'\mathrm{d}\Omega} &= \frac{E'\alpha^2}{EQ^4} 2(2p^\mu p^\nu - g^{\mu\nu} p \cdot p') \\ &\quad \times \left[\left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) W_1 + \left(P_\mu - q_\mu \frac{P \cdot q}{q^2}\right)\left(P_\nu - q_\nu \frac{P \cdot q}{q^2}\right) \frac{W_2}{M_{\mathrm{N}}^2}\right] . \end{aligned} \tag{3.24}$$

Because $q^\mu W_{\mu\nu} = 0$, we were able to replace p'_μ by p_μ, and we have again neglected the electron mass in comparison to p_μ and p'_μ. We shall set it to zero in the following. Taking into account that

$$-Q^2 = q^2 \approx -2p \cdot p' \tag{3.25}$$

and

$$2p \cdot q = 2p \cdot (p - p') = 2m^2 - 2p \cdot p' \approx -Q^2 \quad , \tag{3.26}$$

we obtain

$$
\begin{aligned}
\frac{\mathrm{d}^2\sigma}{\mathrm{d}E'\mathrm{d}\Omega} &= \frac{E'}{E}\frac{\alpha^2}{Q^4}2\left[\left(2\frac{(p\cdot q)^2}{q^2}+3p\cdot p'\right)W_1+2\left(P\cdot p-\frac{q\cdot p}{q^2}q\cdot P\right)^2\frac{W_2}{M_\mathrm{N}^2}\right.\\
&\quad\left.-\left(P_\mu-q_\mu\frac{q\cdot P}{q^2}\right)^2 p\cdot p'\frac{W_2}{M_\mathrm{N}^2}\right]\\
&= \frac{E'}{E}\frac{2\alpha^2}{Q^4}\left[Q^2W_1+2\left(P\cdot p-\frac{1}{2}P\cdot q\right)^2\frac{W_2}{M_\mathrm{N}^2}\right.\\
&\quad\left.-\frac{Q^2}{2}\left(M_\mathrm{N}^2-\frac{(P\cdot q)^2}{q^2}\right)\frac{W_2}{M_\mathrm{N}^2}\right] \quad . \qquad (3.27)
\end{aligned}
$$

We look at the scattering in the laboratory system, in which the nucleon is at rest before the collision, i.e.,

$$
P_\mu=(M_\mathrm{N},0),\qquad \nu=\frac{P\cdot q}{M_\mathrm{N}}=\frac{P\cdot(p-p')}{M_\mathrm{N}}=\frac{M_\mathrm{N}(E-E')}{M_\mathrm{N}}=E-E' \ . \qquad (3.28)
$$

In addition we introduce the scattering angle θ of the electron in the laboratory system; thus

$$
\boldsymbol{p}\cdot\boldsymbol{p}'=|\boldsymbol{p}|\,|\boldsymbol{p}'|\cos(\theta)=EE'\cos(\theta) \qquad (3.29)
$$

and

$$
Q^2=2p\cdot p'=2(EE'-\boldsymbol{p}\cdot\boldsymbol{p}')=4EE'\sin^2\left(\frac{\theta}{2}\right) \quad . \qquad (3.30)
$$

Finally we use definition (3.19):

$$
\begin{aligned}
\frac{\mathrm{d}^2\sigma}{\mathrm{d}E'\mathrm{d}\Omega} &= \frac{E'}{E}\frac{\alpha^2}{Q^4}2\left\{4EE'\sin^2\left(\frac{\theta}{2}\right)W_1+2M_\mathrm{N}^2\left(E-\frac{\nu}{2}\right)^2\frac{W_2}{M_\mathrm{N}^2}\right.\\
&\quad\left.-\left[2EE'\sin^2\left(\frac{\theta}{2}\right)+\frac{\nu^2}{2}\right]W_2\right\}\\
&= \frac{E'}{E}\frac{\alpha^2}{Q^4}2\left\{4EE'\sin^2\left(\frac{\theta}{2}\right)W_1+\left[2E^2-2E\nu-2EE'\sin^2\left(\frac{\theta}{2}\right)\right]W_2\right\} . \qquad (3.31)
\end{aligned}
$$

Because $\nu=(P\cdot q)/M_\mathrm{N}=(P\cdot p-P\cdot p')/M_\mathrm{N}=E-E'$, the last term simplifies to give

$$
2M_\mathrm{N}^2EE'\left[1-\sin^2\left(\frac{\theta}{2}\right)\right]\frac{W_2}{M_\mathrm{N}^2}=2EE'\cos^2\left(\frac{\theta}{2}\right)W_2 \quad , \qquad (3.31a)
$$

and we obtain

$$
\left.\frac{\mathrm{d}^2\sigma}{\mathrm{d}E'\mathrm{d}\Omega}\right|_{\mathrm{eN}}=4E'^2\frac{\alpha^2}{Q^2}\left[2\sin^2\left(\frac{\theta}{2}\right)W_1\left(Q^2,\nu\right)+\cos^2\left(\frac{\theta}{2}\right)W_2\left(Q^2,\nu\right)\right] . \qquad (3.32)
$$

This is the final expression for the inclusive unpolarized electron–nucleon scattering cross section. The functions $W_1\left(Q^2,\nu\right)$ and $W_2\left(Q^2,\nu\right)$ are called the *structure functions* of inclusive electron–nucleon scattering. To avoid confusion, we write them in what follows as $W_1^{\mathrm{eN}}\left(Q^2,\nu\right)$ and $W_2^{\mathrm{eN}}\left(Q^2,\nu\right)$. Experimentally one measures for a definite electron beam energy the direction and energy of the scattered electrons and sums the total reaction cross section for each (θ, E') bin. From this one obtains $W_1^{\mathrm{eN}}\left(Q^2,\nu\right)$ and $W_2^{\mathrm{eN}}\left(Q^2,\nu\right)$.

What now is the advantage of (3.32)? So far we have only managed to eliminate one of the three parameters E, E', and θ. While $\mathrm{d}^2\sigma/\mathrm{d}E'\mathrm{d}\Omega$ can in general be any function of E, E', and θ, we have to express it by arbitrary functions of Q^2 and ν. The importance in (3.32) lies mainly in the fact that the structure functions can be calculated from the microscopic properties of the quark model and that then the variables x and Q^2 are the relevant ones. In leading order of α_s the structure functions turn out to be Q^2 independent and the residual Q^2 dependence can be used as a most sensitive test for the quark interaction, i.e., for QCD.

$W_{\mu\nu}^{\mathrm{eN}}$ can be expressed by

$$W_{\mu\nu}^{\mathrm{eN}}(P,q) = \frac{1}{2\pi}\int \mathrm{d}^4x\ \mathrm{e}^{\mathrm{i}q\cdot x}\frac{1}{2}\sum_{\mathrm{pol.}}\langle \mathrm{N}(P)|\hat{J}_\mu(x)\hat{J}_\nu(0)|\mathrm{N}(P)\rangle \quad , \tag{3.33}$$

where $|\mathrm{N}(P)\rangle$ designates the *state vector of a nucleon* with momentum P and $\hat{J}$ the *electromagnetic current operator*.

We shall discuss the derivation of this relation in Example 3.2. In the laboratory frame $P_\mu = (M,0)$. Also, we can orient the coordinate system such that q points in the z direction. Then from (3.18) it holds that

$$W_{00}^{\mathrm{eN}} = \left(\frac{q_0 q_0}{q^2} - 1\right) W_1\left(Q^2,\nu\right) + \left(1 + \frac{\nu}{Q^2}q_0\right)^2 W_2\left(Q^2,\nu\right) \tag{3.34}$$

and

$$W_{11}^{\mathrm{eN}} = W_1\left(Q^2,\nu\right) \tag{3.34a}$$

with

$$q_0 = E - E' = \nu \quad . \tag{3.35}$$

For example, for $W_1\left(Q^2,\nu\right)$ we obtain

$$W_1\left(Q^2,\nu\right) = \frac{1}{2\pi}\int \mathrm{d}^4x\ \mathrm{e}^{\mathrm{i}q\cdot x}\frac{1}{2}\sum_{\mathrm{pol.}}\langle \mathrm{N}(P)|\hat{J}_1(x)\hat{J}_1(0)|\mathrm{N}(P)\rangle \quad . \tag{3.36}$$

The right-hand side can (at least in principle) be computed for any quark model, and its correctness can thus be tested by comparing the result with the experimental values for W_1 and W_2.

We shall illustrate the meaning of the nucleon structure functions for a very simple, and inadequate, model in Exercise 3.9. This model is indeed so simple that we do not need to use (3.36) but can choose a simpler way.

Before we discuss the experimentally determined properties of structure functions and their meaning, we give the analogous result for *neutrino–nucleon* and *antineutrino–nucleon scattering*. These formulas again hold in the rest system of the nucleon, i.e., the laboratory system

$$\left.\frac{\mathrm{d}^2\sigma}{\mathrm{d}E'\mathrm{d}\Omega}\right|_{\nu\mathrm{N}} = \frac{G_\mathrm{F}^2}{2\pi^2}E'^2\left[2\sin^2\left(\frac{\theta}{2}\right)W_1^{\nu\mathrm{N}}\left(Q^2,\nu\right) + \cos^2\left(\frac{\theta}{2}\right)W_2^{\nu\mathrm{N}}\left(Q^2,\nu\right)\right.$$
$$\left. - \frac{E+E'}{M_\mathrm{N}}\sin^2\left(\frac{\theta}{2}\right)W_3^{\nu\mathrm{N}}\left(Q^2,\nu\right)\right] \tag{3.37}$$

$$\left.\frac{\mathrm{d}^2\sigma}{\mathrm{d}E'\mathrm{d}\Omega}\right|_{\bar{\nu}\mathrm{N}} = \frac{G_\mathrm{F}^2}{2\pi^2}E'^2\left[2\sin^2\left(\frac{\theta}{2}\right)W_1^{\bar{\nu}\mathrm{N}}\left(Q^2,\nu\right) + \cos^2\left(\frac{\theta}{2}\right)W_2^{\bar{\nu}\mathrm{N}}\left(Q^2,\nu\right)\right.$$
$$\left. + \frac{E+E'}{M_\mathrm{N}}\sin^2\left(\frac{\theta}{2}\right)W_3^{\bar{\nu}\mathrm{N}}\left(Q^2,\nu\right)\right] \quad . \tag{3.38}$$

Here G_F is the Fermi coupling constant of weak interactions. The derivation of these relations is to be found in Exercises 3.3 and 3.4.

An important assumption we have made tacitly up to now is that single-photon and single-W exchange, dominate the cross section. This assumption is safe because of the smallness of the coupling constants. For strongly interacting particles the situation becomes more complicated. Such reactions will be analyzed in detail in Chap. 6.

EXAMPLE

3.1 Normalization and Phase Space Factors

We start with the relation

$$\mathrm{d}\sigma = \frac{1}{4E\omega|\boldsymbol{v}|}\frac{1}{2}\sum_{s,s'}|F_{ss'}|^2(2\pi)^4\delta^4(k'+p'-k-p)\frac{\mathrm{d}^3\boldsymbol{k}'}{(2\pi)^3 2\omega'}\frac{\mathrm{d}^3\boldsymbol{p}'}{(2\pi)^3 2E'} \quad , \tag{1}$$

where $(E,\boldsymbol{p})$, $(E',\boldsymbol{p}')$, $(\omega,\boldsymbol{k})$ and $(\omega',\boldsymbol{k}')$ denote the four-momenta of the incoming electron, the outgoing electron, the incoming nucleon, and the outgoing multi-hadron state, respectively. Now we have to integrate this expression over $\boldsymbol{k}'$ and, in contrast to electron–pion–scattering, also over ω', since ω' is no longer fixed by $\boldsymbol{k}'$:

$$\mathrm{d}\sigma = \frac{\mathrm{d}^3\boldsymbol{p}'}{4EE'|\boldsymbol{v}|}\int \mathrm{d}^4k'\left(\frac{1}{2}\frac{1}{4\omega\omega'}\sum_{s,s'}|F_{ss'}|^2\right)(2\pi)^{-2}\delta^4(k'+p'-k-p) \; . \tag{2}$$

Obviously, the dimension of the integrals has changed by one, implying that the structure function has got an additional dimension of 1/energy. This is why we shall later on define the dimensionless structure function F_1 as $F_1 = W_1 M_\mathrm{N}$. The integral on the right-hand side is equal to $e^4 L^{\mu\nu}W_{\mu\nu}/Q^4$. Note that the nucleon normalization factors 2ω and $2\omega'$ are included in $W_{\mu\nu}$ and that $L_{\mu\nu}$ contains the factor 1/2, which is due to spin averaging (see (3.20)).

$$\mathrm{d}\sigma = \frac{\mathrm{d}^3\boldsymbol{p}'}{4EE'|\boldsymbol{v}|}\frac{e^4}{(2\pi)^2Q^4}L^{\mu\nu}W_{\mu\nu} \quad . \tag{3}$$

For our case (a massless electron incident on a resting nucleon)

Example 3.1.

$$|v| \approx c = 1 \quad . \tag{4}$$

Furthermore we employ

$$\mathrm{d}^3p' = (p')^2\,\mathrm{d}\,|\boldsymbol{p}'|\,\mathrm{d}\Omega' \approx E'^2\mathrm{d}E'\,\mathrm{d}\Omega \tag{5}$$

and in this way obtain (3.22):

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}E'\mathrm{d}\Omega} = \frac{E'}{E}\frac{e^4}{(4\pi)^2Q^4}L^{\mu\nu}W_{\mu\nu} = \frac{E'}{E}\frac{\alpha^2}{Q^4}L^{\mu\nu}W_{\mu\nu} \quad . \tag{6}$$

EXAMPLE

3.2 Representation of $W_{\mu\nu}$ by Electromagnetic Current Operators

The coupling of the exchanged photon to the nucleon can be written in a general way as

$$\hat{J}_\mu(x)A^\mu(x) \quad , \tag{1}$$

where for the moment no assumption about the structure of $\hat{J}_\mu(x)$ has been made. Now we consider the corresponding current matrix element between the initial nucleon state with momentum P_μ and an arbitrary hadronic final state X with four-momentum $P_{\mathrm{X},\mu}$:

$$\left\langle \mathrm{X}(P_\mathrm{X})\left|\hat{J}_\mu(x)\right|\mathrm{N}(P)\right\rangle = \left\langle \mathrm{X}(P_\mathrm{X})\left|\hat{J}_\mu(0)\right|\mathrm{N}(P)\right\rangle \mathrm{e}^{-\mathrm{i}(P-P_\mathrm{X})\cdot x} \;. \tag{2}$$

In order to evaluate the inclusive cross section, (2) is squared and summed over all X-states:

$$\sum_{\mathrm{X},P_\mathrm{X}} \left\langle \mathrm{N}(P)\left|\hat{J}_\mu(0)\right|\mathrm{X}(P_\mathrm{X})\right\rangle \left\langle \mathrm{X}(P_\mathrm{X})\left|\hat{J}_\nu(0)\right|\mathrm{N}(P)\right\rangle \quad . \tag{3}$$

In addition we have to average the spin of the incoming nucleon, since this quantity is not observed. Finally energy–momentum conservation is ensured by a δ function, which is due to the integration over x:

$$\begin{aligned} W_{\mu\nu} = \frac{1}{2}\sum_{\mathrm{pol.}}\sum_{\mathrm{X},P_\mathrm{X}} &\left\langle \mathrm{N}(P)\left|\hat{J}_\mu(0)\right|\mathrm{X}(P_\mathrm{X})\right\rangle \left\langle \mathrm{X}(P_\mathrm{X})\left|\hat{J}_\nu(0)\right|\mathrm{N}(P)\right\rangle \\ &\times (2\pi)^3\,\delta^4(P_X - P - q) \quad . \end{aligned} \tag{4}$$

q denotes the momentum of the virtual photon (see Fig. 3.3). At the end of this subsection more detailed arguments for the form of (4) will be given. In particular the factor $(2\pi)^3$, which occurs instead of the usual factor $(2\pi)^4$, deserves an additional explanation. The summation of the projection operator $|\mathrm{X}(P_\mathrm{X})\rangle\langle\mathrm{X}(P_\mathrm{X})|$ in (4) has the form of a complete projection operator. But the completeness relation cannot yet be employed, since the argument of the δ function depends on P_X .

Example 3.2.

Using the integral representation

$$\delta^4(P_X - P - q) = \int \frac{d^4y}{(2\pi)^4} e^{-i(P_X-P-q)\cdot y} \tag{5}$$

we can write (4) in the form

$$\begin{aligned} W_{\mu\nu} &= \frac{1}{2} \sum_{\text{pol.}} \sum_{X,P_X} \frac{1}{2\pi} \int d^4y \, \langle N(P) | \hat{J}_\mu(0) | X(P_X) \rangle \\ &\quad \times \langle X(P_X) | \hat{J}_\nu(0) | N(P) \rangle \, e^{-i(P_X-P)\cdot y} \, e^{iq\cdot y} \\ &= \frac{1}{4\pi} \sum_{\text{pol.}} \sum_{X,P_X} \int d^4y \, \langle N(P) | \hat{J}_\mu(y) | X(P_X) \rangle \langle X(P_X) | \hat{J}_\nu(0) | N(P) \rangle \, e^{iq\cdot y} , \end{aligned} \tag{6}$$

where $\hat{J}_\mu(y)$ has been obtained by means of (2). Now we are able to employ the completeness relation

$$\sum_{X,P_X} |X(P_X)\rangle \langle X(P_X)| = 1 \quad , \tag{7}$$

which yields the expression for $W_{\mu\nu}(P, q)$ given in (3.33):

$$W_{\mu\nu} = \frac{1}{4\pi} \sum_{\text{pol.}} \int d^4y \, e^{iq\cdot y} \langle N(P) | \hat{J}_\mu(y) \hat{J}_\nu(0) | N(P) \rangle \quad . \tag{8}$$

In addition we note that the expectation value of the commutator of the current operators could be used in (8), i.e.,

$$W_{\mu\nu} = \frac{1}{4\pi} \sum_{\text{pol.}} \int d^4y \, e^{iq\cdot y} \langle N(P) | [\hat{J}_\mu(y) \hat{J}_\nu(0)] | N(P) \rangle \, . \tag{9}$$

In order to show the equivalence of (8) and (9) one has to prove that the following expression vanishes:

$$\begin{aligned} &\int d^4y \, e^{iq\cdot y} \langle N(P) | \hat{J}_\nu(0) \hat{J}_\mu(y) | N(P) \rangle \\ &= \sum_{X,P_X} \int d^4y \, e^{iq\cdot y} \langle N(P) | \hat{J}_\nu(0) | X(P_X) \rangle \langle X(P_X) | \hat{J}_\mu(y) | N(P) \rangle \\ &= \sum_{X,P_X} \int d^4y \, e^{i(q+P_X-P)\cdot y} \langle N(P) | \hat{J}_\nu(0) | X(P_X) \rangle \langle X(P_X) | \hat{J}_\mu(0) | N(P) \rangle \\ &= (2\pi)^4 \sum_{X,P_X} \delta^4(P_X - P + q) \langle N(P) | \hat{J}_\nu(0) | X(P_X) \rangle \langle X(P_X) | \hat{J}_\mu(0) | N(P) \rangle \end{aligned} \tag{10}$$

The energy δ function requires that $E_X = P_0 - q_0 = M_N - (E - E') < M_N$ holds in the lab system. However, such a state X does not exist, since the nucleon is the state of lowest energy with baryon number 1. Therefore (10) vanishes and (9) is proved.

EXERCISE

3.3 The Nucleonic Scattering Tensor with Weak Interaction

Problem. Repeat the discussion leading from (3.6) to (3.18) assuming that parity is not conserved, i.e., that Γ_μ consists of Lorentz vectors and Lorentz axial vectors. Take into account that time-reversal invariance still holds.

Solution. In this case the transition current is of the general form

$$\begin{aligned}\bar{u}(P')\Gamma_\mu u(P) = {} & B(P'_\mu + P_\mu)S + \mathrm{i}C(P'_\mu - P_\mu)S + A\,V_\mu + D(P'^\nu + P^\nu)T_{\mu\nu} \\ & + \mathrm{i}E(P'^\nu - P^\nu)T_{\mu\nu} + B'(P'_\mu + P_\mu)P + \mathrm{i}C'(P'_\mu - P_\mu)P + A'\,A_\mu \\ & + D'(P'^\nu + P^\nu)\,\varepsilon_{\mu\nu\alpha\beta}\,T^{\alpha\beta} + \mathrm{i}E'(P'^\nu - P^\nu)\,\varepsilon_{\mu\nu\alpha\beta}\,T^{\alpha\beta} \qquad (1)\end{aligned}$$

with real functions A, B, C, ... and with

$$\begin{aligned} S &= \bar{u}(P')\,u(P) \quad , \\ P &= \bar{u}(P')\,\mathrm{i}\,\gamma_5\,u(P) \quad , \\ V_\mu &= \bar{u}(P')\,\gamma_\mu\,u(P) \quad , \\ A_\mu &= \bar{u}(P')\,\gamma_\mu\,\gamma_5\,u(P) \quad , \\ T_{\mu\nu} &= \bar{u}(P')\,\sigma_{\mu\nu}\,u(P) \quad . \end{aligned} \qquad (2)$$

Writing down (1) we have assumed that the transition current is real. The right-hand side must therefore be invariant under the transformation

$$(\cdots)^*|_{P'_\mu \leftrightarrow P_\mu} \quad . \qquad (3)$$

Time inversion yields

$$S \to S \quad , \quad P \to -P \quad , \quad V_\mu \to V^\mu \quad , A_\mu \to A^\mu \quad , \quad T_{\mu\nu} \to -T^{\mu\nu} \qquad (4)$$

and

$$\boldsymbol{P} \leftrightarrow -\boldsymbol{P}' \quad , \quad P^0 \leftrightarrow P'^0 \quad . \qquad (5)$$

The transition $\boldsymbol{P} \leftrightarrow -\boldsymbol{P}'$, $P^0 \leftrightarrow P'^0$ is due to the complex conjugation of $\bar{u}(P')\,\Gamma_\mu\,u(P)$, which replaces the momentum P'_μ by the negative value of P_μ and vice versa, i.e., $P'_\mu \leftrightarrow -P_\mu$. Because $t \to -t$ there is an additional change in the sign of the zero component. This is easily understood, because under time reversal the direction of motion changes and initial and final states are exchanged. The energies, however, remain positive.

Under combined transformations (4) and (5), (1) assumes the form

$$\begin{aligned}\bar{u}(P')\Gamma_\mu u(P) \to {} & B(P'^\mu + P^\mu)S - \mathrm{i}C(P'^\mu - P^\mu)S + A\,V^\mu - D(P'_\nu + P_\nu)T^{\mu\nu} \\ & + \mathrm{i}E(P'_\nu - P_\nu)T^{\mu\nu} - B'(P'^\mu + P^\mu)P + \mathrm{i}C'(P'^\mu - P^\mu)P + A'\,A^\mu \\ & - D'(P'_\nu + P_\nu)\,\varepsilon_{\mu\nu\alpha\beta}\,T_{\alpha\beta} + \mathrm{i}E'(P'_\nu - P_\nu)\,\varepsilon_{\mu\nu\alpha\beta}\,T_{\alpha\beta} \quad . \qquad (6)\end{aligned}$$

Only the spatial components of the current vector should change sign under time reversal. In order to conserve T invariance (6) must therefore be equal to $\bar{u}(P')\,\Gamma^\mu\,u(P)$. Employing $\varepsilon_{\mu\nu\alpha\beta} = -\varepsilon^{\mu\nu\alpha\beta}$ we have

Exercise 3.3.

$$C = D = B' = E' = 0 \quad . \tag{7}$$

Because of the Gordon decomposition, E can again be replaced by A and B. In an analogous way D' can also be eliminated using A' and C'. In order to derive this identity we consider the expression

$$\begin{aligned}
&\bar{u}(P')\left(-2M_{\mathrm{N}}\not{a}\gamma_5 + \left\{\frac{\not{P}' - \not{P}}{2}, \not{a}\right\}_+ \gamma_5 + \left[\frac{\not{P}' + \not{P}}{2}, \not{a}\right]_- \gamma_5\right) u(P) \\
&\quad = \bar{u}(P')\left(-2M_{\mathrm{N}}\not{a}\gamma_5 + \not{P}'\not{a}\gamma_5 - \not{a}\not{P}\gamma_5\right) u(P) \\
&\quad = \bar{u}(P')\left(-2M_{\mathrm{N}}\not{a}\gamma_5 + M_{\mathrm{N}}\not{a}\gamma_5 + \not{a}\gamma_5 M_{\mathrm{N}}\right) u(P) \\
&\quad = 0 \quad .
\end{aligned} \tag{8}$$

Differentiating this relation with respect to a^μ yields

$$\begin{aligned}
0 &= -2M_{\mathrm{N}}A_\mu + \bar{u}(P')\Big[(P'_\mu - P_\mu)\gamma_5 - \mathrm{i}(P'^\nu + P^\nu)\sigma_{\nu\mu}\,\gamma_5\Big]u(P) \\
&= -2M_{\mathrm{N}}A_\mu - \mathrm{i}(P'_\mu - P_\mu)P - \frac{1}{2}\,\varepsilon_{\mu\nu\alpha\beta}\,T^{\alpha\beta}(P'^\nu + P^\nu) \quad ,
\end{aligned} \tag{9}$$

which represents te desired result. Consequently we have reduced (1) to

$$\bar{u}(P')\Gamma_\mu u(P) = B(P'_\mu + P_\mu)S + A\,V_\mu + \mathrm{i}C'(P'_\mu - P_\mu)P + A'A_\mu \quad . \tag{10}$$

$W_{\mu\nu}$ then becomes

$$\begin{aligned}
W_{\mu\nu} &= \mathrm{tr}\Bigg\{\Big[A\gamma_\mu + B(P'_\mu + P_\mu) \\
&+ A'\gamma_\mu\gamma_5 + C'(P'_\mu - P_\mu)\gamma_5)\Big]\frac{\not{P}' + M_{\mathrm{N}}}{2M_{\mathrm{N}}}\Big[A\gamma_\nu + B(P'_\nu + P_\nu) \\
&+ A'\gamma_\nu\gamma_5 - C'(P'_\nu - P_\nu)\gamma_5\Big]\frac{\not{P} + M_{\mathrm{N}}}{2M_{\mathrm{N}}}\Bigg\} \\
&= \frac{1}{4M_{\mathrm{N}}^2}\mathrm{tr}\Big\{\Big[A\gamma_\mu + B(P'_\mu + P_\mu)\Big](\not{P}' + M_{\mathrm{N}})\Big[A\gamma_\nu + B(P'_\nu + P_\nu)\Big]\not{P} + M_{\mathrm{N}})\Big\} \\
&+ \frac{1}{4M_{\mathrm{N}}^2}\mathrm{tr}\Big\{\,[A'\gamma_\mu + C'(P'_\mu + P_\mu)]\,(-\not{P}' + M_{\mathrm{N}})\,[-A'\gamma_\nu - C'(P'_\nu - P_\nu)]\,(\not{P} + M_{\mathrm{N}})\Big\} \\
&+ \frac{1}{4M_{\mathrm{N}}^2}\mathrm{tr}\,\{A'\gamma_\mu\gamma_5\not{P}'A\gamma_\nu\not{P} + A\gamma_\mu\not{P}'A'\gamma_\nu\gamma_5\not{P}\}\,.
\end{aligned} \tag{11}$$

The first two traces do not have to be evaluated explicitly. It is sufficient to know that because $P'_\mu = q_\mu + P_\mu$ their contribution can only be of the form

$$g_{\mu\nu}V_1 + P_\mu P_\nu V_2 + (P_\mu q_\nu + P_\nu q_\mu)V_3 + q_\mu q_\nu V_4 \quad . \tag{12}$$

In the case of the last trace the identity

$$\mathrm{tr}\{\gamma_5\not{a}\not{b}\not{c}\not{d}\} = 4\mathrm{i}\varepsilon_{\mu\nu\alpha\beta}\,a^\mu\,b^\nu\,c^\alpha\,d^\beta$$

holds. Therefore the hadronic tensor is of the form

$$W_{\mu\nu} = g_{\mu\nu}V_1 + P_\mu P_\nu V_2 + (P_\mu q_\nu + P_\nu q_\mu)V_3 + q_\mu q_\nu V_4 + \mathrm{i}\varepsilon_{\mu\nu\alpha\beta}P^\alpha q^\beta V_5\ . \tag{13}$$

Exercise 3.3.

We have derived (13) for elastic lepton–nucleon scattering. In the case of inelastic processes, including particle creation, $q_\mu = P'_\mu - P_\mu$ is no longer valid. $q \cdot P'$, q^2 and $q \cdot P$ then are independent quantities. However, the results again are quite simple if one sums over all possible final states, i.e., over the final states of the reactions

$$\begin{aligned} \nu_\mathrm{e}(p) + \mathrm{N}(P) &\to \mathrm{e}^-(p-q) + \mathrm{X} \quad , \\ \bar{\nu}_\mathrm{e}(p) + \mathrm{N}(P) &\to \mathrm{e}^+(p-q) + \mathrm{X} \quad . \end{aligned} \tag{14}$$

Therefore the inclusive inelastic scattering tensor for parity-violating lepton–nucleon scattering is

$$\begin{aligned} W_{\mu\nu} =& g_{\mu\nu} V_1(Q^2,\nu) + P_\mu P_\nu V_2(Q^2,\nu) + (P_\mu q_\nu + P_\nu q_\mu) V_3(Q^2,\nu) \\ &+ q_\mu q_\nu V_4(Q^2,\nu) + \mathrm{i}\varepsilon_{\mu\nu\alpha\beta} P^\alpha q^\beta V_5(Q^2,\nu) \quad . \end{aligned} \tag{15}$$

EXERCISE

3.4 The Inclusive Weak Lepton-Nucleon Scattering

Problem. The coupling of the charged vector bosons $W^\pm_\mu$ of the weak interaction to leptons is described by (see Sect. 4.1)

$$\frac{g}{\sqrt{2}} \bar{\nu}(x) \gamma_\mu \frac{1-\gamma_5}{2} \mathrm{e}(x) W^\mu(x) + \mathrm{h.c.} \quad . \tag{1}$$

Derive from this equation the leptonic scattering tensor $L_{\mu\nu}$. Employ the result of Exercise 3.3 to evaluate the differential cross section for the inclusive weak lepton-nucleon scattering.

Solution. Since the neutrino is massless and the electron mass can be neglected, it is helpful to choose the following spinor normalization.

$$\begin{aligned} &u^\dagger_\alpha(p) u_\beta(p) = 2E\delta_{\alpha\beta} \quad , \\ &\sum_{\text{spin}} u(p)\bar{u}(p) = \not{p} + m = \not{p} \quad . \end{aligned} \tag{2}$$

The leptonic scattering tensor is then

$$\begin{aligned} L_{\mu\nu} &= \frac{g^2}{8} \sum_s \mathrm{tr}\Big\{ \bar{\nu}(p) \gamma_\mu (1-\gamma_5) \mathrm{e}(p',s) \bar{\mathrm{e}}(p',s) \gamma_\nu (1-\gamma_5) \nu(p) \Big\} \\ &= \frac{g^2}{8} \mathrm{tr}\Big\{ \not{p} \gamma_\mu (1-\gamma_5) \not{p}' \gamma_\nu (1-\gamma_5) \Big\} \\ &= \frac{g^2}{4} \Big(\mathrm{tr}\Big\{ \not{p} \gamma_\mu \not{p}' \gamma_\nu \Big\} - \mathrm{tr}\Big\{ \gamma_5 \not{p}' \gamma_\nu \not{p} \gamma_\mu \Big\} \Big) \\ &= g^2 \Big(p'_\mu p_\nu + p_\mu p'_\nu - p' \cdot p \, g_{\mu\nu} - \mathrm{i}\varepsilon_{\mu\nu\alpha\beta} p^\alpha p'^\beta \Big) \quad . \end{aligned} \tag{3}$$

Exercise 3.4.

Since the neutrino is always left-handed, there is no averaging of the initial spin. In order to obtain the trace in the second line of (3) we have summed over all neutrino spins and made use of (2). This procedure is correct, because the operator $(1-\gamma_5)$ cancels the contribution of the non physical right-handed neutrino state. In analogy to (3.22) the differential cross section then becomes

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}E'\,\mathrm{d}\Omega}=\frac{E'}{(4\pi)^2E}\left(\frac{1}{q^2-M_{\mathrm{W}}^2}\right)^2 L_{\mu\nu}\frac{1}{2}\left(\frac{g}{2}\right)^2 W_{\mu\nu} \quad . \tag{4}$$

The differences can be explained as follows.

1. The propagator for massive W bosons differs from that for massless photons. For $|q^2|\ll M_{\mathrm{W}}^2$ we can use

$$\left(\frac{1}{q^2-M_{\mathrm{W}}^2}\right)^2\approx\frac{1}{M_{\mathrm{W}}^4} \quad . \tag{5}$$

   This yields Fermi's theory of weak interaction.
2. The coupling constant of the W bosons to the nucleons is usually chosen to be $g/\sqrt{2}$.

In order to evaluate (4) we first show that $q^\mu L_{\mu\nu}$ and $q^\nu L_{\mu\nu}$ vanish:

$$\begin{aligned}q^\mu L_{\mu\nu}&=g^2\left(q\cdot p'\,p_\nu+q\cdot p\,p'_\nu-p'\cdot p\,q_\nu-\mathrm{i}\varepsilon_{\mu\nu\alpha\beta}\,q^\mu p^\alpha p'^\beta\right)\\&=g^2\left[-\frac{q^2}{2}p_\nu+\frac{q^2}{2}p'_\nu+\frac{q^2}{2}(p_\nu-p'_\nu)\right]=0 \quad .\end{aligned} \tag{6}$$

Here we have employed

$$\begin{aligned}&q_\mu=p_\mu-p'_\mu \quad\Rightarrow\quad p\cdot p'=-\frac{q^2}{2};\\&p\cdot q=\frac{q^2}{2}\quad\text{and}\quad p'\cdot q=-\frac{q^2}{2} \quad .\end{aligned} \tag{7}$$

With help of Equation (15) from Exercise 3.3 it therefore follows that

$$\begin{aligned}L^{\mu\nu}W_{\mu\nu}&=L^{\mu\nu}\left(g_{\mu\nu}V_1+P_\mu P_\nu V_2+\mathrm{i}\varepsilon_{\mu\nu\alpha\beta}P^\alpha q^\beta V_5\right)\\&=g^2\Big[(2p'\cdot p-4p'\cdot p)V_1+\left(2p'\cdot P\,p\cdot P-M_N^2\,p\cdot p'\right)V_2\\&\quad+\varepsilon^{\mu\nu}_{\alpha\beta}\,\varepsilon_{\mu\nu\delta\gamma}\,p^\delta q^\gamma P^\alpha q^\beta V_5\Big] \quad .\end{aligned} \tag{8}$$

With

$$p=(E,\boldsymbol{p}) \quad , \quad p'=(E',\boldsymbol{p}') \quad , \quad P=(M,\boldsymbol{0}) \quad , \tag{9}$$

and

$$\varepsilon^{\mu\nu}_{\alpha\beta}\,\varepsilon_{\mu\nu\delta\gamma}=-2\left(g_{\alpha\delta}\,g_{\beta\gamma}-g_{\alpha\gamma}\,g_{\beta\delta}\right) \quad , \tag{10}$$

this assumes the form

$$L^{\mu\nu}W_{\mu\nu} = g^2\left[q^2V_1 + M_{\mathrm{N}}^2\left(2EE' + \frac{1}{2}q^2\right)V_2 - 2\left(p\cdot P\,q^2 - P\cdot q\;p\cdot q\right)V_5\right]$$

$$= g^2\left\{q^2V_1 + 2M_{\mathrm{N}}^2\left(EE' + \frac{1}{4}q^2\right)V_2 - 2M_{\mathrm{N}}\left[Eq^2 - (E-E')\frac{q^2}{2}\right]V_5\right\} \ . \qquad (11)$$

Again using (3.30), we find that

$$q^2 = -Q^2 = -4EE'\sin^2\left(\frac{\vartheta}{2}\right) \qquad (12)$$

leads to

$$L^{\mu\nu}W_{\mu\nu} = g^2 2EE'\left[-2\sin^2\left(\frac{\vartheta}{2}\right)V_1 + M_{\mathrm{N}}^2\cos^2\left(\frac{\vartheta}{2}\right)V_2 + 2M_{\mathrm{N}}\sin^2\left(\frac{\vartheta}{2}\right)(E+E')V_5\right] \ . \qquad (13)$$

In order to provide all structure functions with the same dimension we define

$$W_1 = -V_1 \quad , \quad W_2 = -\frac{V_2}{M_{\mathrm{N}}^2} \quad , \quad W_3 = \frac{V_5}{2M_{\mathrm{N}}^2} \qquad (14)$$

and obtain

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}E'\mathrm{d}\Omega} = \frac{E'^2g^4}{4M_{\mathrm{W}}^4(4\pi)^2}\left[2\sin^2\left(\frac{\vartheta}{2}\right)W_1(Q^2,\nu) + \cos^2\left(\frac{\vartheta}{2}\right)W_2(Q^2,\nu) - \frac{E+E'}{M_{\mathrm{N}}}\sin^2\left(\frac{\vartheta}{2}\right)W_3(Q^2,\nu)\right] \ . \qquad (15)$$

Finally we substitute the so-called Fermi constant of weak interaction:

$$G_{\mathrm{F}} = \frac{g^2}{4\sqrt{2}M_{\mathrm{W}}^2} \quad ; \qquad (16)$$

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}E'\mathrm{d}\Omega} = \frac{G_{\mathrm{F}}^2E'^2}{2\pi^2}\left[2\sin^2\left(\frac{\vartheta}{2}\right)W_1(Q^2,\nu) + \cos^2\left(\frac{\vartheta}{2}\right)W_2(Q^2,\nu) - \frac{E+E'}{M_{\mathrm{N}}}\sin^2\left(\frac{\vartheta}{2}\right)W_3(Q^2,\nu)\right] \ . \qquad (17)$$

This is identical to (3.37). In the reaction $\bar{\nu}+\mathrm{N} \to \mathrm{e}^+ + \mathrm{X}$ the incoming antineutrino corresponds to the outgoing electron in the reaction $\nu+\mathrm{N} \to \mathrm{e}^- + \mathrm{X}$, i.e., p and p' are exchanged. Since the third term in $L_{\mu\nu}$ is antisymmetric under this exchange, the sign of W_3 changes.

We shall now discuss the experimental structure functions of electron–nucleon scattering. These functions will have a particularly simple form when ν is replaced by $x = Q^2/2\nu M_{\mathrm{N}}$. This quantity is called the Bjorken or scaling variable. We shall first consider some of its properties. For elastic scattering (3.19) and (3.7) give

$$x = \frac{Q^2}{2\nu M_{\mathrm{N}}} = \frac{-q^2}{2P\cdot q} = \frac{-q^2}{-q^2} = 1 \ . \qquad (3.39)$$

Since the nucleon is the lightest state with baryon number 1, we know in addition that the invariant mass of the final state

$$M_\mathrm{X} = \sqrt{(P+q)^2} \tag{3.40}$$

must be larger than M_N. If this were not the case, there would be a state X (of arbitrary structure) with baryon number 1 into which the proton could decay, by emitting γ quanta, for example.

From (3.40) it follows that x is always positive and less than 1:

$$x = \frac{Q^2}{(P+q)^2 - P^2 + Q^2} = \frac{Q^2}{M_\mathrm{X}^2 - M_\mathrm{N}^2 + Q^2} = \frac{1}{1 + \frac{M_\mathrm{X}^2 - M_\mathrm{N}^2}{Q^2}} \quad ; \tag{3.41}$$

Q^2 is always positive, as can be seen from (3.30), and

$$0 < x \leq 1 \quad . \tag{3.42}$$

Additionally, it is customary to include a factor of ν in the definition of W_2 and W_3 and a factor of M_N in the definition of W_1. We thus write

$$\begin{aligned} M_\mathrm{N} W_1^\mathrm{eN}(Q^2,\nu) &= F_1^\mathrm{eN}(Q^2,x) \quad , \\ \nu W_2^\mathrm{eN}(Q^2,\nu) &= F_2^\mathrm{eN}(Q^2,x) \quad , \\ \nu W_3^\mathrm{eN}(Q^2,\nu) &= F_3^\mathrm{eN}(Q^2,x) \quad . \end{aligned} \tag{3.43}$$

Frequently, the parameter Q^2 is replaced by a new quantity, the inelasticity parameter y:

$$y = \frac{\nu}{E} = \frac{E - E'}{E} \quad . \tag{3.44}$$

y is dimensionless and specifies which fraction of the lepton energy in the laboratory system is transferred to the nucleon. The value of y thus lies in the range $0 \leq y \leq 1$. The meaning of Q^2, ν, and x is also illustrated in Fig. 3.4. It can easily be checked (see Exercise 3.5) that

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}E'\mathrm{d}\Omega} = \frac{E'}{2\pi M_\mathrm{N} E y} \frac{\mathrm{d}^2\sigma}{\mathrm{d}x\,\mathrm{d}y} \quad . \tag{3.45}$$

In this notation, (3.32) (see Exercise 3.5) becomes

$$\left.\frac{\mathrm{d}^2\sigma}{\mathrm{d}x\,\mathrm{d}y}\right|_\mathrm{eN} = \frac{8\pi M_\mathrm{N} E \alpha^2}{Q^4} \left[xy^2 F_1^\mathrm{eN}(Q^2,x) + \left(1 - y - \frac{M_\mathrm{N} x y}{2E}\right) F_2^\mathrm{eN}(Q^2,x) \right] . \tag{3.46}$$

The first measurements of inelastic electron–proton scattering at high energies were performed in 1968 at the Stanford Linear Accelerator (SLAC). These and subsequent experiments have yielded two important results:

1. For sufficiently large values of Q^2, i.e., for $Q^2 \geq 1\mathrm{GeV}^2$, F_2^eN and F_1^eN do not depend significantly on Q^2, but only on x. This behavior is termed *Bjorken scaling*.
2. In this domain (i.e., for $Q^2 \geq 1\mathrm{GeV}^2$) it holds to good approximation that

$$F_2^\mathrm{eN} \approx 2x F_1^\mathrm{eN} \quad .$$

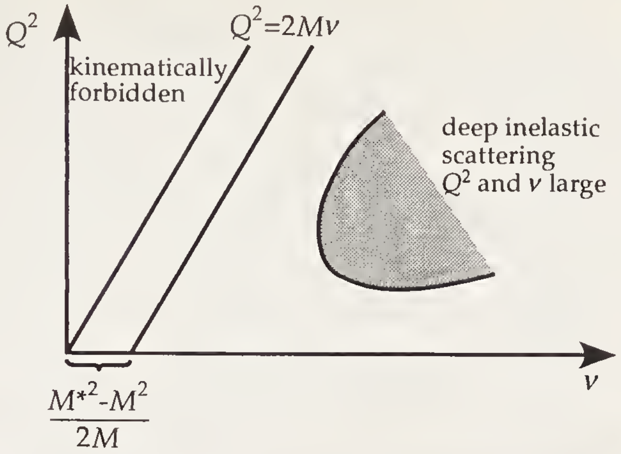

Fig. 3.4. The physical meaning of the different domains of x, Q^2, and ν. For elastic scattering only, $Q^2 = 2M\nu$ holds, otherwise Q^2 and ν are independent

EXERCISE

3.5 The Cross Section as a Function of x and y

Problem. Derive (3.45) and (3.46).

Solution. Equations (3.44), (3.39), and (3.30) yield

$$y = \frac{E-E'}{E} = 1 - \frac{E'}{E} \quad ,$$
$$x = \frac{Q^2}{2(E-E')M_{\mathrm{N}}} = \frac{2EE'}{(E-E')M_{\mathrm{N}}} \sin^2\left(\frac{\theta}{2}\right) \quad . \tag{1}$$

Owing to the cylindrical symmetry of the problem we have in addition

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}E'\mathrm{d}\Omega} = \frac{\mathrm{d}^2\sigma}{\mathrm{d}E' 2\pi \sin(\theta)\mathrm{d}\theta} \quad . \tag{2}$$

Therefore we only have to evaluate the Jacobi determinant

$$\begin{aligned}\frac{d^2\sigma}{dE'd\Omega} &= \frac{1}{2\pi\sin(\theta)} \left|\frac{\partial(x,y)}{\partial(E',\theta)}\right| \frac{\mathrm{d}^2\sigma}{\mathrm{d}x\,\mathrm{d}y} \\ &= \begin{vmatrix} \frac{2E\sin^2\left(\frac{\theta}{2}\right)}{M_{\mathrm{N}}}\left(\frac{E'+(E-E')}{(E-E')^2}\right) & \frac{2EE'}{(E-E')M_{\mathrm{N}}}\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) \\ -\frac{1}{E} & 0 \end{vmatrix} \frac{1}{2\pi\sin(\theta)}\frac{\mathrm{d}^2\sigma}{\mathrm{d}x\,\mathrm{d}y} \\ &= \frac{1}{2\pi\sin(\theta)}\frac{2E'}{(E-E')M_{\mathrm{N}}}\frac{\sin(\theta)}{2}\frac{\mathrm{d}^2\sigma}{\mathrm{d}x\,\mathrm{d}y} \\ &= \frac{E'}{2\pi M_{\mathrm{N}} E y}\frac{\mathrm{d}^2\sigma}{\mathrm{d}x\,\mathrm{d}y} \quad .\end{aligned} \tag{3}$$

This is identical to (3.45). With the help of the definitions introduced in (3.39–3.44) the double-differential cross section for electron–nucleon scattering consequently becomes

Exercise 3.5.

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}x\,\mathrm{d}y} = \frac{8\pi M_\mathrm{N} y E E' \alpha^2}{Q^4}\left[\frac{2}{M_\mathrm{N}}\sin^2\left(\frac{\theta}{2}\right) F_1^{\mathrm{eN}}(Q^2,x) + \frac{\cos^2\left(\frac{\theta}{2}\right)}{yE} F_2^{\mathrm{eN}}(Q^2,x)\right] . \quad (4)$$

Here we have employed (3.32) and (3.43). Now we replace θ by x and y:

$$\sin^2\left(\frac{\theta}{2}\right) = \frac{(E-E')M_\mathrm{N}}{2EE'}x = \frac{M_\mathrm{N}}{2E'}xy \quad , \qquad (5)$$

$$\begin{aligned}\frac{\mathrm{d}^2\sigma}{\mathrm{d}x\,\mathrm{d}y} &= \frac{8\pi M_\mathrm{N} y E E' \alpha^2}{Q^4}\left[\frac{xy}{E'}F_1^{\mathrm{eN}}(Q^2,x) + \frac{1-\frac{M_\mathrm{N}}{2E'}xy}{yE}F_2^{\mathrm{eN}}(Q^2,x)\right] \\ &= \frac{8\pi M_\mathrm{N} E \alpha^2}{Q^4}\left[xy^2 F_1^{\mathrm{eN}}(Q^2,x) + \left(1 - y - \frac{M_\mathrm{N}}{2E}xy\right) F_2^{\mathrm{eN}}(Q^2,x)\right] \quad , \qquad (6)\end{aligned}$$

which yields (3.46).

These results are illustrated in Figs. 3.5 and 3.6. In Fig. 3.5 it can be seen that at least for medium values of x the measured points are practically horizontal. In the following we shall discuss the physical meaning of this. In Chap. 5 we shall discuss how the Q^2 dependence of the structure functions for large and small x can be understood in the framework of quantum chromodynamics.

These two unusual properties can be easily reduced to the common physical cause that the scattering is off pointlike spin-$\frac{1}{2}$ constituents of the nucleon. These constituents are termed, following R. P. Feynman, *partons*. What do we know about this kind of scattering?

The scattering cross section for the scattering of electrons off a pointlike parton of mass M_i and charge $q_i e$ can be constructed from the results for electron–muon scattering, namely (2.165), Equation (11) in Exercise 2.13, and the fact that

$$2\left(E - E'\right)M = 2\nu M = Q^2 = +4EE'\sin^2\left(\frac{\theta}{2}\right) \quad ,$$

yielding

$$\frac{E}{E'} = 1 + \frac{2E}{M}\sin^2\left(\frac{\theta}{2}\right) \quad .$$

Putting all this together gives

$$\begin{aligned}\frac{\mathrm{d}\sigma'}{\mathrm{d}\Omega} &= \frac{\alpha^2 q_i^2}{4E^2\sin^4\left(\frac{\theta}{2}\right)}\frac{1}{1+\frac{2E}{M}\sin^2\left(\frac{\theta}{2}\right)}\cos^2\left(\frac{\theta}{2}\right)\left[1 - \frac{q^2\tan^2\left(\frac{\theta}{2}\right)}{2M^2}\right] \\ &= \frac{\alpha^2 q_i^2}{4E^2}\frac{1}{\sin^4\left(\frac{\theta}{2}\right)}\frac{\left[\cos^2\left(\frac{\theta}{2}\right) - \frac{q^2}{2M_i^2}\sin^2\left(\frac{\theta}{2}\right)\right]}{1+\frac{2E}{M_i}\sin^2\left(\frac{\theta}{2}\right)} \quad . \qquad (3.47)\end{aligned}$$

This cross section is computed in the laboratory system as the rest system of the parton. The laboratory system, however, is quite unsuitable for our needs since here partons cannot be treated as free particles because they are held together by the interaction to which this volume is dedicated. Thus we do not know at all the

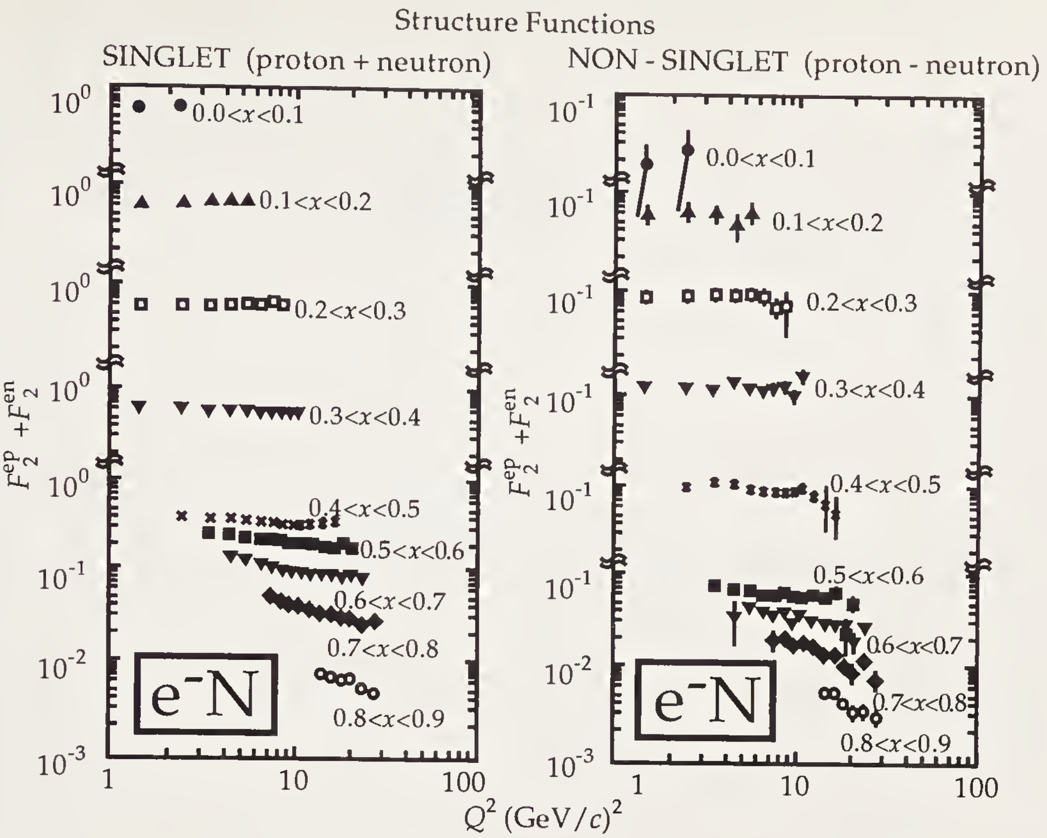

Fig. 3.5. The F_2 structure functions for electron–proton and electron–neutron scattering have been combined into $F_2^{\mathrm{ep}} \pm F_2^{\mathrm{en}}$ and plotted against Q^2 The Bjorken variable x is defined as $x = Q^2/2\nu M_{\mathrm{N}}$ (see Review of Particle Properties: Phys. Lett. **170** (1986))

initial momenta of the partons; they are definitely not at rest. There are, however, frames of reference where the initial momentum is known. These are all frames of reference in which the nucleon is moving very fast. The momentum of the ith parton is then

$$P_{i,\mu} = \xi_i \cdot P_\mu + \Delta P_{i\mu} \quad ; \quad \max_\mu(\Delta P_\mu) \ll \max_\mu(P_\mu) \quad . \tag{3.48}$$

The unknown momentum fraction $\Delta P_{i\mu'}$, which comes from the interactions of the partons with each other, is much smaller than the momentum from the collective motion. To describe electron–parton scattering sensibly, (3.47) must be transformed into a different Lorentz system.[5] This is done by rewriting it using only the invariant variables

$$\begin{aligned}(p-p')^2 &=: t = -Q^2 = (P-P')^2 \quad , \\ (p+P)^2 &=: s = (p'+P')^2 \quad , \\ (P'-P)^2 &=: u = (p-P')^2 \quad .\end{aligned} \tag{3.49}$$

These quantities are called *Mandelstam variables*. In the laboratory system

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[5] Here and in the following we omit the parton index i of the mass and momentum variables.

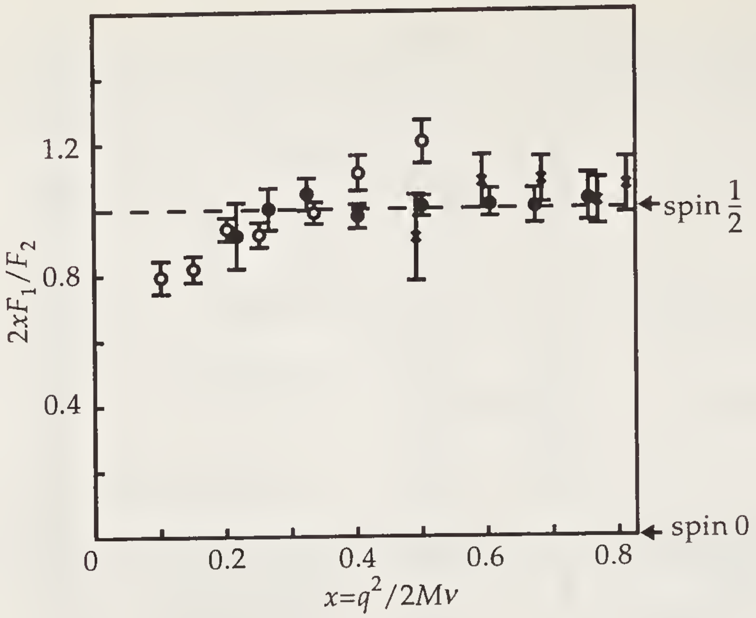


Fig. 3.6. The ratio of the structure functions F_1 and F_2 within the scaling region provides a test of the so-called Callan–Gross relations (see (3.82))

$$Q^2 = 4EE' \sin^2\left(\frac{\theta}{2}\right) \quad , \tag{3.50a}$$

$$s = 2p \cdot P + M^2 = M(2E + M) \quad , \tag{3.50b}$$

$$u = -2p' \cdot P + M^2 = -M(2E' - M) \quad . \tag{3.50c}$$

We thus use (note the cylindrical symmetry)

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{2\pi} \frac{\mathrm{d}\sigma}{\sin(\theta)\mathrm{d}\theta} = \frac{1}{2\pi \sin(\theta)} \frac{\mathrm{d}Q^2}{\mathrm{d}\theta} \frac{\mathrm{d}\sigma}{\mathrm{d}Q^2} \quad . \tag{3.51}$$

From $P'_\mu = P_\mu + p_\mu - p'_\mu$, it follows that

$$M^2 = P'^2_\mu = (P_\mu + p_\mu - p'_\mu)^2 = M^2 + (2E - E')M - Q^2 \quad , \tag{3.52}$$

or, after solving for E',

$$E' = E\left[1 + 2\frac{E}{M}\sin^2\left(\frac{\theta}{2}\right)\right]^{-1} \quad . \tag{3.53}$$

For the derivative $\mathrm{d}Q^2/\mathrm{d}\theta$ we obtain, with (3.50a) and (3.53),

$$\begin{aligned}
\frac{\mathrm{d}Q^2}{\mathrm{d}\theta} &= 4EE' \sin\left(\frac{\theta}{2}\right) \cos\left(\frac{\theta}{2}\right) - \frac{E\, 2\frac{E}{M} \sin\left(\frac{\theta}{2}\right) \cos\left(\frac{\theta}{2}\right)}{\left[1 + 2\frac{E}{M}\sin^2\left(\frac{\theta}{2}\right)\right]^2} 4E \sin^2\left(\frac{\theta}{2}\right) \\
&= 2EE' \sin\theta \left\{1 - \frac{2E \sin^2\left(\frac{\theta}{2}\right)}{M\left[1 + 2\frac{E}{M}\sin^2\left(\frac{\theta}{2}\right)\right]^2}\right\} \\
&= \frac{2EE' \sin\theta}{\left[1 + 2\frac{E}{M}\sin^2\left(\frac{\theta}{2}\right)\right]} \quad .
\end{aligned} \tag{3.54}$$

For $\mathrm{d}\sigma/\mathrm{d}Q^2$ we therefore find according to (3.51) that

$$\frac{d\sigma}{dQ^2} = \frac{\pi}{EE'}\frac{\alpha^2 q_i^2}{4E^2}\frac{1}{\sin^4\left(\frac{\theta}{2}\right)}\left[\cos^2\left(\frac{\theta}{2}\right) + \frac{Q^2}{2M_i^2}\sin^2\left(\frac{\theta}{2}\right)\right] \quad . \tag{3.55}$$

We further insert, according to (3.50a,b,c),

$$\begin{aligned} E' &= \frac{u - M_i^2}{-2M_i} \quad , \quad E = \frac{s - M_i^2}{2M_i} \quad , \\ \sin^2\left(\frac{\theta}{2}\right) &= \frac{Q^2}{4EE'} = -\frac{Q^2 M_i^2}{\left(u - M^2\right)\left(s - M_i^2\right)} \quad , \\ \cos^2\left(\frac{\theta}{2}\right) &= 1 + \frac{Q^2 M_i^2}{\left(u - M_i^2\right)\left(s - M_i^2\right)} \quad , \end{aligned} \tag{3.56}$$

so that after appropriate substitutions, (3.55) becomes

$$\frac{d\sigma_i}{dQ^2} = \frac{4\pi\alpha^2 q_i^2}{Q^4}\left(-\frac{u_i - M_i^2}{s_i - M_i^2}\right)\left[1 + \left(1 - \frac{Q^2}{2M_i^2}\right)\frac{Q^2 M_i^2}{\left(u_i - M_i^2\right)\left(s_i - M_i^2\right)}\right] . \tag{3.57}$$

The quantities u_i and s_i refer to the ith parton. With this, we have succeeded in the first step. We must now choose an appropriate laboratory system and substitute the parton masses. It is now a major statement of the parton model that one obtains a good description of experimental results when one assumes that partons are massless. For $M_i \longrightarrow 0$, (3.57) simplifies to

$$\frac{d\sigma_i}{dQ^2} = \frac{4\pi\alpha^2 q_i^2}{Q^4}\left(\frac{Q^4}{2u_i s_i} - 1\right)\frac{u_i}{s_i} \quad . \tag{3.58a}$$

In addition, it holds that

$$\begin{aligned} Q^2 - s_i - u_i &= 2p \cdot p' - 2p \cdot P_i + 2p \cdot P_i' \\ &= 2p \cdot (p' - P_i + P_i') = 2p^2 = 0 \end{aligned} \tag{3.59}$$

and therefore

$$\left(\frac{Q^4}{2u_i s_i} - 1\right)\frac{u_i}{s_i} = \frac{s_i^2 + u_i^2}{2s_i^2} \quad . \tag{3.60}$$

Thus (3.58a) becomes

$$\frac{d\sigma_i}{dQ^2} = \frac{4\pi\alpha^2 q_i^2}{Q^4}\left(\frac{s_i^2 + u_i^2}{2s_i^2}\right) \quad . \tag{3.58b}$$

For our frame of reference, we now choose the so-called *Breit system*. This is characterized by the property that q_μ is purely spacelike and points in the z direction (see Exercise 3.6):

$$\begin{aligned} q_\mu &= \left(0, 0, 0, \sqrt{-q^2}\right) = \left(0, 0, 0, \sqrt{Q^2}\right) \quad , \\ P_\mu &= \left(\tilde{P}, 0, 0, -\tilde{P}\right) \quad , \quad \tilde{P} \gg M_N \quad . \end{aligned} \tag{3.61}$$

Since we neglect the momentum fraction of the partons stemming from the parton–parton interaction, for consistency we must also neglect the nucleon masses:

$$P_{i,\mu} = \xi_i \; (\tilde{P}, 0, 0 - \tilde{P}) \quad . \tag{3.62}$$

The cross section for the scattering off a parton in the Breit system thus becomes

$$\begin{aligned} \frac{\mathrm{d}\sigma_i}{\mathrm{d}Q^2} &= \frac{4\pi\alpha^2 q_i^2}{Q^4} \left(\frac{(p+P_i)^4 + (p'-P_i)^4}{2(p+P_i)^4} \right) \\ &= \frac{4\pi\alpha^2 q_i^2}{Q^4} \left(\frac{(\xi_i 2p \cdot P)^2 + (\xi_i 2p' \cdot P)^2}{2(2\xi_i p \cdot P)^2} \right) \\ &= \frac{4\pi\alpha^2 q_i^2}{Q^4} \left(\frac{(p+P)^4 + (p'-P)^4}{2(p+P)^4} \right) \\ &= \frac{4\pi\alpha^2 q_i^2}{Q^4} \left(\frac{s^2+u^2}{2s^2} \right) \quad . \end{aligned} \tag{3.63}$$

s and u are now Mandelstam variables for the whole nucleon. We recognize that $\mathrm{d}\sigma/\mathrm{d}Q^2$ is completely independent of ξ_i. To compare this expression with (3.46), we must write in the Breit system not only the elastic scattering cross section for pointlike particles but also the inelastic scattering cross section for particles with an internal structure. To this end, (3.46) is rewritten such that only Lorentz-invariant quantities appear. For $M_{\mathrm{N}}^2 \ll u, s, Q^2$ (see (3.56))

$$\begin{aligned} E' &= -\frac{u}{2M_{\mathrm{N}}} \quad , \quad E = \frac{s}{2M_{\mathrm{N}}} \quad , \\ y &= \frac{E - E'}{E} \equiv \frac{s+u}{s} \end{aligned} \tag{3.64}$$

and

$$x = \frac{Q^2}{2\nu M_{\mathrm{N}}} = \frac{Q^2}{s+u} \quad . \tag{3.65}$$

Equation (3.46) is thus nearly in invariant form. One only has to substitute s for E:

$$\left.\frac{\mathrm{d}^2\sigma}{\mathrm{d}x\,\mathrm{d}y}\right|_{\mathrm{eN}} = \frac{4\pi s\alpha^2}{Q^4} \left[xy^2 F_1^{\mathrm{eN}}(Q^2,x) + \left(1 - y - \frac{xyM_{\mathrm{N}}^2}{s}\right) F_2^{\mathrm{eN}}(Q^2,x) \right] . \tag{3.66}$$

Because $M_{\mathrm{N}} \ll s$ the second term can be further simplified. This expression must now be compared to the sum over partons i of (3.63). To this end, we must first rewrite (3.63) as a double-differential cross section.

To do this, we start from the momentum conservation $P_i' = P_i + q$ of the scattering parton, i.e.,

$$0 \approx P_i'^2 = (P_i + q)^2 = (\xi_i P + q)^2 = \xi_i^2 M_i^2 + 2\xi_i P \cdot q + q^2 \quad . \tag{3.67}$$

When the parton mass vanishes, this implies that

$$\xi_i = \frac{Q^2}{2P \cdot q} = x \quad . \tag{3.68}$$

If we observe scattering characterized by a Bjorken variable x, this means in the parton model that the interacting parton carried a fraction $\xi_i = x$ of the nucleon's total momentum! So the variable x has a simple meaning (however, only in reference frames with very large nucleon momentum).

Owing to (3.68), (3.65) can immediately be written as a double differential:

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}x\,\mathrm{d}Q^2} = \frac{4\pi\alpha^2 q_i^2}{Q^4}\frac{s^2+u^2}{2s^2}\delta(\xi_i - x) \quad . \tag{3.69}$$

The transition from Q^2 to the desired variable y occurs with the aid of (3.39), (3.44), and (3.56), according to which

$$Q^2 = 2\,P\cdot q\,x = 2M_\mathrm{N}(E-E')x = 2M_\mathrm{N}Eyx = syx \quad , \tag{3.70}$$

and we thus obtain

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}x\,\mathrm{d}y} = \left.\frac{\mathrm{d}Q^2}{\mathrm{d}y}\right|_x \frac{\mathrm{d}^2\sigma}{\mathrm{d}x\,\mathrm{d}Q^2} = \frac{2\pi\alpha^2 q_i^2}{Q^4}\frac{s^2+u^2}{s^2}\delta(\xi_i - x)\frac{Q^2 s}{s+u} \quad . \tag{3.71}$$

Here, in the last step, we have put $x = Q^2/(s+u)$, according to (3.65). To obtain the total cross section in the parton model, we must sum (3.71) over all partons i and all possible momentum fractions ξ_i. The single summands must be weighted with their proper probabilities $f_i(\xi_i)$:

$$\begin{aligned}\left.\frac{\mathrm{d}^2\sigma}{\mathrm{d}x\,\mathrm{d}y}\right|_{\mathrm{eN}} &= \sum_i \int_0^1 \mathrm{d}\xi_i f_i(\xi_i)\frac{2\pi\alpha^2 q_i^2}{Q^4}\frac{s^2+u^2}{s^2}\delta(\xi_i - x)\frac{Q^2 s}{s+u} \\ &= \frac{2\pi\alpha^2 s}{Q^4}\frac{s^2+u^2}{s^2}\sum_i f_i(x)q_i^2 x \\ &= \frac{2\pi\alpha^2 s}{Q^4}\left[(y-1)^2+1\right]\sum_i f_i(x)q_i^2 x \end{aligned} \tag{3.72}$$

with

$$\begin{aligned}(y-1)^2+1 = y^2-2y+2 &= \left(\frac{s+u}{s}\right)^2 - 2\frac{s+u}{s}+2 \\ &= \frac{s^2+u^2}{s^2}+\frac{2su}{s^2}-2\frac{s+u}{s}+2 \\ &= \frac{s^2+u^2}{s^2}\end{aligned}$$

according to (3.64). In addition the normalization

$$\int_0^1 \mathrm{d}\xi_i f_i(\xi_i) = 1 \tag{3.73}$$

holds. Comparing (3.72) with (3.66) shows that in the parton model it must hold that

$$2\left[xy^2F_1^{\mathrm{eN}}(Q^2,x)+(1-y)F_2^{\mathrm{eN}}(Q^2,x)\right] = \left[(y-1)^2+1\right]\sum_i f_i(x)q_i^2 x \quad , \qquad s = \mathrm{const} \quad . \tag{3.74}$$

Equation (3.74) holds for all scattering reactions, i.e., for any value of E, E', and θ. We use this fact by letting the parameter $s = 2M_{\mathrm{N}}E$, which was fixed up to now, vary:

$$s \rightarrow s' \quad . \tag{3.75}$$

Simultaneously, we substitute

$$u \rightarrow \frac{s'}{s}u = u' \quad , \quad Q^2 \rightarrow \frac{s'}{s}Q^2 = Q'^2 \quad . \tag{3.76}$$

x and y are not changed under the simultaneous transformations

$$y \rightarrow \frac{s' + u'}{s'} = \frac{s + u}{s} \quad , \quad x \rightarrow \frac{Q'^2}{s' + u'} = \frac{Q^2}{s + u} \quad . \tag{3.77}$$

Since the right-hand side of (3.74), computed in the parton model, depends only on the variables x and y, it follows for the left-hand side that

$$xy^2 F_1^{\mathrm{eN}}(Q^2,x) + (1-y)F_2^{\mathrm{eN}}(Q^2,x) = \mathrm{const} \quad \text{for all } Q'^2 \tag{3.78}$$

or

$$F_1^{\mathrm{eN}}(Q^2,x) = F_1^{\mathrm{eN}}(x) \quad , \quad F_2^{\mathrm{eN}}(Q^2,x) = F_2^{\mathrm{eN}}(x) \quad . \tag{3.79}$$

The structure functions in the parton picture thus no longer depend on Q^2. Since we have neglected here both the momenta coming from the binding of the partons and the nucleon mass, (3.79) should hold for sufficiently large values for Q^2. This has indeed been experimentally observed as discussed above. Equation (3.74) then becomes

$$2\left[xy^2 F_1^{\mathrm{eN}}(x) + (1-y)F_2^{\mathrm{eN}}(x)\right] = \left[y^2 + 2(1-y)\right]\sum_i f_i(x)q_i^2 x \quad . \tag{3.80}$$

By comparing powers of y, we find that

$$F_1^{\mathrm{eN}}(x) = \frac{1}{2}\sum_i f_i(x)q_i^2 \quad , \quad F_2^{\mathrm{eN}}(x) = \sum_i f_i(x)q_i^2 x \quad , \tag{3.81}$$

and

$$F_2^{\mathrm{eN}}(x) = 2xF_1^{\mathrm{eN}}(x) \quad . \tag{3.82}$$

This is the *Callan–Gross relation.*

We have thus seen that deep-inelastic scattering processes can be explained by the basic assumptions of the parton model: the hadron is built of pointlike constituents whose interaction energy with each other and whose mass is small compared to $\sqrt{Q^2} > 1\mathrm{GeV}$. The physical foundations of (3.81) and (3.82) can also be illustrated from a different point of view: we can assume that partons are massless. There are no dimensional quantities in electron–parton scattering except for the momenta. On the other hand, we know from (3.46) that F_1^{eN} and F_2^{eN} are dimensionless. Expanding $F_1^{\mathrm{eN}}(Q^2,x)$ in powers of Q^2,

$$F_1^{\mathrm{eN}}(Q^2,x) = F_1^{\mathrm{eN}}(0,x) + \frac{\mathrm{d}F_1^{\mathrm{eN}}(0,x)}{\mathrm{d}Q^2}Q^2 + \frac{1}{2}\frac{\mathrm{d}^2F_1^{\mathrm{eN}}(0,x)}{\mathrm{d}(Q^2)^2}Q^4 + \dots \quad , \qquad (3.83\mathrm{a})$$

it follows that $(\mathrm{d}F_1^{\mathrm{eN}}/\mathrm{d}Q^2)(0,x)$ must, on the one hand, be dimensionless (since it does not depend on Q^2, x is dimensionless, and there are no other dimensional quantities), on the other hand, it must have dimension $[\text{length}]^2$ to cancel the dimension of Q^2. It thus follows that $(\mathrm{d}F_1^{\mathrm{eN}}/\mathrm{d}Q^2)(0,x) = 0$. Repeating the same arguments for higher derivatives, it follows that $F_1^{\mathrm{eN}}(Q^2,x) = F_1^{\mathrm{eN}}(0,x)$ and thus we get (3.79). While the scaling, i.e., the sole dependence on x, must hold for all massless partons, completely independent of their properties, the appearance of two structure functions and (3.82) of course depend on the Feynman rules. We have already seen that three structure functions are present in a parity-violating interaction (see (3.37) and (3.38)). On the other hand, we obtain only one structure function for scalar particles, i.e., $F_1 = 0$ (see Exercise 3.7). In addition to deep-inelastic electron–nucleon scattering, photon–nucleon scattering also yields information about structure functions. In Exercise 3.8, we shall show that the relation of the cross sections for transverse and for scalar photons, in particular, is of interest, since it tests the Callan–Gross relation (3.82). As can be seen from Fig. 3.6, it holds very well experimentally. Therefore it suffices to investigate scattering reactions as a function of $F_2(x)$. For this reason, Fig. 3.5 shows only $F_2(x) \sim (F_2^{\mathrm{ep}} + F_2^{\mathrm{en}})$. In Exercise 3.9 we shall finally compute the structure functions following from a simple constituent-quark model. Independent of the special assumption of each model, all such computations yield the following results.

1. If one assumes that the nucleon is composed of three quarks, the one-particle distributions $f(x)$ are typically maximal for $x = 1/3$ and, accordingly, the structure function $F_2(x)$ for somewhat higher values of x.
2. Designating the structure functions averaged over protons and nucleons by $F_2(x) = \frac{1}{2}[F_2^{\mathrm{ep}}(x) + F_2^{\mathrm{en}}(x)]$, where

$$\begin{aligned} F_2^{\mathrm{en}}(x) &= \sum_i f_i(x) q_i^2 x \\ &= f(x)\cdot x \sum_i q_i^2 \end{aligned}$$

(here it is assumed that all three quarks in the nucleon have, owing to their negligible masses, the same wave function), it follows that the integral

$$\begin{aligned} \frac{18}{5}\int \mathrm{d}x\, F_2(x) &= \frac{18}{5}\frac{1}{2}\int \mathrm{d}x\, f(x)x \left(\frac{4}{9}+\frac{4}{9}+\frac{1}{9}+\frac{4}{9}+\frac{1}{9}+\frac{1}{9}\right) \\ &= 3\int \mathrm{d}x f(x)x = \int \mathrm{d}x \sum_i f_i(x)x \end{aligned} \qquad (3.83\mathrm{b})$$

is unity, i.e., the whole momentum of the nucleon is carried by the quarks.

Compare here Fig. 3.11. Obviously, the experiments yield different results.

1. $F_2(x)$ is maximal for small x, i.e., there must be charged partons carrying only a small momentum fraction. It is natural to identify them with virtual quark–antiquark pairs. A careful analysis of electromagnetic and electroweak structure functions indeed justifies this assumption, as we shall see below.

2. The integral $F_2(x)$ is experimentally much smaller than unity, namely approximately 0.45. Half of the nucleon momentum is thus carried by electrically neutral particles. This is an important point since it is the most direct evidence for the existence of gluons. We shall return to this point in Sect. 4.2.

It may be added that more refined models are also not able to give a satisfying description of the experimental data in Fig. 3.10 starting from the quark–quark interaction without ad hoc assumptions. It is possible, however, to describe the relationship between two structure functions. We can, for example, calculate very well the relative change of F_2 for different values of Q^2. We shall discuss this in Chap. 5. The reason for this only partial success is that the calculation of structure functions itself is a completcly nonperturbative problem and thus very difficult, while the Q^2 dependence is calculable by summing a few classes of graphs, i.e., in perturbation theory. One usually introduces distribution functions for the various quarks and antiquarks:

$$f_{\mathrm{u}}(x) =: \mathrm{u}(x) \quad , \quad f_{\bar{\mathrm{u}}}(x) =: \bar{\mathrm{u}}(x) \quad , \quad \text{etc.} \tag{3.83c}$$

Equation (3.81) then becomes

$$\begin{aligned} F_2^{\mathrm{ep}}(x) = x \left\{ \frac{4}{9}\left[\mathrm{u}(x) + \bar{\mathrm{u}}(x)\right] + \frac{1}{9}\left[\mathrm{d}(x) + \bar{\mathrm{d}}(x)\right] \right. \\ \left. + \frac{4}{9}\left[\mathrm{c}(x) + \bar{\mathrm{c}}(x)\right] + \frac{1}{9}\left[\mathrm{s}(x) + \bar{\mathrm{s}}(x)\right] \right\} \quad . \end{aligned} \tag{3.83d}$$

The crucial point is now that the structure functions for the reactions $\nu_{\mathrm{e}} + \mathrm{n} \longrightarrow \mathrm{e}^- + \mathrm{p}$, $\nu_{\mathrm{e}} + \mathrm{n} \longrightarrow \nu_{\mathrm{e}} + \mathrm{n}$, etc. involve different combinations of $\mathrm{u}(x)$, $\bar{\mathrm{u}}(x)$, $\mathrm{d}(x)$, and $\bar{\mathrm{d}}(x)$. (The weak charges of the quarks are not proportional to their electric charges.) Therefore quark distributions can be deduced from the different structure functions. We shall not pursue this further here the basic idea should be clear. In Fig. 3.7, however, we have sketched approximately what the parton distributions look like in a proton. Most interesting is the fact that quark and antiquark distributions coincide for $x \longrightarrow 0$. This can be understood if both are virtual excitations (so-called sea quarks), generated by the quark–quark interaction. The obvious divergence of the functions $\mathrm{u}(x)$, $\mathrm{d}(x)$, and $\bar{\mathrm{q}}(x)$ for $x \longrightarrow 0$ thus indicates that the interaction is large for small momentum transfers.

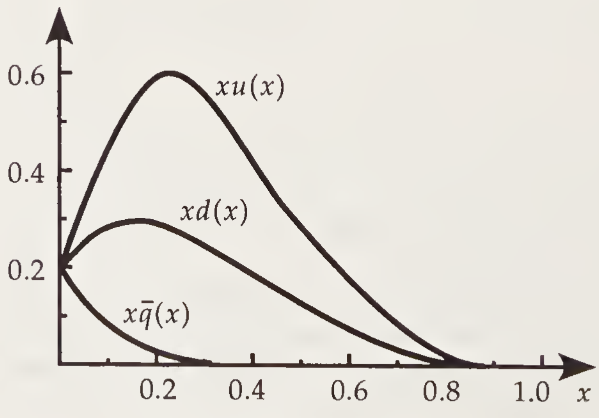


Fig. 3.7. A sketch of the quark distribution functions for protons with $\overline{q}(x) = \frac{1}{2}(\overline{u}(x) + \overline{d}(x))$. We recognize that the contributions of the valence quarks $x(u(x) - \overline{q}(x))$ and $x(d(x) - \overline{q}(x))$ vanish for $x \longrightarrow 0$. Hence virtual quark–antiquark pairs dominate for small values of x

EXERCISE

3.6 The Breit System

Problem. Prove the existence of a Lorentz reference frame where (3.61) holds.

Solution. We start with the laboratory system. Here

$$\begin{aligned} q_\mu &= (q_0, \boldsymbol{q}) \quad , \quad q_0 > 0 \quad , \\ P_\mu &= (M, \boldsymbol{0}) \quad . \end{aligned} \tag{1}$$

First we perform a rotation in such a way that q is parallel to the z direction.

$$\begin{aligned} q'_\mu &= (q'_0, 0, 0, q'_3) \quad , \quad q'_0, q'_3 > 0 \\ P'_\mu &= (M_\mathrm{N}, 0, 0, 0) \quad . \end{aligned} \tag{2}$$

Now we boost the system in the z direction, i.e., we transform to a reference frame moving with the velocity

$$v_z = \frac{q'_0}{q'_3} c \quad . \tag{3}$$

This is possible because the momentum transfer q_μ is spacelike, i.e.,

$$Q^2 = q_3'^2 - q_0'^2 > 0 \to q'_3 > q'_0 \quad . \tag{4}$$

Then in the new reference frame

$$\begin{aligned} q''_\mu &= \gamma(q'_0 - \beta q'_3, 0, 0, q'_3 - \beta q'_0) \\ &= \gamma\left(0, 0, 0, q'_3 - \beta^2 q'_3\right) \\ &= \left(0, 0, 0, q'_3\sqrt{1 - \frac{q_0'^2}{q_3'^2}}\right) = \left(0, 0, 0, \sqrt{Q^2}\right) \quad , \end{aligned} \tag{5}$$

with

$$\beta = \frac{v_z}{c} = \frac{q'_0}{q'_3} \quad \text{and} \quad \gamma = \left(1 - \beta^2\right)^{-1/2} \quad . \tag{6}$$

Under this tranformation the nucleon momentum becomes

$$P''_\mu = \gamma(M_\mathrm{N}, 0, 0, -\beta M_\mathrm{N}) \quad . \tag{7}$$

Since we consider reactions where Q^2 and

$$\nu = \frac{q' \cdot P'}{M_\mathrm{N}} = q'_0 = \frac{q'' \cdot P''}{M_\mathrm{N}} \tag{8}$$

get very large, but their ratio $x = Q^2/2M_\mathrm{N}\nu$ remains constant,

Exercise 3.6.

$$\nu = \frac{q'' \cdot P''}{M_{\mathrm{N}}} = \beta\gamma\sqrt{Q^2} = \frac{Q^2}{2M_{\mathrm{N}}x} \Rightarrow \beta\gamma = \frac{\nu}{\sqrt{Q^2}} = \frac{\sqrt{Q^2}}{2M_{\mathrm{N}}x} \quad . \tag{9}$$

must hold. Equation (9) can only be fulfilled for $\beta \approx 1$, $\gamma \gg 1$. Hence

$$\tilde{P} = \gamma M_{\mathrm{N}} = \frac{\sqrt{Q^2}}{2x} \quad , \quad P''_\mu \approx (\tilde{P}, 0, 0, -\tilde{P}) \quad . \tag{10}$$

This is (3.62). In the Breit system the nucleon (and with it all partons) and the photon move in the z direction towards each other. In the final state the scattered parton carries the momentum (x denotes the initial momentum fraction of the parton, i.e., $P^{\mathrm{i}}_\mu = xP''_\mu$)

$$\begin{aligned} P^{\mathrm{f}}_\mu = xP''_\mu + q''_\mu &= \left(\frac{\sqrt{Q^2}}{2}, 0, 0, -\sqrt{\frac{Q^2}{2}}\right) + q''_\mu \\ &= \left(\frac{\sqrt{Q^2}}{2}, 0, 0, \sqrt{\frac{Q^2}{2}}\right) \quad . \end{aligned} \tag{11}$$

Obviously the spatial momentum of the parton is flipped to its opposite direction by the reaction (see Fig. 3.8). In the Breit system the parton is simply reflected. We shall come back to this in Example 3.8.

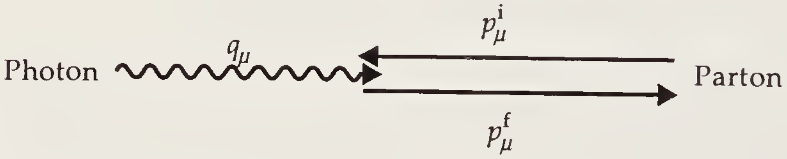


Fig. 3.8. Photon–parton interaction in the Breit system. The initial photon and parton carry the four-momenta $q_\mu = (0, 0, 0, \sqrt{Q^2})$ and $P^{\mathrm{i}}_\mu = \frac{1}{2}(\sqrt{Q^2}, 0, 0, -\sqrt{Q^2})$, respectively. The final parton then has the momentum $P^{\mathrm{f}}_\mu = \frac{1}{2}(\sqrt{Q^2}, 0, 0, \sqrt{Q^2})$

EXERCISE

3.7 The Scattering Tensor for Scalar Particles.

Problem. Repeat the steps leading to (3.18) for scalar particles.

Solution. We start with (3.6). In the case of scalar particles all terms containing γ-matrices vanish:

$$\Gamma_\mu = BP'_\mu + CP_\mu \quad . \tag{1}$$

Equation (3.8) then yields

$$q^\mu \Gamma_\mu = (P'^\mu - P^\mu)\Gamma_\mu = 0 \quad \Rightarrow C = B \quad , \tag{2}$$

$$\Gamma_\mu = B(P'_\mu + P_\mu) \quad . \tag{3}$$

This is rewritten by replacing P'_μ by P_μ and q_μ:

Exercise 3.7.

$$\begin{aligned}\Gamma_\mu &= 2B\left(P_\mu + \frac{1}{2}q_\mu\right)\\ &= 2B\left(P_\mu - q_\mu\frac{-q^2/2}{q^2}\right)\\ &= 2\left(P_\mu - q_\mu\frac{P\cdot q}{q^2}\right)B(Q^2) \quad . \end{aligned} \tag{4}$$

Relation (3.7) has been employed in the last step. For a free scalar field one simply has

$$u^*(P)u(P) = \text{const} \quad . \tag{5}$$

Now $W_{\mu\nu}$ follows directly from Γ_μ:

$$W_{\mu\nu} = \text{const} \times \left(P_\mu - q_\mu\frac{P\cdot q}{q^2}\right)\left(P_\nu - q_\nu\frac{P\cdot q}{q^2}\right)B^2(Q^2) \quad . \tag{6}$$

Up to now we have considered the elastic process. To obtain the equation equivalent to (6) for the inclusive inelastic case, we again have to replace $B^2(Q^2)$ by $B^2(Q^2,\nu) =: W_2(Q^2,\nu)$.

$$W_{\mu\nu} = \text{const} \times \left(P_\mu - q_\mu\frac{P\cdot q}{q^2}\right)\left(P_\nu - q_\nu\frac{P\cdot q}{q^2}\right)W_2(Q^2,\nu) \quad . \tag{7}$$

Comparing this result with (3.18) shows that there is only a W_2 function but no W_1 function for scalar particles. Consequently (3.66) then becomes

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}x\,\mathrm{d}y} = \frac{4\pi s\alpha^2}{Q^4}(1-y)F_2(Q^2,x) \quad , \tag{8}$$

with a corresponding structure function $F_2(Q^2,x)$.

EXAMPLE

3.8 Photon–Nucleon Scattering Cross Sections for Scalar and Transverse Photon Polarization

For scalar as well as for transverse polarized photons the spin 4 vector is perpendicular to the photon momentum. With

$$q_\mu = (\nu, 0, 0, q_3) \tag{1}$$

the spin unit vectors can in the laboratory system be chosen as follows. For the transverse case

$$\varepsilon_{1,\mu} = (0,1,0,0) \quad , \quad \varepsilon_{2,\mu} = (0,0,1,0) \quad . \tag{2}$$

Example 3.8.

Scalar polarization is defined by

$$\varepsilon_{0,\mu} = (q_3, 0, 0, \nu)\frac{1}{\sqrt{Q^2}} \quad . \tag{3}$$

Clearly

$$\varepsilon_{\lambda,\mu}\, q^{\mu} = 0 \tag{4}$$

and

$$\left|\varepsilon_{\lambda,\mu}\varepsilon^{\lambda,\mu}\right| = 1 \quad \text{for} \quad \lambda = 1, 2, 3 \quad . \tag{5}$$

In the case of photon–nucleon scattering we get

$$\sigma(\gamma \mathrm{N} \to \mathrm{X}) = \mathrm{const} \times \varepsilon_{\lambda,\mu}\, \varepsilon_{\lambda,\nu}\, W^{\mu\nu} \tag{6}$$

for the inclusive cross section, where $\varepsilon_{\lambda,\mu}$ denotes the polarization vector of the incoming photon. By means of (3.18)

$$P_{\mu} = (M_{\mathrm{N}}, 0, 0, 0) \quad , \tag{7}$$

$$\sigma_{\mathrm{T}}(\gamma \mathrm{N} \to \mathrm{X}) = \mathrm{const} \times W_1\left(Q^2, \nu\right) \tag{8}$$

can be derived in the laboratory system for transversely polarized photons and

$$\begin{aligned} &\sigma_{\mathrm{S}}(\gamma \mathrm{N} \to \mathrm{X}) \\ &= \mathrm{const} \times \left[\left(\frac{-q_3^2 + \nu_0^2}{Q^2} + 0\right) W_1\left(Q^2, \nu\right) + \left(\frac{M_{\mathrm{N}} q_3}{\sqrt{Q^2}}\right)^2 \frac{W_2\left(Q^2, \nu\right)}{M_{\mathrm{N}}^2}\right] \\ &= \mathrm{const} \times \left[-W_1\left(Q^2, \nu\right) + \frac{q_3^2}{Q^2} W_2\left(Q^2, \nu\right)\right] \end{aligned} \tag{9}$$

for scalar photons. Inserting (3.43) now yields

$$\begin{aligned} \sigma_{\mathrm{T}}(\gamma \mathrm{N} \to \mathrm{X}) &= \mathrm{const} \times \frac{1}{M_{\mathrm{N}}} F_1^{\mathrm{eN}}\left(Q^2, x\right) \quad , \\ \sigma_{\mathrm{S}}(\gamma \mathrm{N} \to \mathrm{X}) &= \mathrm{const} \times \frac{1}{M_{\mathrm{N}}}\left[-F_1^{\mathrm{eN}}\left(Q^2, \nu\right) + \frac{q_3^2 M_{\mathrm{N}}}{Q^2 \nu} F_2^{\mathrm{eN}}\left(Q^2, \nu\right)\right] \quad . \end{aligned} \tag{10}$$

Because

$$Q^2 = q_3^2 - q_0^2 = q_3^2 - \nu^2 \tag{11}$$

the factor in front of $F_2^{\mathrm{eN}}\left(Q^2, \nu\right)$ can be simplified to

$$\frac{q_3^2 M_{\mathrm{N}}}{Q^2 \nu} = \frac{\left(Q^2 + \nu^2\right) M_{\mathrm{N}}}{Q^2 \nu} = \frac{M_{\mathrm{N}}}{\nu} + \frac{1}{2x} \quad . \tag{12}$$

In the scaling region $\nu \gg M_{\mathrm{N}}$ we therefore obtain

Example 3.8.

$$\sigma_{\mathrm{T}}(\gamma \mathrm{N} \to \mathrm{X})|_{\nu^2,Q^2 \gg (1\mathrm{GeV})^2} = \mathrm{const} \times \frac{1}{M_{\mathrm{N}}} F_1^{\mathrm{eN}}(x) \quad , \tag{13}$$

$$\sigma_{\mathrm{S}}(\gamma \mathrm{N} \to \mathrm{X})|_{\nu^2,Q^2 \gg (1\mathrm{GeV})^2} = \mathrm{const} \times \frac{1}{M_{\mathrm{N}}} \left[\frac{1}{2x} F_2^{\mathrm{eN}}(x) - F_1^{\mathrm{eN}}(x)\right] \quad . \tag{14}$$

Because of the Callan–Gross relation (3.82) σ_{S} should therefore vanish. In fact Fig. 3.6 was obtained by analyzing photon–nucleon scattering. It should be noted that on the other hand $F_1(x) = 0$ holds for scalar particles (see Exercise 3.7). We would therefore expect $\sigma_{\mathrm{T}}/\sigma_{\mathrm{S}} \to 0$ for scalar partons. But this contradicts experimental observations, i.e., all models are non-physical, which represents the charged constituents of the nucleon by scalar particles. These results can easily be understood within the Breit system. To that end one only has to remember that for transverse photon polarization the spin is either parallel to the direction of motion or opposite to it. More precisely this means that

$$\varepsilon_\mu^+ = (-\varepsilon_{1,\mu} - \mathrm{i}\varepsilon_{2,\mu})\frac{1}{\sqrt{2}} \quad , \quad \boldsymbol{S} \cdot \boldsymbol{q} = +|\boldsymbol{q}| \tag{15}$$

and

$$\varepsilon_\mu^- = (\varepsilon_{1,\mu} - \mathrm{i}\varepsilon_{2,\mu})\frac{1}{\sqrt{2}} \quad , \quad \boldsymbol{S} \cdot \boldsymbol{q} = -|\boldsymbol{q}| \quad . \tag{16}$$

Here $\boldsymbol{S}$ denotes the photon spin. According to Fig. 3.8 the partons do not carry angular momentum and consequently the photon can only be absorbed by the parton, if the spin component in the z direction of the latter particle is changed by 1. But this is impossible for scalar particles and leads to $\sigma_{\mathrm{T}} = 0$. Massless spin-$\frac{1}{2}$ particles, however, encounter a completely different situation. For these particles the spin component parallel to the direction of motion can only assume the values +1/2 and −1/2. The corresponding spin states are known as positive and negative helicity or as right-handed and left-handed particles. Helicity states are defined by the projection operator $(1 \pm \gamma_5)/2$, left-handed and right-handed states by the projection of the spin onto the momentum axis. For ultrarelativistic particles positive helicity corresponds to right-handed particles, that is the spin points in the direction of motion, and negative helicity corresponds to the spin pointing in the opposite direction. The vector γ_μ conserves the helicity, i.e., left-handed particles, for example couple only to other left-handed particles (see Sec. 4.1). Since the direction of motion of a parton is changed into its opposite in the Breit system, the spin must consequently be flipped at the same time (for sufficiently fast partons, i.e., for sufficiently large Q^2). Therefore spin-$\frac{1}{2}$ partons are only able to absorb a photon if S_z is equal to ±1, i.e., if the photon is transverse. In this case σ_{T} is nonzero (see Fig. 3.9).

Scalar photons (the Coulomb-field, for example, consists of such photons) have zero spin projection, i.e., here $\sigma_{\mathrm{S}} = 0$ for spin-$\frac{1}{2}$ partons and $\sigma_{\mathrm{S}} \neq 0$ for spin-0 partons, which is the opposite of the situation encountered by their transverse counterparts.

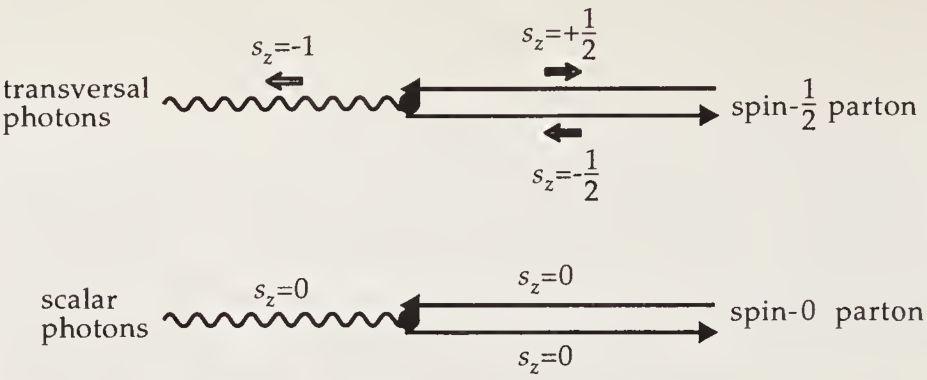


Fig. 3.9. The role of the spin in photon–parton interactions. Owing to the conservation of angular momentum, spin-$\frac{1}{2}$ partons can absorb only transverse photons and spin-0 partons only scalar photons (for relativistic partons, i.e., up to corrections of the order of m^2/Q^2)

EXAMPLE

3.9 A Simple Model Calculation for the Structure Functions of Electron–Nucleon Scattering

We have already stated that the structure functions $W_1\left(Q^2,\nu\right)$ and $W_2\left(Q^2,\nu\right)$ of (3.36) can be derived from any microscopic model. This procedure, however, is quite cumbersome, but it allows predictions for values of Q^2 and ν^2 smaller than $(1\ \mathrm{GeV})^2$, i.e., for values beyond the scaling region. If we are interested only in the restricted information provided by $F_2(x)$, then there is an easier way. In this case we can start directly from (3.81):

$$F_2^{\mathrm{eN}}(x) = \sum_i f_i(x) q_i^2 x \qquad (1)$$

The only input needed is the functions $f_i(x)$, i.e., the probabilities that the parton with index i carries a fraction x of the total momentum. We are going to eveluate these probabilities in an extremely simple model by making the following assumptions:

1. The nucleon consists of quarks.
2. The wave functions of up and down quarks are identical, which yields

$$F_2^{\mathrm{eN}}(x) = \left(\sum_i q_i^2\right) f(x)x \quad . \qquad (2)$$

Averaging the structure function over the proton and neutron gives

$$\begin{aligned} F_2^{\mathrm{eN}}(x) &= \frac{1}{2}\left[F_2^{\mathrm{ep}}(x) + F_2^{\mathrm{en}}(x)\right] \\ &= \frac{3}{2}\left[\left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2\right] f(x)x \\ &= \frac{5}{6} f(x)x \quad . \end{aligned} \qquad (3)$$

3. We employ Gaussian distributions for the internal wave functions in position space, i.e., we set

Example 3.9.

$$\Psi_P(r,s) = \frac{\alpha^2}{3\pi^2}\exp\left\{\frac{\alpha}{2}\left[\frac{1}{2}\left(r^2 - 2\left(\frac{P\cdot r}{M}\right)^2\right)\right.\right.$$
$$\left.\left. + \frac{2}{3}\left(s^2 - 2\left(\frac{P\cdot s}{M}\right)^2\right)\right]\right\} \tag{4}$$

with Jacobi coordinates

$$\begin{aligned} r_\mu &= (z_2 - z_3)_\mu \quad , \\ s_\mu &= \left[z_1 - \frac{1}{2}(z_2 - z_3)\right]_\mu \quad , \\ &= \frac{1}{2}(2z_1 - z_2 - z_3)_\mu \quad . \end{aligned} \tag{5}$$

Here $\Psi_P(r,s)$ is the three-quark wave function, while the quark positions and the proton momentum are denoted by the four-vectors z_1, z_2, z_3, and P_μ, respectively.

Statements 1 and 2 are quite clear but (4) needs further explanation. First one has to take into account that the center-of-mass motion separates, i.e., the center-of-mass coordinate $Z_\mu = \frac{1}{3}(z_1 + z_2 + z_3)_\mu$ must not occur in (4). Then the differences between the quark coordinates are assumed to be Gaussian distributed. This can be archieved by employing Gaussian distributions for r and s. Formula (4) can be better understood by writing down a part of the exponent in its explicit form:

$$\begin{aligned} \frac{r^2}{2} + \frac{2s^2}{3} &= \frac{1}{6}\left(3z_2^2 - 6z_2z_3 + 3z_3^2 + 4z_1^2 + z_2^2 + z_3^2 - 4z_1z_2 - 4z_1z_3 + 2z_2z_3\right) \\ &= \frac{2}{3}\left(z_1^2 + z_2^2 + z_3^2 - z_1z_2 - z_2z_3 - z_3z_2\right) \\ &= \frac{1}{3}\left[(z_1 - z_2)^2 + (z_1 - z_3)^2 + (z_2 - z_3)^2\right] \quad . \end{aligned} \tag{6}$$

We could think of employing spatial Gaussians only and plane waves with respect to the time coordinate. This represents the completely noninteracting case and ensures energy conservation at any time t. Consequently each quark carries exactly one third of the total energy. In an interacting theory the energy of every parton is a function of time and varies in general around a mean value. One can simulate this effect by using another Gaussian like

$$\exp\left(-\frac{\alpha}{4}\left(r_0^2 + r^2\right)\right) \quad , \tag{7}$$

which is clearly a useful assumption. Expression (7) ensures that the three quarks populate neighboring positions at neighboring times and therefore describe a bound state. Unfortunately a function that is localized in space and time is not Lorentz invariant. In order to restore Lorentz invariance in (7) we now use the fact that the proton momentum in the rest system is simply

$$P_\mu = (M, 0, 0, 0) \quad . \tag{8}$$

We therefore obtain in the rest system

Example 3.9.

$$\exp\left(-\frac{\alpha}{4}\left(r_0^2+r^2\right)\right)=\exp\left(\frac{\alpha}{4}\left[r^2-\frac{2}{M^2}(P\cdot r)^2\right]\right) \quad , \tag{9}$$

where Ψ has been normalized in such a way that

$$\int \mathrm{d}^4r\,\mathrm{d}^4s\,\Psi_P^2(r,s)=1 \tag{10}$$

holds. Starting with (4) we now determine the mean square charge radius of the proton, which is given by ($P_\mu=(M,0,0,0)$):

$$\langle r^2\rangle_\mathrm{c}=\int \mathrm{d}^4r\,\mathrm{d}^4s\sum_i q_i(z_i-Z)^2\,\Psi_P^2(r,s) \quad . \tag{11}$$

With the help of the center-of-mass coordinate $Z=\frac{1}{3}(z_1+z_2+z_3)$ we get

$$\begin{aligned}\sum_i q_i(z_i-Z)^2 &= -\frac{1}{3}(z_1-Z)^2+\frac{2}{3}(z_2-Z)^2+\frac{2}{3}(z_3-Z)^2\\ &= -\frac{1}{27}(2z_1-z_2-z_3)^2+\frac{2}{27}(2z_2-z_1-z_3)^2+\frac{2}{27}(2z_3-z_1-z_2)^2\\ &= \frac{1}{27}\Big(-4z_1^2-z_2^2-z_3^2+4z_1\cdot z_2\\ &\quad +4z_1\cdot z_3-2z_2\cdot z_3+2z_1^2+8z_2^2+2z_3^2-8z_2\cdot z_1-8z_2\cdot z_3\\ &\quad +4z_1\cdot z_3+2z_1^2+2z_2^2+8z_3^2-8z_3\cdot z_1-8z_3\cdot z_2+4z_1\cdot z_2\Big)\\ &= \frac{1}{27}\left(9z_2^2+9z_3^2-18z_2\cdot z_3\right)\\ &= \frac{1}{3}r^2 \quad .\end{aligned} \tag{12}$$

In the center-of-mass system this yields

$$\begin{aligned}\langle r^2\rangle_\mathrm{c} &= \frac{1}{3}\left(\frac{\alpha^2}{3\pi^2}\right)^2\int \mathrm{d}^4s\,\exp\left[-\frac{2\alpha}{3}\left(s_0^2+s^2\right)\right]\int \mathrm{d}r_0\,d^3r\,\mathrm{e}^{-\frac{\alpha}{2}r_0^2}\,\mathrm{e}^{-\frac{\alpha}{2}r^2}\,r^2\\ &= \frac{\alpha\sqrt{\alpha}}{6\pi\sqrt{3\pi}}4\pi\int_0^\infty \mathrm{d}r\,r^4\,\mathrm{e}^{-\frac{\alpha}{2}r^2}=\frac{1}{\alpha} \quad .\end{aligned} \tag{13}$$

Thus we have to insert the value

$$\alpha=\frac{1}{\langle r^2\rangle_\mathrm{c}}\approx 1.5\,\mathrm{fm}^{-2} \tag{14}$$

for α. From (4) we then get

$$\begin{aligned}\Psi_P(r,s,Z) &= \frac{\alpha^2}{3\pi^2}\frac{\mathrm{e}^{-\mathrm{i}P\cdot Z}}{\sqrt{VT}}\exp\left\{\frac{\alpha}{2}\left[\frac{1}{2}\left(r^2-2\left(\frac{P\cdot r}{M}\right)^2\right)\right.\right.\\ &\quad \left.\left.+\frac{2}{3}\left(s^2-2\left(\frac{P\cdot s}{M}\right)^2\right)\right]\right\}\end{aligned} \tag{15}$$

and

Example 3.9.

$$\int \mathrm{d}^4 r\, \mathrm{d}^4 s\, \mathrm{d}^4 Z\, \Psi_P^\dagger(r,s,Z)\Psi_P(r,s,Z) = 1 \tag{16}$$

for the total wave function, where VT denotes the integration volume for Z. In order to understand the following note that the *one-particle Wigner function*

$$f(Z,p) := \int \frac{\mathrm{d}^4 v}{(2\pi)^4}\, \mathrm{e}^{-\mathrm{i}p_\mu v^\mu} \Psi^\dagger\left(Z_\mu + \frac{1}{2}v_\mu\right) \Psi\left(Z_\mu + \frac{1}{2}v_\mu\right) \tag{17}$$

yields the probability for finding a particle at a position Z_μ with a momentum P_μ. Here $\Psi(x)$ denotes the one-particle wave function. Now it is easy to evaluate from (15) the distribution function $f(x)$ of, for example, the first parton:

$$\begin{aligned} f_1(Z,p_\mu) = N \int \mathrm{d}^4 v\, \mathrm{e}^{-\mathrm{i}p_\mu v^\mu} \int \mathrm{d}^4 z_2 \int \mathrm{d}^4 z_3 \\ \times \Psi_P^\dagger(r,s,Z)\Big|_{z_{1,\mu}\to s_{1,\mu}+\frac{1}{2}v_\mu} \Psi_P(r,s,Z)\Big|_{z_{1,\mu}\to s_{1,\mu}-\frac{1}{2}v_\mu} \quad . \end{aligned} \tag{18}$$

The normalization constant N is determined later. The probability of finding a parton with momentum $p = xP$ at any center-of-mass coordinate Z is

$$\begin{aligned} f_1(x) &= \int \mathrm{d}^4 Z\, f_1(Z,xP) \\ &= N \int \mathrm{d}^4 v\, \mathrm{e}^{-\mathrm{i}P_\mu v^\mu x} \int \mathrm{d}^4 Z\, \mathrm{d}^4 z_2\, \mathrm{d}^4 z_3 \\ &\quad \times \Psi_P^\dagger\left(r, s + \frac{1}{2}v, Z + \frac{1}{6}v\right) \Psi_P\left(r, s - \frac{1}{2}v, Z - \frac{1}{6}v\right) \\ &= N \int \mathrm{d}^4 v\, \mathrm{e}^{-\mathrm{i}P_\mu v^\mu \left(x - \frac{1}{3}\right)} \int \mathrm{d}^4 Z\, \mathrm{d}^4 z_2\, \mathrm{d}^4 z_3\, \frac{\alpha^4}{9\pi^4 VT} \\ &\quad \times \exp\left\{\alpha\left[\frac{1}{2}\left(r^2 - 2\left(\frac{P\cdot r}{M}\right)^2\right)\right.\right. \\ &\quad \left.\left. + \frac{2}{3}\left(s^2 - 2\left(\frac{P\cdot s}{M}\right)^2\right) + \frac{1}{6}\left(v^2 - 2\left(\frac{P\cdot v}{M}\right)^2\right)\right]\right\} \quad . \end{aligned} \tag{19}$$

The functional determinant

$$\left|\frac{\partial(r,s)}{\partial(z_2,z_3)}\right| = \left\|\begin{matrix} 1 & -1 \\ -\frac{1}{2} & -\frac{1}{2} \end{matrix}\right\|^4 = 1 \tag{20}$$

is equal to one. By employing (16) we can reduce (19) to

$$f_1(x) = N \int \mathrm{d}^4 v\, \exp\left(-\mathrm{i}P^\mu v_\mu \left(x - \frac{1}{3}\right)\right) \exp\left(\frac{\alpha}{6}\left[v^2 - 2\left(\frac{P\cdot v}{M}\right)^2\right]\right) . \tag{21}$$

Since this expression is Lorentz invariant, we can again evaluate it in the rest system:

Example 3.9.

$$\begin{aligned} f_1(x) &= N \int \mathrm{d}^4 v \exp\left(-\mathrm{i} M v_0 \left(x - \tfrac{1}{3}\right)\right) \exp\left(\tfrac{\alpha}{6}\left(v_0^2 + \boldsymbol{v}^2\right)\right) \\ &= N \left(\frac{6\pi}{\alpha}\right)^{3/2} \int \mathrm{d}v_0 \exp\left(-\tfrac{\alpha}{6}\left[v_0 + \mathrm{i}\tfrac{3M}{\alpha}\left(x - \tfrac{1}{3}\right)\right]^2\right) \exp\left(-\tfrac{3M^2}{2\alpha}\left(x - \tfrac{1}{3}\right)^2\right) \\ &= N \left(\frac{6\pi}{\alpha}\right)^2 \exp\left(-\tfrac{3M^2}{2\alpha}\left(x - \tfrac{1}{3}\right)^2\right) \quad . \end{aligned} \tag{22}$$

Now we recognize the following. For the simple quark model considered here the distribution functions are peaked around $x = 1/3$. This can immediately be understood, because it only means that each quark carries on average one third of the total momentum. Figure 3.10 shows, however, that the experimentally observed distribution functions increase monotonically for small values of x. This means that every charged particle has less momentum than one would expect from our simple quark model. Consequently there must be more partons in a nucleon than those three, which correspond to valence quarks.

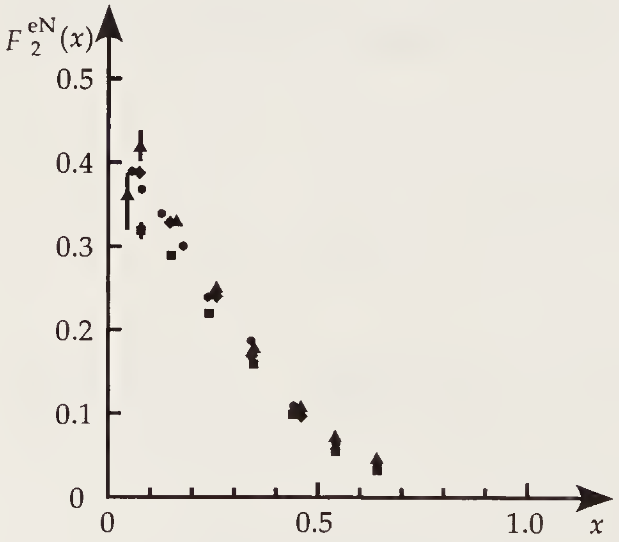


Fig. 3.10. The experimentally observed F_2 structure function of deep inelastic electron–nucleon scattering

Because of the approximations leading to (22) x can become larger than 1 and even assume negativ values. In this case, however, $f(x)$ is more or less zero. These difficulties are mainly due to assumption (4), which allows for arbitrarily large parton momenta (being the Fourier transform of a Gaussian distribution, (22) is again a Gaussian). Consequently $f_1(x)$ should be normalized according to

$$1 = \int_{-\infty}^{\infty} f_1(x)\mathrm{d}x = N \left(\frac{6\pi}{\alpha}\right)^2 \sqrt{\frac{\pi 2\alpha}{3M^2}} \quad , \tag{23}$$

$$f_1(x) = \sqrt{\frac{3M^2}{2\pi\alpha}} \exp\left(-\frac{3M^2}{2\alpha}\left(x - \frac{1}{3}\right)^2\right) \quad . \tag{24}$$

Example 3.9.

The function

$$F_2^{\mathrm{eN}}(x) = \frac{1}{2}\left[F_2^{\mathrm{ep}}(x) + F_2^{\mathrm{en}}(x)\right]$$

$$= 3 \times \frac{5}{18} \times f_1(x)x \qquad (25)$$

obtained in this way is depicted in Fig. 3.11. The qualitative deviation from experimental results is considerable. The failure of our model, however, yields valuable information about specific features of the quark–quark interaction, which help to obtain better results. In addition note that

$$3\int\limits_{-\infty}^{\infty} xf_1(x)\mathrm{d}x = 3\int\limits_{-\infty}^{\infty}\left(x + \frac{1}{3}\right)\sqrt{\frac{3M^2}{2\pi\alpha}}\exp\left(\frac{3M^2}{2\alpha}x^2\right)\mathrm{d}x$$

$$= 3\left(0 + \frac{1}{3}\right) = 1 \qquad (26)$$

holds. Therefore the three quarks together carry the whole proton momentum. Owing to our assumptions, this must obviously be true, and therefore (26) represents a test of our calculation.

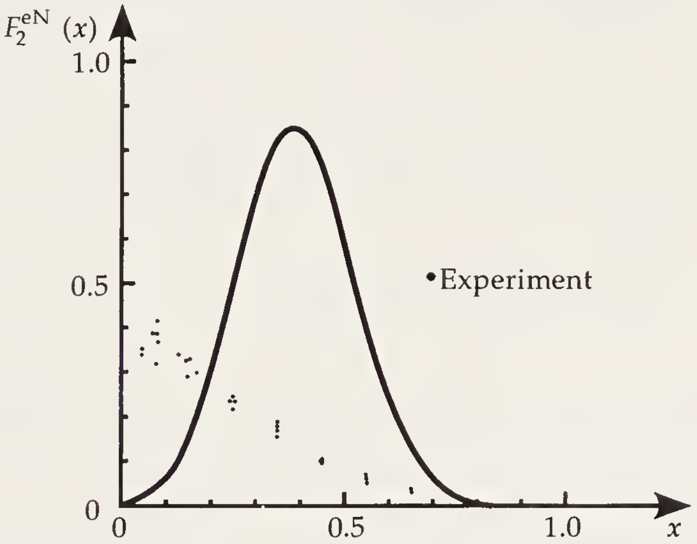


Fig. 3.11. Comparison of the experimentally observed F_2 structure function of deep inelastic electron–nucleon scattering with our simple model calculation

In the next section we shall discuss a more refined model for the nucleon, the MIT bag model. In Sect. 5.6 we shall see what the structure functions of this model look like. In fact they do not agree much better with the data than our results. We shall then also see how by adding Q^2 evolution from QCD ad hoc the results become much more acceptable.

3.3 The MIT Bag Model

Before we turn in the following chapters to today's prevailing theory of quark–quark interaction, we shall first discuss a specific 'bag model', the MIT bag model, in more detail. Since no free quarks have experimentally been observed, one imagines that the quarks are tightly confined inside the hadrons. Inside of this confinement volume they behave mainly as free particles. All bag models must be regarded as pure phenomenology. It is at present unclear how strong are any relations of such models to QCD. Should the confinement problem one day be solved from the QCD equations (which we shall discuss in the next chapter), it might turn out that the model assumptions of the MIT bag are unphysical. Besides this basic problem, there are also problems inherent to the MIT bag model. The rigid boundary condition can lead to spurious motions, e.g., oscillations of all quarks with respect to the bag, and is not Lorentz invariant.

These disadvantages are set off by the great advantage that nearly all interesting processes and quantities can be calculated in a bag-model framework. Sometimes there are quite far-reaching approximations involved, but in total these recipes allow for quite a good phenomenological understanding of subhadronic physics. This might be found more satisfactory than a strictly formal theory of quark–quark interaction which fails to predict many physically interesting quantities owing to mathematical problems.

We shall now formulate the MIT bag model. We start from the fact that the quark–quark interaction makes it impossible to separate colored quarks. This is most easily implemented by specifying some surface and demanding that the color current through it vanishes. This color current is analogous to the electromagnetic current:

$$\hat{J}_\mu^\alpha = (\bar{\mathrm{q}}_r, \bar{\mathrm{q}}_b, \bar{\mathrm{q}}_g)\lambda^\alpha\gamma_\mu \begin{pmatrix} \mathrm{q}_r \\ \mathrm{q}_b \\ \mathrm{q}_g \end{pmatrix} \tag{3.84}$$

with the eight color-SU(3) matrices λ^a. The indices stand for the three colors ('red', 'blue', and 'green'). Let the chosen surface be characterized by a normal vector n_μ. Then the desired condition can be written as

$$n^\mu \cdot \hat{J}_\mu^\alpha = 0\Big|_{\text{surface}} \quad , \quad \alpha = 1, 2, 3, \ldots, 8 \quad . \tag{3.85}$$

By introducing a four-dimensional normal vector we have reached a covariant form, but this does not correspond to a general covariance of the model because we must still specify the bag surface. Indeed we restrict ourselves to purely spatial surfaces, i.e., $n_\mu \to \boldsymbol{n}$:

$$-\boldsymbol{n} \cdot \boldsymbol{J}^\alpha\Big|_{R=R(\theta,\varphi)} = 0 \quad . \tag{3.86}$$

To simplify (3.86) further, color independence is demanded of the internal quark wave function, i.e., $\mathrm{q}_i(x) = \mathrm{q}(x)$, $i = r, b, g$. In this way we obtain the 'quadratic bag boundary condition'

$$\boldsymbol{n} \cdot \bar{\mathrm{q}}\boldsymbol{\gamma}\mathrm{q}\Big|_{R=R(\theta,\varphi)} = 0 \quad . \tag{3.87}$$

The expression $\boldsymbol{n}\cdot\boldsymbol{\gamma}$ has the property that its square is the negative unit matrix:

$$(\boldsymbol{n}\cdot\boldsymbol{\gamma})^2 = n_i n_j \frac{1}{2}(\gamma_i\gamma_j + \gamma_j\gamma_i) = -n_i n_i \mathbb{1} = -(\boldsymbol{n})^2 \cdot \mathbb{1} = -\mathbb{1} \quad . \tag{3.88}$$

Its eigenvalues are accordingly ±i. Each quark state can be expanded into the corresponding eigenvectors, and the nice thing is that for just these eigenstates (3.87) is satisfied. From

$$\boldsymbol{n}\cdot\boldsymbol{\gamma}\,\mathrm{q}_+ = \mathrm{i}\mathrm{q}_+ \tag{3.89}$$

it follows by Hermitian conjugation that

$$\begin{aligned} &(\mathrm{q}_+^\dagger \boldsymbol{n}\cdot\boldsymbol{\gamma}^\dagger)\gamma_0 = -\mathrm{i}\bar{\mathrm{q}}_+ \quad , \\ \Rightarrow\quad &\bar{\mathrm{q}}_+ \boldsymbol{n}\cdot\boldsymbol{\gamma} = -\mathrm{i}\bar{\mathrm{q}}_+ \quad . \end{aligned} \tag{3.90}$$

If one multiplies (3.89) by $\bar{\mathrm{q}}_+$ from the left and (3.90) by q_+ from the right and adds both equations, then it follows that

$$\bar{\mathrm{q}}_+ \boldsymbol{n}\cdot\boldsymbol{\gamma}\,\mathrm{q}_+ = 0 \tag{3.91}$$

and correspondingly for the eigenvalue −i. By restricting ourselves to eigenvectors of $\boldsymbol{n}\cdot\boldsymbol{\gamma}$, we can thus always guarantee that (3.87) is satisfied and must only solve the much easier linear equations

$$\boldsymbol{n}\cdot\boldsymbol{\gamma}\,\mathrm{q} = \pm\mathrm{i}\mathrm{q} \quad . \tag{3.92}$$

The disadvantage of this procedure is that other solutions of (3.87) that are not eigenvectors of $\boldsymbol{n}\cdot\boldsymbol{\gamma}$ are in this way excluded.

We shall show in Exercise 3.10 that eigenvectors to the eigenvalue −i are just the antiparticle solutions to those with the eigenvalue +i. We can therefore restrict ourselves to one sign. Normally we choose

$$\mathrm{i}\boldsymbol{n}(\theta,\varphi)\cdot\boldsymbol{\gamma}\,\mathrm{q}(x) = -\mathrm{q}(x)\Big|_{R=R(\theta,\varphi)} \tag{3.93}$$

as the particle solution.

In principle we can choose any bag shape, i.e., an arbitrary function $R = R(\theta,\varphi)$ and a corresponding normal vector $\boldsymbol{n}(\theta,\varphi)$. The simplest shape is naturally a sphere, i.e.,

$$\begin{aligned} &R(\theta,\varphi) = R = \text{const} \quad , \qquad \boldsymbol{n}(\theta,\varphi) = \boldsymbol{e}_r \quad , \\ \Rightarrow\quad &-\mathrm{i}\,\boldsymbol{e}_r\cdot\boldsymbol{\gamma}\,\mathrm{q}(|\boldsymbol{x}| = R) = \mathrm{q}(|\boldsymbol{x}| = R) \quad . \end{aligned} \tag{3.94}$$

Obviously there now remains only one parameter that is not fixed by the model assumptions, the bag radius R. Since it is unsatisfactory to choose R arbitrarily for every hadron, we introduce another boundary condition. This is obtained by demanding that the pressure of the quarks on the bag surface be constant. The model assumption leading to this is that the vacuum around the bag is a complex state exerting some kind of pressure on the bag. If this exceeds the interior pressure, the bag shrinks; otherwise it inflates further. How we can think of this pressure as arising is discussed in Sect. 7.3.

EXERCISE

3.10 Antiquark Solutions in a Bag

Problem. Show that a quark wave function obeying the equation

$$\mathrm{i}n\cdot\gamma(x)\mathrm{q}(x) = -\mathrm{q}(x)\Big|_{R=R(\vartheta,\phi)} \tag{1}$$

corresponds to antiquark solutions that fulfill

$$\mathrm{i}n\cdot\gamma(x)\tilde{\mathrm{q}}(x) = \tilde{\mathrm{q}}(x) \quad . \tag{2}$$

Solution. We have to recall that antiparticle wave functions are $\hat{C}\hat{P}\hat{T}$ transforms of the corresponding particle solutions:

$$\tilde{\mathrm{q}}(x) = \hat{C}\hat{P}\hat{T}\mathrm{q}(x) \quad . \tag{3}$$

In the standard representation this is

$$\hat{C}\hat{P}\hat{T}\ldots = \gamma_0\mathrm{i}\gamma_2\Big(\mathrm{i}\gamma_1\gamma_3(\ldots)^*\Big)^* \quad . \tag{4}$$

Applying (4) to (1) we obtain

$$\begin{aligned}\tilde{\mathrm{q}}(x) &= \hat{C}\hat{P}\hat{T}\Big[-\mathrm{i}n\cdot\gamma\mathrm{q}(x)\Big]\\ &= \gamma_0\mathrm{i}\gamma_2\Big[\mathrm{i}\gamma_1\gamma_3(\mathrm{i}n\cdot\gamma^*)\big(\mathrm{q}(x)\big)^*\Big]^*\\ &= \gamma_0\mathrm{i}\gamma_2\Big[(-\mathrm{i}n\cdot\gamma)\mathrm{i}\gamma_1\gamma_3\big(\mathrm{q}(x)\big)^*\Big]^*\\ &= \gamma_0\mathrm{i}\gamma_2(\mathrm{i}n\cdot\gamma^*)\Big[\mathrm{i}\gamma_1\gamma_3\big(\mathrm{q}(x)\big)^*\Big]^*\\ &= \mathrm{i}n\cdot\gamma\gamma_0\mathrm{i}\gamma_2\Big[\mathrm{i}\gamma_1\gamma_3\big(\mathrm{q}(x)\big)^*\Big]^*\\ &= \mathrm{i}n\cdot\gamma\tilde{\mathrm{q}}(x) \quad .\end{aligned} \tag{5}$$

Here we have repeatedly made use of the fact that γ_1 and γ_3 are real and that γ_2 is purely imaginary.

The antiparticle solutions therefore fulfill the modified boundary condition (2), i.e. we have only to determine the solutions of (1) in order to obtain the quark spectrum in the MIT bag.

To calculate the bag pressure we start from the energy momentum tensor for Dirac particles

$$T_{\mu\nu} = \sum_{\mathrm{q}}\frac{\mathrm{i}}{2}\left(\bar{\mathrm{q}}\gamma_\mu\frac{\overrightarrow{\partial}}{\partial x^\nu}\mathrm{q} - \bar{\mathrm{q}}\frac{\overleftarrow{\partial}}{\partial x^\nu}\gamma_\mu\mathrm{q}\right) \quad . \tag{3.95}$$

Using the boundary condition (3.93), which in its most general form is

$$\begin{aligned} n_\mu \gamma^\mu \mathrm{q} &= -\mathrm{i}\mathrm{q}\Big|_{\text{surface}} \quad , \\ \bar{\mathrm{q}} n_\mu \gamma^\mu &= \mathrm{i}\bar{\mathrm{q}}\Big|_{\text{surface}} \quad , \end{aligned} \tag{3.96}$$

it follows that

$$\begin{aligned} n^\mu T_{\mu\nu}\Big|_{\text{surface}} &= -\sum_{\mathrm{q}} \frac{1}{2}\left(\bar{\mathrm{q}}\frac{\overrightarrow{\partial}}{\partial x^\nu}\mathrm{q} + \bar{\mathrm{q}}\frac{\overleftarrow{\partial}}{\partial x^\nu}\mathrm{q}\right)\Bigg|_{\text{surface}} \\ &= -\frac{1}{2}\frac{\partial}{\partial x^\nu}\sum_{\mathrm{q}} \bar{\mathrm{q}}\mathrm{q}\Big|_{\text{surface}} \end{aligned} \tag{3.97}$$

With (3.91) it also holds from (3.89) that

$$\bar{\mathrm{q}}\mathrm{q}\Big|_{\text{surface}} = 0 \quad . \tag{3.98}$$

The gradient in (3.97) therefore points in the direction of the outward normal. According to the usual definition the absolute value of $n^\mu T_{\mu\nu}$ is just the pressure, so we conclude that

$$n^\mu T_{\mu\nu}\Big|_{\text{surface}} = -n_\nu P_{\mathrm{D}} \quad . \tag{3.99}$$

Since $n_\nu = (0, -\boldsymbol{n})$, the force exerted on the volume element by the pressure is directed outward. The Dirac pressure $(\boldsymbol{n}^2 = -1)$

$$P_{\mathrm{D}} = n^\mu n^\nu T_{\mu\nu} = -\frac{1}{2}n^\nu \frac{\partial}{\partial x^\nu}\sum_{\mathrm{q}} \mathrm{q}\bar{\mathrm{q}}\Big|_{\text{surface}} \tag{3.100}$$

is positive. For $n^\nu = (0, \boldsymbol{n})$ this becomes with $n^\nu \partial/\partial x^\nu = (0, \boldsymbol{n})\left(\partial/\partial x^0, \nabla\right) = \boldsymbol{n}\cdot\nabla$

$$P_{\mathrm{D}} = \frac{1}{2}\boldsymbol{n}\cdot\nabla\sum_{\mathrm{q}} \bar{\mathrm{q}}\mathrm{q}\Big|_{\text{surface}} \quad . \tag{3.101}$$

According to the model, this quark pressure must equal a constant exterior pressure B,

$$B = \frac{1}{2}\boldsymbol{n}\cdot\nabla\sum_{\mathrm{q}} \bar{\mathrm{q}}\mathrm{q}\Big|_{R=R(\theta,\varphi)} \quad . \tag{3.102}$$

Equations (3.102) and (3.94), together with the convention that the quarks are to be considered free inside the bag, define the MIT bag model. By introducing the new parameter B, the radii of all hadron bags are fixed.

In the following we shall investigate *spherical bags*. This is for sure the most obvious assumption for hadron ground states and has the additional advantage that the solutions can be found mostly analytically. We are thus looking for solutions of the stationary free and (for the time being) massless Dirac equation. The finite-mass case is discussed in Exercise 3.11.

$$\not{p}\,\Psi = 0 \tag{3.103}$$

with the boundary conditions

$$-\boldsymbol{e}_r \cdot \boldsymbol{\gamma}\Psi = \Psi\Big|_{|\boldsymbol{x}|=R} \quad \text{or} \quad \begin{pmatrix} 0 & -\mathrm{i}\sigma_r \\ \mathrm{i}\sigma_r & 0 \end{pmatrix} \Psi = \Psi\Big|_{|\boldsymbol{x}|=R} \tag{3.104}$$

and

$$-\frac{1}{2}\frac{\partial}{\partial r}\sum_{\mathrm{q}} \bar{\mathrm{q}}\mathrm{q}\Big|_{|\boldsymbol{x}|=R} = B \quad . \tag{3.105}$$

Owing to the spherical symmetry of the problem, an expansion into spherical spinors suggests itself, for example, as in the solution of the hydrogen problem:

$$\Psi = \begin{pmatrix} g(r)\,\chi^{\mu}_{\kappa}(\theta,\varphi) \\ -\mathrm{i}f(r)\,\chi^{\mu}_{-\kappa}(\theta,\varphi) \end{pmatrix} \mathrm{e}^{-\mathrm{i}Et} \quad . \tag{3.106}$$

The operator $\boldsymbol{\alpha}\cdot\boldsymbol{p}$ is then

$$\boldsymbol{\alpha}\cdot\boldsymbol{p} = -\mathrm{i}\alpha_r\frac{\partial}{\partial r} + \mathrm{i}\frac{\alpha_r}{r}(\beta\hat{K} - 1) \tag{3.107}$$

with

$$\hat{K}\chi^{\mu}_{\kappa} = -\kappa\chi^{\mu}_{\kappa} \quad . \tag{3.107a}$$

The operator $\hat{K} = \beta(\boldsymbol{\Sigma}\cdot\boldsymbol{L} + 1)$ has eigenvalues $-\kappa$ and κ, with κ being defined by

$$\begin{aligned} \kappa &= -j - \frac{1}{2} \quad \text{for} \quad \ell = j - \frac{1}{2} \quad , \\ \kappa &= j + \frac{1}{2} \quad \text{for} \quad \ell = j + \frac{1}{2} \quad . \end{aligned} \tag{3.108}$$

With $\alpha_r = \begin{pmatrix} 0 & \sigma_r \\ \sigma_r & 0 \end{pmatrix}$ we obtain from (3.103)

$$-\mathrm{i}\begin{pmatrix} 0 & \sigma_r \\ \sigma_r & 0 \end{pmatrix}\left(\frac{\partial}{\partial r} + \frac{1}{r} - \frac{\beta}{r}\hat{K}\right)\Psi = E\,\Psi \quad . \tag{3.109}$$

Ansatz (3.106) then yields by virtue of

$$\sigma_r\,\chi^{\mu}_{\kappa} = -\chi^{\mu}_{-\kappa} \tag{3.110}$$

the two equations

$$\left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{1}{r} - \frac{\kappa}{r}\right)f(r) = E\,g(r) \tag{3.111a}$$

and

$$\left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{1}{r} + \frac{\kappa}{r}\right)g(r) = -E\,f(r) \tag{3.111b}$$

or

$$\left(\frac{\mathrm{d}}{\mathrm{d}(Er)} + \frac{1-\kappa}{Er}\right) f = g \tag{3.112a}$$

and

$$\left(\frac{\mathrm{d}}{\mathrm{d}(Er)} + \frac{1+\kappa}{Er}\right) g = -f \quad . \tag{3.112b}$$

We replace $Er =: z$ and take the derivative of (3.112a) with respect to z:

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - \frac{1-\kappa}{z^2} + \frac{1-\kappa}{z}\frac{\mathrm{d}}{\mathrm{d}z}\right) f = \frac{\mathrm{d}}{\mathrm{d}z} g = -\frac{1+\kappa}{z} g - f \quad . \tag{3.113}$$

From (3.112a) it follows that

$$\frac{1+\kappa}{z} g = \left[\frac{1+\kappa}{z}\frac{\mathrm{d}}{\mathrm{d}z} + \frac{1-\kappa^2}{z^2}\right] f \tag{3.114}$$

and consequently

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}z^2} + \frac{2}{z}\frac{\mathrm{d}}{\mathrm{d}z} + \frac{\kappa(1-\kappa)}{z^2} + 1\right] f = 0 \quad . \tag{3.115}$$

This is exactly the differential equation satisfied by the Bessel function: as f and g may not diverge more strongly than $1/r$ at the origin, we must choose j_n functions:

$$f(r) = \mathrm{const} \times j_\ell(Er) \quad , \quad \ell = \begin{cases} -\kappa & \text{for } \kappa < 0 \\ \kappa - 1 & \text{for } \kappa > 0 \end{cases} \quad . \tag{3.116}$$

Analogously it follows that

$$g(r) = \mathrm{const} \times j_\ell(Er) \quad , \quad \ell = \begin{cases} -\kappa - 1 & \text{for} \quad \kappa < 0 \\ \kappa & \text{for} \quad \kappa > 0 \end{cases} \quad . \tag{3.117}$$

The factor from (3.117) can be absorbed into overall normalization. It remains to determine the relative factor from (3.112):

$$f(r) = j_{\bar{\ell}}(Er) \quad , \quad g(r) = c\, j_\ell(Er) \quad , \tag{3.118}$$

$$\Rightarrow \quad \left(\frac{\mathrm{d}}{\mathrm{d}z} + \frac{1-\kappa}{z}\right) j_{\bar{\ell}}(z) = c\, j_\ell(z) \quad , \tag{3.119a}$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}z} + \frac{1+\kappa}{z}\right) c\, j_\ell(z) = -j_{\bar{\ell}}(z) \quad . \tag{3.119b}$$

For $\kappa < 0$, (3.119) gives for $c = 1$ only the usual recursion relations between spherical Bessel functions of order ℓ and $\bar{\ell} = \ell - \mathrm{sgn}(\kappa)$, while for $\kappa > 0$, (3.119) leads to $c = -1$.

The wave function of a massless quark in an MIT bag is thus

$$\Psi = N \begin{pmatrix} j_\ell(Er)\, \chi^\mu_\kappa(\theta, \phi) \\ \mathrm{i}\, \mathrm{sgn}(\kappa) j_{\bar{\ell}}(Er)\, \chi^\mu_{-\kappa}(\theta, \phi) \end{pmatrix} \mathrm{e}^{-\mathrm{i}Et} \quad . \tag{3.120}$$

The normalization will be determined later. First we must check if a wave function of the structure (3.120) can satisfy the boundary conditions at all. We start with the linear boundary condition (3.104). Using (3.120) and (3.110) it reduces to

$$\begin{pmatrix} -\mathrm{sgn}(\kappa) j_{\bar{\ell}}(ER)\, \chi^{\mu}_{\kappa}(\theta,\phi) \\ -\mathrm{i} j_{\ell}(ER)\, \chi^{\mu}_{-\kappa}(\theta,\phi) \end{pmatrix} = \begin{pmatrix} j_{\ell}(ER)\, \chi^{\mu}_{\kappa}(\theta,\phi) \\ \mathrm{i}\, \mathrm{sgn}(\kappa) j_{\bar{\ell}}(ER)\, \chi^{\mu}_{-\kappa}(\theta,\phi) \end{pmatrix} \tag{3.121}$$

or

$$j_{\ell}(ER) = -\mathrm{sgn}(\kappa) j_{\bar{\ell}}(ER) \quad . \tag{3.122}$$

Thus the linear MIT boundary condition can indeed be satisfied and yields an eigenvalue equation for ER. For $\kappa = -1$ we obtain for example the solutions

$$ER|_{\kappa=-1} \approx 2.043,\ 5.396,\ 8.578,\ \dots \ . \tag{3.123}$$

From the asymptotic expansions of the Bessel function, some interesting properties of the quark spectrum can be read off.

1. For a given κ, e.g., $\kappa = -1$, the levels become equidistant for large energies. This follows immediately from

   $$j_{\nu}(z) \simeq \frac{1}{z} \cos\left(z - \frac{\nu+1}{2}\pi\right) \quad \text{for}\, z \to \infty \quad . \tag{3.124}$$

   For $\kappa = -1$, (3.122) therefore asymptotically becomes

   $$\cos\left(ER - \frac{\pi}{2}\right) = \cos(ER - \pi) \quad \text{or} \quad \tan(ER) = -1 \quad , \tag{3.125}$$

   which is obviously periodic in ER.
2. The smaller $|\kappa|$ becomes, the higher the lowest eigenvalue lies. For very large values of ν,

   $$j_{\nu}(z) \simeq \frac{1}{2\sqrt{z\nu}} \left(\frac{ez}{2\nu}\right)^{\nu} \quad , \quad \nu \to \infty \quad . \tag{3.126}$$

   Condition (3.122) can only be satisfied if

   $$\begin{aligned} 1 = \left|\frac{j_{\ell}(ER)}{j_{\ell-1}(ER)}\right| &\simeq \frac{z(\ell-1)^{\ell-1}\sqrt{\ell-1}}{\ell^{\ell}\sqrt{\ell}} < \frac{z}{\ell} \\ \Rightarrow \quad z > \ell > |\kappa| - 1 \quad . \end{aligned} \tag{3.127}$$

   The smallest eigenvalue is thus always larger than $|\kappa| - 1$. These two properties can also be seen in Fig. 3.12, which shows the lowest-lying energy states.

What about the quadratic boundary condition (3.105): can it also be satisfied? Inserting (3.120) into (3.105), we obtain

$$\begin{aligned} B = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}R} \sum_{\mathrm{q}} N_{\mathrm{q}}^{2} \Big[& j^{2}_{\ell_{\mathrm{q}}}(E_{\mathrm{q}}R)\, \chi^{\mu_{\mathrm{q}}}_{\kappa_{\mathrm{q}}}(\theta,\phi)^{+}\, \chi^{\mu_{\mathrm{q}}}_{\kappa_{\mathrm{q}}}(\theta,\phi) \\ & - j^{2}_{\bar{\ell}_{\mathrm{q}}}(E_{\mathrm{q}}R)\, \chi^{\mu_{\mathrm{q}}}_{-\kappa_{\mathrm{q}}}(\theta,\phi)^{+}\, \chi^{\mu_{\mathrm{q}}}_{-\kappa_{\mathrm{q}}}(\theta,\phi) \Big] \quad . \end{aligned} \tag{3.128}$$

Equation (3.128) can only be satisfied if the right-hand side no longer depends on θ and ϕ. This is only the case for $\kappa = \pm 1$, or when for higher κ states the q sum runs over all μ quantum numbers (shells). As we are primarily interested in $\mathrm{q}\bar{\mathrm{q}}$ and qqq systems (mesons and baryons), we can neglect this last possibility.

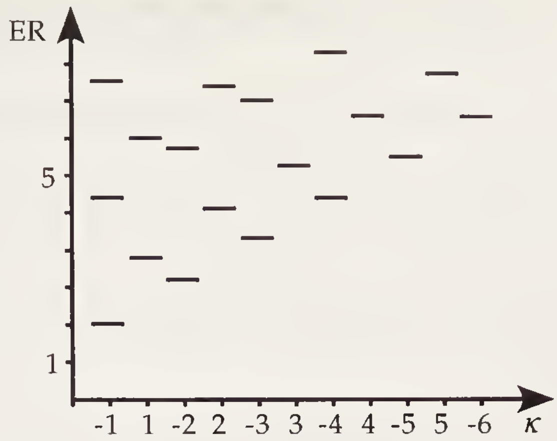


Fig. 3.12. The lowest-energy eigenvalues of the quark states in the MIT bag. *R* denotes the bag radius

The quadratic boundary condition can therefore only be satisfied for $|\kappa| = 1$. For $|\kappa| = 1$ it holds that

$$\chi_{\kappa}^{\mu+}\chi_{\kappa}^{\mu} = \chi_{-\kappa}^{\mu+}\chi_{-\kappa}^{\mu} = \frac{1}{4\pi} \quad , \tag{3.129}$$

and from this (3.128) becomes

$$\begin{aligned} B = -\frac{1}{4\pi}\sum_{\mathrm{q}} N_{\mathrm{q}}^2 \Bigg(& j_{\ell_{\mathrm{q}}}(E_{\mathrm{q}}R)E_{\mathrm{q}}\frac{\mathrm{d}}{\mathrm{d}(E_{\mathrm{q}}R)}j_{\ell_{\mathrm{q}}}(E_{\mathrm{q}}R) \\ & - j_{\bar{\ell}_{\mathrm{q}}}(E_{\mathrm{q}}R)E_{\mathrm{q}}\frac{\mathrm{d}}{\mathrm{d}(E_{\mathrm{q}}R)}j_{\bar{\ell}_{\mathrm{q}}}(E_{\mathrm{q}}R)\Bigg) \quad . \end{aligned} \tag{3.130}$$

In the case that all quarks are in the lowest state with $\kappa = -1$, it follows from (3.122) that

$$\frac{\mathrm{d}}{\mathrm{d}(ER)}j_{\ell_{\mathrm{q}}}(ER) = \frac{\mathrm{d}}{\mathrm{d}(ER)}j_0(ER) = -j_1(ER) = -j_0(ER) \quad , \tag{3.131}$$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}(ER)}j_{\bar{\ell}_{\mathrm{q}}}(ER) &= \frac{\mathrm{d}}{\mathrm{d}(ER)}j_1(ER) = j_0(ER) - \frac{2}{3}j_2(ER) - \frac{2}{3}j_0(ER) \\ &= j_0(ER) - \frac{2}{ER}j_1(ER) = j_0(ER) - \frac{2}{ER}j_0(ER) \quad , \end{aligned} \tag{3.132}$$

and therefore

$$B = \frac{1}{4\pi}N_{\kappa=-1}^2 N_{\mathrm{q}} 2E\, j_0^2(ER)\left(1 - \frac{1}{ER}\right) \quad , \tag{3.133}$$

where N_q indicates the number of quarks in the bag.

In Exercise 3.11 we shall show that

$$N_{\kappa=-1}^2 = \frac{ER}{2R^3(ER-1)j_0^2(ER)} \quad , \tag{3.134}$$

and consequently

$$B = \frac{1}{4\pi}N_{\mathrm{q}}\frac{E}{R^3} \quad \text{or} \quad R^4 = \frac{N_{\mathrm{q}}ER}{4\pi B} \quad . \tag{3.135}$$

Equation (3.128) thus yields for any quark content the corresponding bag radius.

EXERCISE

3.11 The Bag Wave Function for Massive Quarks

Problem. Show that the bag wave function of massive quarks is

$$\Psi = N \begin{pmatrix} j_{\ell_\kappa}(pr)\,\chi_\kappa^\mu \\ \mathrm{i}\frac{p}{E+m}\ \mathrm{sgn}(\kappa)\, j_{\bar{\ell}_\kappa}(pr)\,\chi_{-\kappa}^\mu \end{pmatrix} \mathrm{e}^{-\mathrm{i}Et} \tag{1}$$

with

$$E = \sqrt{p^2 + m^2} \quad ,$$

$$\ell = \begin{cases} -\kappa - 1 & \text{for} \quad \kappa < 0 \\ \kappa & \text{for} \quad \kappa > 0 \end{cases} \quad ,$$

$$\bar{\ell} = \begin{cases} -\kappa & \text{for} \quad \kappa < 0 \\ \kappa - 1 & \text{for} \quad \kappa > 0 \end{cases} \quad .$$

Show furthermore that the normalization condition

$$\int_{\text{bag}} \mathrm{d}^3r\ \Psi^\dagger \Psi = 1 \tag{2}$$

leads to

$$N = \frac{p}{\sqrt{2E(ER + \kappa) + m}}\ \frac{1}{|j_{\ell_\kappa}(pR)|R} \quad , \qquad R = \text{bag radius} \quad . \tag{3}$$

Solution. Adding the mass term to (3.103) yields

$$(\not{p} - m)\Psi = 0 \quad . \tag{4}$$

Correspondingly (3.109) becomes

$$\left[-\mathrm{i}\begin{pmatrix} 0 & \sigma_r \\ \sigma_r & 0 \end{pmatrix}\left(\frac{\partial}{\partial r} + \frac{1}{r} - \frac{\beta}{r}\hat{K}\right) + \beta m\right]\Psi = E\,\Psi \quad . \tag{5}$$

(3.111) then becomes

$$\left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{1}{r} - \frac{\kappa}{r}\right) f(r) - (E - m) g(r) = 0 \tag{6a}$$

and

$$\left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{1}{r} + \frac{\kappa}{r}\right) g(r) + (E + m) f(r) = 0 \quad . \tag{6b}$$

Exercise 3.11.

Now we insert $g(r) = j_{\ell_\kappa}(pr)$ and $f(r) = -[p/(E+m)])\ \mathrm{sgn}(\kappa)\, j_{\bar{\ell}_\kappa}(pr)$ and evaluate the left-hand sides of (6a) and (6b), respectively:

$$\frac{-p^2}{E+m}\,\mathrm{sgn}(\kappa)\left(\frac{\mathrm{d}}{\mathrm{d}(pr)}+\frac{1-\kappa}{pr}\right) j_{\bar{\ell}_\kappa}(pr) - (E-m)\, j_{\ell_\kappa}(pr) = 0 \tag{7a}$$

and

$$p\left(\frac{\mathrm{d}}{\mathrm{d}(pr)}+\frac{1+\kappa}{pr}\right) j_{\ell_\kappa}(pr) - p\ \mathrm{sgn}(\kappa)\, j_{\bar{\ell}_\kappa}(pr) = 0 \tag{7b}$$

With the help of recursion relations (3.119) ($c = -\mathrm{sgn}(\kappa)$) we then obtain

$$\frac{p^2}{E+m}\, j_{\ell_\kappa}(pr) - \frac{p^2}{E+m}\, j_{\ell_\kappa}(pr) = 0 \tag{8a}$$

and

$$p\ \mathrm{sgn}(\kappa)\, j_{\bar{\ell}_\kappa}(pr) - p\ \mathrm{sgn}(\kappa)\, j_{\bar{\ell}_\kappa}(pr) = 0 \quad . \tag{8b}$$

In addition one sees directly that (1) transforms into (3.120) for $m \to 0$. The modified linear boundary condition (3.122) is now

$$j_{\ell_\kappa}(pr) = -\mathrm{sgn}(\kappa)\frac{p}{E+m}\, j_{\bar{\ell}_\kappa}(pr) \quad . \tag{9}$$

By means of (9) we then evaluate the normalization integral:

$$\begin{aligned} 1 &= \int\limits_{\text{bag}} \mathrm{d}^3 r\ \Psi^\dagger \Psi \\ &= \int\limits_0^R \mathrm{d}r\ r^2 \left[j^2_{\ell_\kappa}(pr) + \frac{p^2}{(E+m)^2}\, j^2_{\bar{\ell}_\kappa}(pr)\right] N^2 \,. \end{aligned} \tag{10}$$

Employing the differentiation and recursion formulas of the Bessel functions

$$\begin{aligned} j_n'(z) &= \frac{n}{z} j_n(z) - j_{n+1}(z) \,, \\ j_{n-1}'(z) &= \frac{n-1}{z} j_{n-1}(z) - j_n(z) \,, \end{aligned} \tag{11}$$

$$\begin{aligned} j_{n+1}'(z) &= -\frac{n+2}{z} j_{n+1}(z) + j_n(z) \,, \\ j_{n+1}(z) + j_{n-1}(z) &= \frac{2n+1}{z} j_n(z) \end{aligned} \tag{12}$$

we get

Exercise 3.11.

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}z}\left[z^2\left(j_n^2(z)-j_{n-1}(z)\,j_{n+1}(z)\right)\right]\\
&\quad = 3z^2\left(j_n^2(z)-j_{n-1}(z)\,j_{n+1}(z)\right)+z^3\left[2\,j_n(z)\left(-j_{n+1}(z)+\frac{n}{z}j_n(z)\right)\right.\\
&\qquad \left. -j_{n-1}(z)\left(-\frac{n+2}{z}j_{n+1}(z)+j_n(z)\right)-\left(\frac{n-1}{z}j_{n-1}(z)-j_n(z)\right)j_{n+1}(z)\right]\\
&\quad = 3z^2\left(j_n^2(z)-j_{n-1}(z)\,j_{n+1}(z)\right)\\
&\qquad +z^3\left[\frac{2n}{z}j_n^2(z)-j_n(z)\left(j_{n+1}(z)+j_{n-1}(z)\right)+\frac{3}{z}j_{n-1}(z)\,j_{n+1}(z)\right]\\
&\quad = 3z^2\,j_n^2(z)+z^3\left[\frac{2n}{z}j_n^2(z)-\frac{2n+1}{z}j_n^2(z)\right]\\
&\quad = 2z^2\,j_n^2(z) \quad .
\end{aligned}
\tag{13}
$$

This relation allows a direct evaluation of integral (10):

$$
\begin{aligned}
1 = N^2\frac{R^3}{2}&\left[j_{\ell_\kappa}^2(pR)-j_{\ell_\kappa-1}(pR)\,j_{\ell_\kappa+1}(pR)\right.\\
&\left.+\frac{p^2}{(E+m)^2}\left(j_{\bar{\ell}_\kappa}^2(pR)-j_{\bar{\ell}_\kappa-1}(pR)\,j_{\bar{\ell}_\kappa+1}(pR)\right)\right] \quad .
\end{aligned}
\tag{14}
$$

Equation (14) can be simplified using (9). For $\kappa < 0$ we have (see (12)):

$$
\begin{aligned}
\bar{\ell}_\kappa &= \ell_\kappa+1 \quad , \quad \ell_\kappa = -\kappa-1 \quad ,\\
j_{\ell_\kappa+1}(pR) &= \frac{E+m}{p}j_{\ell_\kappa}(pR) \quad ,\\
j_{\ell_\kappa-1}(pR) &= \frac{2\ell_\kappa+1}{pR}j_{\ell_\kappa}(pR)-j_{\ell_\kappa+1}(pR)\\
&= \left(\frac{2\ell_\kappa+1}{pR}-\frac{E+m}{p}\right)j_{\ell_\kappa}(pR) \quad ,\\
j_{\ell_\kappa+2}(pR) &= \frac{2\ell_\kappa+3}{pR}j_{\ell_\kappa+1}(pR)-j_{\ell_\kappa}(pR)\\
&= \left(\frac{2\ell_\kappa+3}{pR}\frac{E+m}{p}-1\right)j_{\ell_\kappa}(pR) \quad .
\end{aligned}
\tag{15}
$$

Equation (14) therefore becomes

$$
\begin{aligned}
1 = N^2\frac{R^3}{2}j_{\ell_\kappa}^2(pR)&\left[2-\left(\frac{2\ell_\kappa+1}{pR}-\frac{E+m}{p}\right)\frac{E+m}{p}\right.\\
&\left.+\frac{p^2}{(E+m)^2}\left(1-\frac{2\ell_\kappa+3}{pR}\frac{E+m}{p}\right)\right]
\end{aligned}
$$

Exercise 3.11.

$$
\begin{aligned}
&= N^2 \frac{R^3}{2} j^2_{\ell_\kappa}(pR) \left[2 - 2\ell_\kappa \left(\frac{E+m}{p^2R} + \frac{1}{(E+m)R} \right) \right. \\
&\qquad \left. - \frac{E+m}{p^2R} - \frac{3}{(E+m)R} + \frac{E+m}{E-m} + \frac{E-m}{E+m} \right] \\
&= N^2 \frac{R^3}{2} j^2_{\ell_\kappa}(pR) \left(-2\ell_\kappa \frac{2E}{p^2R} - \frac{4E-2m}{p^2R} + \frac{4E^2}{p^2} \right) \\
&= N^2 \frac{R^2}{p^2} j^2_{\ell_\kappa}(pR) \left(2\kappa E + m + 2E^2R \right) \quad .
\end{aligned}
\tag{16}
$$

Hence

$$
N = \frac{p}{R|j_{\ell_\kappa}(pr)|\sqrt{2\kappa E + m + 2E^2R}} \quad . \tag{17}
$$

For $\kappa > 0$ an analogous calculation yields

$$
\begin{aligned}
&\bar{\ell}_\kappa = \ell_\kappa - 1 \quad , \quad \ell_\kappa = \kappa \quad , \\
j_{\ell_\kappa - 1}(pR) &= -\frac{E+m}{p} j_{\ell_\kappa}(pR) \quad , \\
j_{\ell_\kappa + 1}(pR) &= \frac{2\ell_\kappa + 1}{pR} j_{\ell_\kappa}(pR) - j_{\ell_\kappa - 1}(pR) \\
&= \left(\frac{2\ell_\kappa + 1}{pR} + \frac{E+m}{p} \right) j_{\ell_\kappa}(pR) \quad , \\
j_{\ell_\kappa - 2}(pR) &= \frac{2\ell_\kappa - 1}{pR} j_{\ell_\kappa - 1}(pR) - j_{\ell_\kappa}(pR) \\
&= -\left(\frac{E+m}{p} \frac{2\ell_\kappa - 1}{pR} + 1 \right) j_{\ell_\kappa}(pR) \quad .
\end{aligned}
\tag{18}
$$

Equation (14) now has the form

$$
\begin{aligned}
1 &= N^2 \frac{R^3}{2} j^2_{\ell_\kappa}(pr) \left[2 + \left(\frac{2\ell_\kappa + 1}{pR} + \frac{E+m}{p} \right) \frac{E+m}{p} \right. \\
&\qquad \left. + \frac{p^2}{(E+m)^2} \left(\frac{E+m}{p} \frac{2\ell_\kappa - 1}{pR} + 1 \right) \right] \\
&= N^2 \frac{R^3}{2} j^2_{\ell_\kappa}(pr) \left(2\ell_\kappa \frac{2E}{p^2R} + \frac{2m}{p^2R} + \frac{4E^2}{p^2} \right) \\
&= N^2 \frac{R^2}{p^2} j^2_{\ell_\kappa}(pr) \left(2\kappa E + m + 2E^2R \right) \quad .
\end{aligned}
\tag{19}
$$

Therefore also for $\kappa > 0$ N is also given by (17). The corresponding normalization factors for massless quarks are obtained by setting $p = E$ and $m = 0$ (see (3.134)).

Even in the simplest case, however, (3.135) leads to difficulties. From this equation a nucleon (three almost massless quarks in the ground state) and a pion (one quark and one antiquark) should have almost the same radius

$$\frac{R_\mathrm{N}}{R_\pi} = \left(\frac{3}{2}\right)^{1/4} = 1.107 \quad . \tag{3.136}$$

This result does not agree with physical reality. The prediction for the pion mass is correspondingly much too large. In order to obtain a more realistic model we therefore have to go beyond the simple assumptions made so far. Before turning to MIT bag-model applications we want to discuss the necessary additional assumptions. The main problem is the nonphysical equation (3.135). This can be changed by introducing additional pressure terms with different R dependence. Let us first consider the total energy, where the effect of these new terms can be seen most clearly. So far the total energy consists of two terms: the volume energy due to the external pressure B and the single-particle energies:

$$E = B\,\frac{4\pi}{3}R^3 + \sum_\mathrm{q}\frac{\omega_\mathrm{q}}{R} \quad , \quad \omega_\mathrm{q} = E_\mathrm{q}\,R = \mathrm{const} \quad . \tag{3.137}$$

For $\sum_\mathrm{q}\omega_\mathrm{q} \longrightarrow N_\mathrm{q}\omega_\mathrm{q}$ we immediately recognize that a variation with respect to R leads to (3.135). The first term added is

$$E_0 = -\frac{Z}{R} \quad , \quad Z = \mathrm{const} \quad . \tag{3.138}$$

Equation (3.135) then assumes the form

$$R^4 = \frac{N_\mathrm{q}\omega_\mathrm{q} - Z}{4\pi B} \quad . \tag{3.139}$$

A sufficiently large value of Z therefore allows us to adjust the mass and radius difference between mesons and baryons. Physically this new term is interpreted as the *Casimir effect.*[6]The bag yields a lower bound for the zero-point energy of the gluon field. The smaller the bag, the bigger the zero-point energy of the vacuum oscillations. It is therefore clear that E_0 vanishes for $R \to \infty$ and diverges for $R \to 0$. (For very small values of R, however, the bag boundary condition no longer makes any sense, because it is only an effective description of very complicated microscopic processes. Hence R must not assume values smaller than the typical length scale of the reactions considered.) Strictly speaking, we know neither the explicit functional form nor the sign of E_0. The results of simple model calculations do not justify (3.138) and are in addition completely unreliable, since the self-interaction of the gluons (i.e., the non-Abelian structure of QCD, see Chap. 4) has only been treated in a rough approximation. Equation (3.138) should therefore be considered a phenomenological correction term with unknown physical origin.

The next problem is to describe the *mass splitting* within the baryon multiplet. Since all strange particles are considerable heavier than those with $S = 0$, the introduction of quark masses seems to be a reasonable first step. Therefore in the following we make use of the massive eigenfunctions and the corresponding *energy eigenvalues* derived in Exercise 3.11. The masses of u, d and s quarks are treated as free paramcters. This generalization, however, is not sufficient to describe both the relatively small splitting between Σ and N and the huge splitting between K and π. A further correction is necessary that assumes different values for mesons and baryons. We therefore introduce the following interaction between the quarks:

---

[6] See G. Plunien, B. Müller, and W. Greiner: Phys. Rep. **134**, 87 (1986).

$$E_{\mathrm{q}G} = \alpha_{\mathrm{c}} N \cdot \sum_{i<j} (\sigma_i \cdot \sigma_j) \frac{\mu_i \mu_j}{R^3} \left(1 + 2 \int_0^R \frac{\mathrm{d}r}{r^4} \mu_i \mu_j \right) \tag{3.140}$$

with

$$N = \begin{cases} 2 & \text{for a baryon} \\ 4 & \text{for a meson} \end{cases} \quad .$$

μ_i and α_{c} denote the magnetic moment of the quark with index i and the coupling constant of the color interaction.

Equation (3.140) is obtained by taking the color magnetic interaction between the quarks into account. For a more detailed treatment of these matters we refer to Chap. 4, where the QCD equations are discussed. At this point it need only be mentioned that the derivation of (3.140) is not consistent. Again the gluonic self-interaction has been neglected. The electric and magnetic parts of the remaining interaction (which then look exactly like the electromagnetic interaction) are treated in a different way. In order to justify (3.140) we must therefore postulate that all contributions except the one-gluon exchange are described by the bag boundary condition. Since α_{c} is treated more or less as a free parameter (within certain limits), (3.140) can also be interpreted as a phenomenological correction. Summarizing the above arguments we find that the total energy assumes the form

$$E = \frac{4\pi}{3} R^3 B + \frac{\sum_{\mathrm{q}} \omega_{\mathrm{q}} - Z}{R} + E_{\mathrm{q}G} \quad . \tag{3.141}$$

The bag radius is determined by

$$\frac{\partial E}{\partial R} = 0 \quad , \quad \frac{\partial^2 E}{\partial R^2} > 0 \Rightarrow R \tag{3.142}$$

and all features of the specific bags can be obtained by using the wave functions introduced in Exercise 3.11.

A total of about 25 to 30 experimentally observed values for masses, magnetic moments, averaged charge radii, axial coupling constants, and so on is available for fitting the six parameters B, Z, α_{c}, m_{s}, m_{u}, and m_{d}. The agreement achieved in such a fit is in general better than 30%.

Let us start with the masses. Figure 3.13 depicts both the theoretical and the experimental values for the following set of parameters:[7]

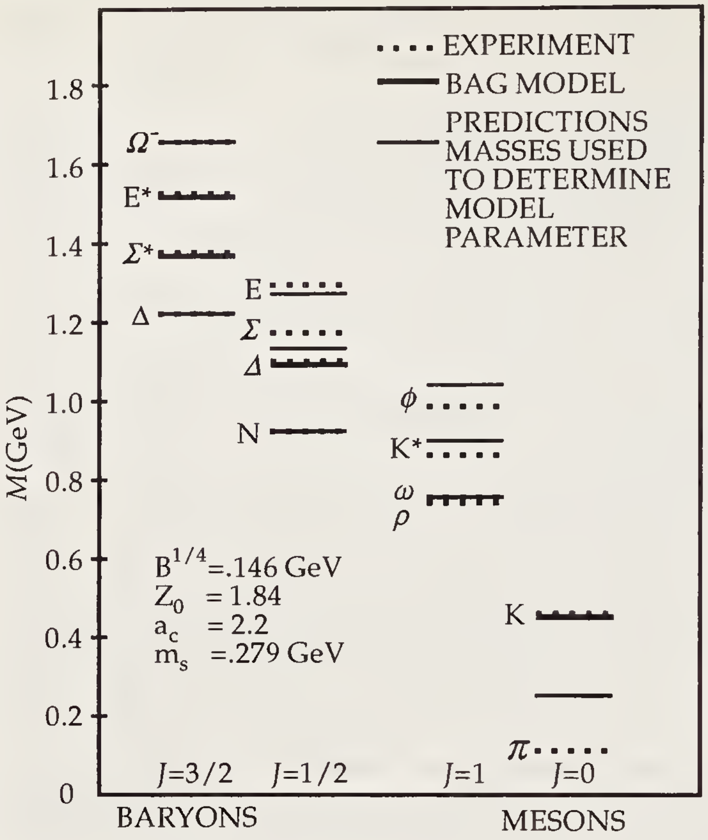
$$\begin{gathered} B = (146\ \mathrm{MeV})^4 \quad , \quad Z = 1.84 \quad , \quad \alpha_{\mathrm{c}} = 2.2 \quad , \\ m_{\mathrm{u}} = 0\ \mathrm{MeV} \quad , \quad m_{\mathrm{d}} = 0\ \mathrm{MeV} \quad , \quad m_{\mathrm{s}} = 279\ \mathrm{MeV} \quad . \end{gathered} \tag{3.143}$$

Clearly all the masses except that of the pion are quite well described. In general the modified MIT model, which has just been introduced, provides a satisfactory description of the light hadrons except for the pion. One possible way out of this

---

[7] See de Grand et al.: Phys. Rev. **D 12**, 2060 (1975)

Fig. 3.13. The MIT-bag-model fit for the lightest mesons and baryons. (from P. Hasenfratz and J. Kuti: The Quark Bag Model, Phys. Rep. **40**, 75 (1978))



problem is a combination of the bag model with a specific treatment of the pions. But we are not going to discuss these so-called hybrid bag models.

Instead we evaluate the averaged charge radius and the magnetic moment of the proton within the context of the MIT bag model. The squared charge radius is defined as

$$\langle r^2\rangle_{\mathrm{ch}} = \frac{1}{e}\int \mathrm{d}^3r_1 \int \mathrm{d}^3r_2 \int \mathrm{d}^3r_3\; \Psi_{\mathrm{p}}^\dagger \left(\hat{Q}\hat{r}^2\right) \Psi_{\mathrm{p}} \quad . \tag{3.144}$$

Here Ψ_{p} denotes the proton wave function, which can be decomposed into quark wave functions as follows:

$$\begin{aligned}\Psi_{\mathrm{p}}\left(\mu=\frac{1}{2}\right) = \frac{1}{3\sqrt{2}}\Big(&2\mathrm{u}^\uparrow(1)\mathrm{u}^\uparrow(2)\mathrm{d}^\downarrow(3) - \mathrm{u}^\uparrow(1)\mathrm{u}^\downarrow(2)\mathrm{d}^\uparrow(3) - \mathrm{u}^\downarrow(1)\mathrm{u}^\uparrow(2)\mathrm{d}^\uparrow(3)\\ &- \mathrm{u}^\uparrow(1)\mathrm{d}^\uparrow(2)\mathrm{u}^\downarrow(3) + \mathrm{u}^\uparrow(1)2\mathrm{d}^\downarrow(2)\mathrm{u}^\uparrow(3) - \mathrm{u}^\downarrow(1)\mathrm{d}^\uparrow(2)\mathrm{u}^\uparrow(3)\\ &- \mathrm{d}^\uparrow(1)\mathrm{u}^\uparrow(2)\mathrm{d}^\downarrow(3) - \mathrm{d}^\uparrow(1)\mathrm{u}^\downarrow(2)\mathrm{d}^\uparrow(3) + 2\mathrm{d}^\downarrow(1)\mathrm{u}^\uparrow(2)\mathrm{d}^\uparrow(3)\Big) \quad .\end{aligned} \tag{3.145}$$

$\mathrm{u}^\uparrow(1)$ denotes the wave function of a u quark with $\mu = +1/2$ attached to quark number one. The structure of (3.145) follows from the flavor symmetry group. Taking the spin into account we have to deal with SU(6). Inserting (3.145) into (3.144) and employing the *quark wave functions* (3.120) (owing to (3.143), the up and down quarks are assumed to be massless) leads to (see Exercise 3.12)

$$\langle r^2\rangle_{\mathrm{ch}} = 0.53\,R^2 \quad . \tag{3.146}$$

The bag radius R follows by minimizing (3.141), yielding $R = 1$ fm (the parameters have been correspondingly adjusted). Hence

$$\sqrt{\langle r^2\rangle_{\mathrm{ch}}} = 0.73\ \mathrm{fm} \quad , \tag{3.147}$$

which agrees with the experimental value to within 20%:

$$\sqrt{\langle r^2\rangle}(\mathrm{exp}) = 0.88 \pm 0.03\ \mathrm{fm} \quad . \tag{3.148}$$

The magnetic moment is obtained by evaluating the expectation value of the corresponding operator:

$$\hat{\mu} = \frac{\hat{Q}}{2}\hat{r} \times \hat{\alpha} \quad , \tag{3.149}$$

$$\mu_{\mathrm{p}} = \int \mathrm{d}^3 r_1 \int \mathrm{d}^3 r_2 \int \mathrm{d}^3 r_3\, \Psi_{\mathrm{p}}^\dagger \sum_i \left(\frac{\hat{Q}_i}{2}\,\hat{r}_i \times \hat{\alpha}_i\right) \Psi_{\mathrm{p}} \quad . \tag{3.150}$$

We again insert the wave functions (3.145) and (3.120). A detailed calculation, which will be performed in Exercise 3.13, yields

$$\frac{|\mu_{\mathrm{p}}|}{\mu_{\mathrm{K}}} = 1.9 \quad , \tag{3.151}$$

where $\mu_{\mathrm{K}} = e/2m_{\mathrm{p}}$ denotes the nuclear magneton and m_{p} the proton mass.

EXERCISE

3.12 The Mean Charge Radius of the Proton

Problem. Evaluate the right-hand side of (3.144):

$$\langle r^2\rangle_{\mathrm{ch}} = \frac{1}{e}\int \mathrm{d}^3 r_1 \int \mathrm{d}^3 r_2 \int \mathrm{d}^3 r_3\, \Psi_{\mathrm{p}}^\dagger(\hat{Q}r^2)\Psi_{\mathrm{p}} \quad . \tag{1}$$

Solution. Since r^2 does not change the quantum number μ, all spin orientations yield the same value. Therefore we can restrict our calculation to one specific case,

$$\begin{aligned}\Psi_{\mathrm{p}}\left(\mu = \frac{1}{2}\right) \to \frac{1}{\sqrt{6}}\,&(2\mathrm{u}^\uparrow(1)\mathrm{u}^\uparrow(2)\mathrm{d}^\downarrow(3)\\ &-\mathrm{u}^\uparrow(1)\mathrm{d}^\uparrow(2)\mathrm{u}^\downarrow(3) - \mathrm{d}^\uparrow(1)\mathrm{u}^\uparrow(2)\mathrm{u}^\downarrow(3)) \quad .\end{aligned} \tag{2}$$

The charge operator $\hat{Q}r^2$ acts on one quark at a time. For the proton it can be replaced by

$$\hat{Q}r^2 \to Q_1 r_1^2 + Q_2 r_2^2 + Q_3 r_3^2 \quad . \tag{3}$$

Equation (1) then becomes

Exercise 3.12.

$$\begin{aligned}\langle r^2\rangle_{\mathrm{ch}} = \frac{1}{e}\int \mathrm{d}^3r_1 \int \mathrm{d}^3r_2 \int \mathrm{d}^3r_3 \frac{1}{6}\,&\big(2\mathrm{u}^{\uparrow}(1)\mathrm{u}^{\uparrow}(2)\mathrm{d}^{\downarrow}(3)\\ &-\mathrm{u}^{\uparrow}(1)\mathrm{d}^{\uparrow}(2)\mathrm{u}^{\downarrow}(3) - \mathrm{d}^{\uparrow}(1)\mathrm{u}^{\uparrow}(2)\mathrm{u}^{\downarrow}(3)\big)^{\dagger}\\ &\times (Q_1r_1^2 + Q_2r_2^2 + Q_3r_3^2)\,\big(2\mathrm{u}^{\uparrow}(1)\mathrm{u}^{\uparrow}(2)\mathrm{d}^{\downarrow}(3)\\ &-\mathrm{u}^{\uparrow}(1)\mathrm{d}^{\uparrow}(2)\mathrm{u}^{\downarrow}(3) - \mathrm{d}^{\uparrow}(1)\mathrm{u}^{\uparrow}(2)\mathrm{u}^{\downarrow}(3)\big) \quad .\end{aligned} \tag{4}$$

All quarks are assumed to be massless and in the same state (namely the 1s state), i.e., we do not distinguish between up and down quarks. Therefore r_2^2 and r_3^2 can in the integrand be replaced by r_1^2:

$$Q_1r_1^2 + Q_2r_2^2 + Q_3r_3^2 \to (Q_1 + Q_2 + Q_3)r_1^2 = er_1^2 \quad . \tag{5}$$

Now the integrals over r_2 and r_3 can easily be evaluated by using the orthogonality relations

$$\int \mathrm{d}^3r_2\, \mathrm{q}^{s\dagger}(2)\, \mathrm{q}'^{s'}(2) = \delta_{qq'}\,\delta_{ss'} \quad , \quad \begin{cases} \mathrm{q},\ \mathrm{q}' = \mathrm{u},\ \mathrm{d} \\ \mathrm{s},\ \mathrm{s}' = \uparrow,\ \downarrow \end{cases} \quad . \tag{6}$$

$$\langle r^2\rangle_{\mathrm{ch}} = \int \mathrm{d}^3r_1\, r_1^2\, \frac{1}{6}\left(4\mathrm{u}^{\uparrow\dagger}(1)\mathrm{u}^{\uparrow}(1) + \mathrm{u}^{\uparrow\dagger}(1)\mathrm{u}^{\uparrow}(1) + \mathrm{d}^{\uparrow\dagger}(1)\mathrm{d}^{\uparrow}(1)\right) \quad . \tag{7}$$

Owing to the assumption made above, the wave functions of u and d yield the same contributions, and (7) simplifies to

$$\langle r^2\rangle_{\mathrm{ch}} = \int \mathrm{d}\Omega_1 \int \mathrm{d}r_1\, r_1^4\, \mathrm{u}^{\uparrow\dagger}(1)\mathrm{u}^{\uparrow}(1) \quad . \tag{8}$$

Now we insert the explicit form of the wave function (3.120) and skip the index 1 in the remaining calculation:

$$\langle r^2\rangle_{\mathrm{ch}} = N^2 \int \mathrm{d}r \int \mathrm{d}\Omega r^4 \left[j_0^2(Er)\,\chi_1^{\frac{1}{2}\dagger}(\Omega)\,\chi_1^{\frac{1}{2}}(\Omega) + j_1^2(Er)\,\chi_{-1}^{\frac{1}{2}\dagger}(\Omega)\,\chi_{-1}^{\frac{1}{2}}(\Omega)\right] \quad . \tag{9}$$

Because of the orthogonality of the spherical spinors and because of (3.134), (9) assumes the form

$$\langle r^2\rangle_{\mathrm{ch}} = \frac{ER}{2R^3(ER-1)j_0^2(ER)} \int_0^R \mathrm{d}r\, r^4 \left(j_0^2(Er) + j_1^2(Er)\right) \quad . \tag{10}$$

The remaining integral can again be evaluated by means of the recursion relations or simply by inserting the explicit expressions

$$\begin{aligned} j_0(r) &= \frac{\sin z}{z} \quad , \\ j_1(z) &= \frac{\sin z}{z^2} - \frac{\cos z}{z} \quad . \end{aligned} \tag{11}$$

Hence

Exercise 3.12.

$$\begin{aligned}
\langle r^2\rangle_{\mathrm{ch}} &= \frac{ER(ER)^2}{2R^3(ER-1)\sin^2(ER)} \\
&\quad \times \int\limits_0^R \mathrm{d}r\, r^4\left[\frac{\sin^2(Er)}{E^4r^4} - 2\frac{\sin(Er)\cos(Er)}{E^3r^3} + \frac{1}{E^2r^2}\right] \\
&= \frac{1}{2E(ER-1)\sin^2(ER)} \\
&\quad \times \left[-r\sin^2(Er) + r - \frac{1}{E}\sin(Er)\cos(Er) + \frac{1}{3}r^3E^2\right]_0^R \quad .
\end{aligned} \tag{12}$$

The boundary condition

$$\begin{aligned}
& j_0(ER) = j_1(ER) \\
\Rightarrow \quad & ER\cos(ER) = (1-ER)\sin(ER)
\end{aligned} \tag{13}$$

simplifies (12) to give

$$\begin{aligned}
\langle r^2\rangle_{\mathrm{ch}} &= \frac{1}{2E(ER-1)\sin^2(ER)} \\
&\quad \times \left[\frac{1}{E}(1-ER)\sin(ER)\,\cos(ER) - \frac{1}{E}\sin(ER)\cos(ER) + \frac{1}{3}R^3E^2\right] \\
&= \frac{-ER\sin(ER)\cos(ER) + \frac{1}{3}R^3E^3}{2E^2(ER-1)\sin^2(ER)} \\
&= 0.53\;R^2 \quad .
\end{aligned} \tag{14}$$

In the last step we have inserted the numerical value for ER, which is equal to 2.0428.

EXERCISE

3.13 The Magnetic Moment of the Proton

Problem. Evaluate the right-hand side of (3.150) with the help of the wave functions (3.145) and (3.120).

Solution. Since the SU(6) wave function of the proton is symmetric under permutations of the indices 1, 2, and 3, we have

$$\begin{aligned}
&\int \Psi_{\mathrm{p}}^{\dagger}\left(\frac{\hat{Q}_1}{2}\hat{\boldsymbol{r}}_1\times\hat{\boldsymbol{\alpha}}_1\right)\Psi_{\mathrm{p}}\,\mathrm{d}^3r_1\,\mathrm{d}^3r_2\,\mathrm{d}^3r_3 \\
&\quad = \int \Psi_{\mathrm{p}}^{\dagger}\left(\frac{\hat{Q}_2}{2}\hat{\boldsymbol{r}}_2\times\hat{\boldsymbol{\alpha}}_2\right)\Psi_{\mathrm{p}}\,\mathrm{d}^3r_1\,\mathrm{d}^3r_2\,\mathrm{d}^3r_3 \\
&\quad = \int \Psi_{\mathrm{p}}^{\dagger}\left(\frac{\hat{Q}_3}{2}\hat{\boldsymbol{r}}_3\times\hat{\boldsymbol{\alpha}}_3\right)\Psi_{\mathrm{p}}\,\mathrm{d}^3r_1\,\mathrm{d}^3r_2\,\mathrm{d}^3r_3 \quad .
\end{aligned} \tag{1}$$

Therefore (3.150) can in a first step be simplified to

Exercise 3.13.

$$
\begin{aligned}
\mu_{\mathrm{p}} &= 3\int \mathrm{d}^3r_1\, \mathrm{d}^3r_2\, \mathrm{d}^3r_3\, \Psi_{\mathrm{p}}^{\dagger}\left(\frac{\hat{Q}_1}{2}\hat{r}_1 \times \hat{\alpha}_1\right)\Psi_{\mathrm{p}} \\
&= \frac{1}{6}\int \mathrm{d}^3r_1\left[10\mathrm{u}^{\uparrow}(1)^{\dagger}\left(\frac{\hat{Q}_1}{2}\hat{r}_1 \times \hat{\alpha}_1\right)\mathrm{u}^{\uparrow}(1) + 2\mathrm{u}^{\downarrow}(1)^{\dagger}\left(\frac{\hat{Q}_1}{2}\hat{r}_1 \times \hat{\alpha}_1\right)\mathrm{u}^{\downarrow}(1)\right. \\
&\quad \left. + 4\mathrm{d}^{\downarrow}(1)^{\dagger}\left(\frac{\hat{Q}_1}{2}\hat{r}_1 \times \hat{\alpha}_1\right)\mathrm{d}^{\downarrow}(1) + 2\mathrm{d}^{\uparrow}(1)^{\dagger}\left(\frac{\hat{Q}_1}{2}\hat{r}_1 \times \hat{\alpha}_1\right)\mathrm{d}^{\uparrow}(1)\right] \quad . \qquad (2)
\end{aligned}
$$

Here we have drawn on the orthogonality of the quark wave function

$$
\int \mathrm{d}^3r\; \mathrm{q}^{s\,\dagger}\mathrm{q}'^{s'} = \delta_{\mathrm{qq}'}\,\delta_{\mathrm{ss}'} \quad , \quad \mathrm{q} = \mathrm{u},\ \mathrm{d} \quad , \quad s = \uparrow, \downarrow \quad . \qquad (3)
$$

Next we insert the quark charges and make use of the fact that up and down quarks are described by the same spatial wave function. In addition we omit the index 1:

$$
\begin{aligned}
\mu_{\mathrm{p}} &= \frac{e}{6}\int \mathrm{d}^3r\left[\frac{10}{3}\mathrm{u}^{\uparrow\dagger}(\hat{r} \times \hat{\alpha})\mathrm{u}^{\uparrow} + \frac{2}{3}\mathrm{u}^{\downarrow\dagger}(\hat{r} \times \hat{\alpha})\mathrm{u}^{\downarrow}\right. \\
&\quad \left. - \frac{2}{3}\mathrm{d}^{\downarrow\dagger}(\hat{r} \times \hat{\alpha})\mathrm{d}^{\downarrow} - \frac{1}{3}\mathrm{d}^{\uparrow\dagger}(\hat{r} \times \hat{\alpha})\mathrm{d}^{\uparrow}\right] \\
&= \frac{e}{2}\int \mathrm{d}^3r\mathrm{u}^{\uparrow\dagger}(\hat{r} \times \hat{\alpha})\mathrm{u}^{\uparrow} \quad . \qquad (4)
\end{aligned}
$$

In order to evaluate this expression we insert into it (3.120) in the form

$$
\mathrm{u}^{\uparrow} = N\begin{pmatrix} j_0(Er)\,\chi_{-1}^{\frac{1}{2}} \\ \mathrm{i}\,j_1(Er)\,\hat{\sigma}_r\,\chi_{-1}^{\frac{1}{2}} \end{pmatrix} \qquad (5)
$$

and obtain

$$
\begin{aligned}
\mu_{\mathrm{p}} &= \frac{e}{2}N^2\int_0^R \mathrm{d}r\, r^2\int \mathrm{d}\Omega\left(j_0(Er)\chi_{-1}^{\frac{1}{2}\dagger},\ -\mathrm{i}\,j_1(Er)\chi_{-1}^{\frac{1}{2}\dagger}\hat{\sigma}_r\right) \\
&\quad \times \begin{pmatrix} 0 & \hat{r} \times \hat{\sigma} \\ \hat{r} \times \hat{\sigma} & 0 \end{pmatrix}\begin{pmatrix} j_0(Er)\,\chi_{-1}^{\frac{1}{2}} \\ \mathrm{i}\,j_1(Er)\,\hat{\sigma}_r\,\chi_{-1}^{\frac{1}{2}} \end{pmatrix} \\
&= \frac{e}{2}N^2\int_0^R \mathrm{d}r\, r^2\int \mathrm{d}\Omega\;\mathrm{i}\,j_0(Er)\,j_1(Er)\chi_{-1}^{\frac{1}{2}\dagger}\left[(\hat{r} \times \hat{\sigma})\,\hat{\sigma}_r - \hat{\sigma}_r\,(\hat{r} \times \hat{\sigma})\right]\chi_{-1}^{\frac{1}{2}} \qquad (6)
\end{aligned}
$$

By means of the commutation relations of the σ-matrices the term inside the brackets simplifies to

$$
\begin{aligned}
\left[\varepsilon_{ijk}r_j\hat{\sigma}_k,\ \hat{\sigma}_{\ell}r_{\ell}\right] &= \varepsilon_{ijk}\left[\hat{\sigma}_k,\ \hat{\sigma}_{\ell}\right]r_jr_{\ell} \\
&= \varepsilon_{ijk}2\mathrm{i}\varepsilon_{k\ell m}\hat{\sigma}_m r_jr_{\ell} \\
&= 2\mathrm{i}\left(\delta_{\ell i}\delta_{mj} - \delta_{\ell j}\delta_{im}\right)r_jr_{\ell}\hat{\sigma}_m \\
&= 2\mathrm{i}r_i\hat{\sigma} \cdot r - 2\mathrm{i}r^2\hat{\sigma}_i \quad . \qquad (7)
\end{aligned}
$$

Hence

Exercise 3.13.

$$\mu_{\mathrm{p}} = \frac{e}{2} N^2 \int\limits_0^R \mathrm{d}r\, r^2 \int \mathrm{d}\Omega\, \mathrm{i} j_0(Er) j_1(Er) \frac{1}{4\pi} (1 \quad 0)\left[2\mathrm{i}r\sigma_r - 2\mathrm{i}r\sigma\right] \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \frac{eN^2}{4\pi} \int\limits_0^R \mathrm{d}r\, r^3 j_0(Er) j_1(Er) \int \mathrm{d}\Omega \left[e_3 - \cos\theta \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix} \right] \quad , \tag{8}$$

where we have used

$$(\sigma_j)_{11} = \delta_{j3} \tag{9}$$

and

$$\left(\sigma \cdot \frac{r}{r}\right)_{11} = \frac{r_3}{r} = \cos\theta \quad . \tag{10}$$

Now we can easily perform the ϕ, θ and r integrations:

$$\mu_{\mathrm{p}} = \frac{eN^2}{2} e_3 \int\limits_0^R \mathrm{d}r\, r^3 j_0(Er) j_1(Er) \int\limits_{-1}^{1} \mathrm{d}\cos\theta \left(1 - \cos^2\theta\right)$$

$$= e_3 \frac{2}{3} N^2 e \int\limits_0^R \mathrm{d}r\, r^3 \frac{\sin(Er)}{Er} \left(\frac{\sin(Er)}{E^2 r^2} - \frac{\cos(Er)}{Er} \right)$$

$$= e_3 \frac{2}{3} \frac{N^2}{E^3} e \left(-\frac{r}{2} \sin^2(Er) - \frac{3}{4E} \sin(Er)\cos(Er) + \frac{3}{4} r \right)_0^R$$

$$= e_3 \frac{2}{3} \frac{N^2}{E^4} e \left(-\frac{1}{2} \omega \sin^2\omega - \frac{3}{4} \sin\omega\cos\omega + \frac{3}{4}\omega \right) \quad . \tag{11}$$

Here $\omega = ER$. With the boundary condition

$$\omega\cos\omega = (1 - \omega)\sin\omega \tag{12}$$

the expression inside the brackets can further be simplified:

$$-\frac{3}{4}\left(\omega\sin^2\omega + \sin\omega\cos\omega - \omega\right) + \frac{1}{4}\omega\sin^2\omega$$

$$= -\frac{3}{4}\left(\sin^2\omega - \omega\cos\omega\sin\omega + \sin\omega\cos\omega - \omega\right) + \frac{1}{4}\omega\sin^2\omega$$

$$= -\frac{3}{4}\left(\sin^2\omega + \omega\cos^2\omega - \omega\right) + \frac{1}{4}\omega\sin^2\omega$$

$$= -\frac{3}{4}(1 - \omega)\sin^2\omega + \frac{1}{4}\omega\sin^2\omega$$

$$= \frac{1}{4}(4\omega - 3)\sin^2\omega \quad . \tag{13}$$

Finally we insert N^2 from (3.134)

$$\mu_{\mathrm{p}} = e_3 \frac{\omega e}{3R^3(\omega - 1)} \frac{\omega^2}{\sin^2\omega} \frac{1}{E^4} \frac{1}{4} (4\omega - 3)\sin^2\omega$$

$$= e_3 \frac{eR}{\omega(\omega - 1)} \frac{4\omega - 3}{12} = 0.203\, eRe_3 \tag{14}$$

with $\omega = 2.04$.

Exercise 3.13.

Clearly μ_p has to be parallel to the z direction, because in the spherical bag model the quantization axis of the angular momentum is the only direction of special importance.

Since μ_p defines a direction, this special direction can be chosen to be identical with the quantization axis mentioned. In units of the nuclear magneton $\mu_k = e/2m_p$ the absolute value of the magnetic moment of the proton is (with $R = 1$ fm)

$$\begin{aligned} \frac{|\mu_p|}{\mu_K} &= 0.203 \times 1\,\text{fm} \times 2 \times 938.28\,\text{MeV} = 1.9 \quad , \\ \mu_K &= \frac{e}{2m_p} \quad . \end{aligned} \tag{15}$$

The value (3.151) differs considerably from the experimental observation,

$$|\mu_p|\,(\text{exp}) = 2.79\,\mu_K \quad . \tag{3.152}$$

The magnetic moments predicted for the other hadrons are too small as well. The ratios of these moments, however, are quite well described.

Table 3.3 compares the MIT model results for the charge radii and the magnetic moments with the corresponding experimental data.

Table 3.3. Comparison of charge radii and magnetic moments with the corresponding results of the MIT bag model

Particles	$\langle r^2\rangle_{ch}$ (exp.)	$\langle r^2\rangle_{ch}$ (theor.)
p	0.88 ± 0.03 fm	0.73 fm
n	−0.12 ± 0.01 fm	0
π	0.78 ± 0.10 fm	0.5 fm

Particles	$\frac{\mu}{\mu_K}$ (exp.)	$\frac{\mu}{\mu_K}$ (theor.)	$\frac{\mu}{\mu_p}$ (exp.)	$\frac{\mu}{\mu_p}$ (theor.)
p	2.793	1.90	–	–
n	−1.913	−1.27	−0.68	2/3
Λ	0.613 ± 0.004	−0.48	−0.219 ± 0.001	−0.26
Σ^+	2.38 ± 0.02	1.84	0.85 ± 0.01	0.97
Σ^0	–	0.59	–	0.31
Σ^-	−1.14 ± 0.05	−0.68	−0.36 ± 0.02	−0.36
Ξ^0	−1.25 ± 0.014	−1.06	−0.448 ± 0.005	−0.56
Ξ^-	−0.69 ± 0.04	−0.44	−0.25 ± 0.01	−0.23

There are other interesting quantities that can be evaluated in the context of the MIT bag model. One can usually achieve an agreement similar to that for the masses, charge radii, and magnetic moments. Later we shall discuss at a much higher technical level the predictions made by the bag model for the proton structure functions. Let us now summarize the advantages and disadvantages of the MIT bag model.

Advantages: The MIT bag model provides a phenomenological description of hadrons that is quantitatively correct and therefore allows for rough estimates for all quantities not yet experimentally observed. We can furthermore estimate what phenomenological consequences follow from additional quark interactions. For example, the bag model allows us to connect the known weak interaction of hadrons to the weak interaction of quarks.

Disadvantages: The bag model is conceptually unsatisfying. The physical meaning of the specific correction terms is unclear and the model is neither renormalizable nor Lorentz invariant. In addition we obtain similarly good results with completely different approaches (see Sect. 3.1). The basic physical assumptions of these models, i.e., for the MIT bag the rigid boundary condition and the free motion of the quarks inside the bag, differ drastically. Therefore at least some of them should be wrong.

4. Gauge Theories and Quantum-Chromodynamics

Nowadays the common model of quark–quark interactions is an SU(3) gauge theory in a degree of freedom, arbitrarily called "color". There are by now many experimental facts supporting this model, as we shall discuss later in more detail. It seems by now also to be proven that QCD (quantum chromodynamics) is able to correctly describe the most pronounced feature of quark–quark interactions, i.e., confinement, and that it will generate, for example the correct hadron masses. The results obtained so far are all compatible with the phenomenological properties, but in many cases the accuracy of theoretical calculations is still rather low. Typical uncertainties for hadron masses, for example, are 10 percent. In fact, the general acceptance of QCD is based not only on its own achievements, but also to a large extent on the outstanding success of the gauge theory of weak and electromagnetic interactions. For this reason we begin with a brief overview of the Glashow–Salam–Weinberg model. (The Glashow–Salam–Weinberg model combined with QCD is usually called the standard model). The typical features of gauge theories are discussed in the context of this specific model.

4.1 The Standard Model – A Typical Gauge Theory

The ideas of local internal symmetries and gauge transformations were introduced quite a long time ago. But the actual reason for developing the modern schemes for general non-Abelian gauge theories was the lack of a renormalizable field theory for massive spin-1 particles.

Spin-0 and spin-$\frac{1}{2}$ particles are described by the well known method of field quantization employing the Klein–Gordon and the Dirac Lagrangians, respectively. This procedure also applies to massless spin-1 particles, despite some complications owing to the four-potentials A_μ not being physical observables. The theory obtained in this way proved to be excellent, in particular the part known as quantum electrodynamics. However, a consistent field theory for massive vector particles, especially a field theory of weak interactions could not be formulated. This problem even led some physicists to doubt the whole concept of local field theories. On the other hand, having overcome that difficulty, quantum field theory is today considered the correct theory of elementary particles (the validity of a possible alternative, the so-called "string-models", is still heavily disputed).

We shall not analyze the properties of gauge theories in this volume. A deeper understanding requires a lot of formal knowledge, which we shall present in another book. At this point we restrict ourselves to describing the general structure of gauge theories and we simply state that all theories built on these principles are renormalizable.

The standard model contains the well-known electromagnetic interaction. Here the photon is described by the free Lagrangian

$$L_0 = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad , \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{4.1}$$

and its coupling to, for example, the electron via the interaction term

$$L_{\text{int}} = \overline{\Psi}(p_\mu - eA_\mu)\gamma^\mu \Psi \quad . \tag{4.2}$$

These equations already include the most basic assumption of quantum field theory: elementary particles have to be described by local fields and point interactions. As a matter of fact, only the four-potentials A_μ and not the "physical" fields $\boldsymbol{E}$ and $\boldsymbol{B}$ can fulfill these requirements. If $\boldsymbol{E}$ and $\boldsymbol{B}$ are regarded as elementary, then the electromagnetic interaction has to be nonlocal. The best proof of this is the Aharonov–Bohm effect. The phase of a charged particle's wave function is influenced by magnetic fields located in an area where the wave function itself is zero.

As is well known, Lagrangians (4.1) and (4.2) are invariant under the transformation

$$\begin{aligned} A_\nu(x) &\to A_\nu(x) - \partial_\nu \theta(x) \quad , \\ \psi(x) &\to \mathrm{e}^{\mathrm{i}e\theta(x)} \Psi(x) \quad . \end{aligned} \tag{4.3}$$

Equation (4.3) expresses the fact that not all of the four fields A_μ correspond to physical degrees of freedom. Physical quantities must not depend on the arbitrary "gauge angle" $\theta(x)$. This requirement leads to relations between the propagator and the vertex function, for example, which is important to prove the renormalizability of QED (the so called Ward identity).

From the gauge symmetry of QED we are led directly to that of a general gauge group by replacing the complex-valued functions A_ν by matrix functions. If $\hat{\lambda}^j, j = 1, 2, \ldots, N$ is a basis in the chosen matrix space, we correspondingly have

$$\begin{aligned} A_\nu(x) &\to \sum_j \hat{\lambda}^j A_\nu^j(x) =: \overline{A}_\nu(x) \quad , \\ \theta(x) &\to \sum_j \hat{\lambda}^j \theta^j(x) =: \overline{\theta}(x) \quad . \end{aligned} \tag{4.4}$$

The specific choice of matrices determines the underlying symmetry group. If one chooses, for example, traceless Hermitian 3 × 3 matrices, then the transformations

$$\exp\left(\mathrm{i}e \sum_j \hat{\lambda}^j \theta^j(x) \right) \tag{4.5}$$

are nothing other than a three-dimensional representation of the SU(3) transformations. On the other hand it is clear that any other representation of SU(3) instead of the chosen matrices would also have been possible. Since the $\hat{\lambda}$ matrices in general do not commute, the equations have to be slightly changed in the case of (4.4) (in addition in (4.1 – 4.3) we replace e by g):

$$L_0 = \frac{1}{4}\,\mathrm{tr}\left\{\overline{F}_{\mu\nu}\overline{F}^{\mu\nu}\right\} \quad , \quad \overline{F}_{\mu\nu} = \partial_\mu \overline{A}_\nu - \partial_\nu \overline{A}_\mu + \mathrm{i}g\left[\overline{A}_\mu, \overline{A}_\nu\right] \tag{4.6}$$

$$L_{\mathrm{int}} = \overline{\Psi}\left(p_\mu - g\overline{A}_\mu\right)\gamma^\mu \Psi \quad , \tag{4.7}$$

$$\begin{aligned} \overline{A}_\nu(x) &\rightarrow \mathrm{e}^{\mathrm{i}g\overline{\theta}(x)}\left(\overline{A}_\nu(x) - \frac{\mathrm{i}}{g}\partial_\nu\right)\mathrm{e}^{-\mathrm{i}g\overline{\theta}(x)} \quad , \\ \bar{\Psi}(x) &\rightarrow \mathrm{e}^{\mathrm{i}g\overline{\theta}(x)}\Psi(x) \quad . \end{aligned} \tag{4.8}$$

Now $\Psi(x)$ must be defined as a vector corresponding to the choice of $\bar{A}_\nu(x)$. (In principle Ψ could also be a matrix; then the trace of L_{int} would have to be evaluated. Such cases occur in models trying to unify the electroweak and strong interaction with a single symmetry group.) The main content of gauge field theories is contained in the quantized equations (4.6) and (4.7).

With these few remarks we have already formulated the basic ideas of gauge theories. But, like many others, this particular physical concept can be further investigated in two different directions. The first is to study the consequences of (4.6) and (4.7) in more and more detail. The second deals with the basics of these equations and tries to discover deeper-lying foundations. Persuing the latters we could e.g. give a general geometric formulation of the principles of gauge theories. Being geometric this formulation shows the same structure as the basic equations in the theory of general relativity. Such a correspondence gives rise to the hope that the unification of the theory of gravitation and quantum theory, which deals with all other kinds of interaction, is no longer completely unlikely. However, it is still not clear whether this correspondence expresses a real similarity of both theories or whether it just follows from the generality of geometric considerations. Some elements of the geometric formulation of gauge theories are discussed in Example 4.1. Here we will discuss only the phenomenological cosequences of (4.6) and (4.7).

EXAMPLE

4.1 The Geometric Formulation of Gauge Symmetries

For the sake of simplicity we consider a specific example, namely a set of spinor fields $\Psi_i(x), i = 1, 2, \ldots, N$, combined as a vector

$$\Psi = \begin{pmatrix} \Psi_1(x) \\ \Psi_2(x) \\ \vdots \\ \Psi_N(x) \end{pmatrix} \quad . \tag{1}$$

Furthermore we assume a local symmetry such that

$$\Psi'(x) = \exp\left(-\mathrm{i}g\bar{\theta}(x)\right)\Psi(x) \tag{2}$$

Example 4.1.

is equivalent to $\Psi(x)$. This situation is analogous to an arbitrary, e.g., contravariant, vector and its Lorentz transform:

$$V^\mu(x) \quad , \quad V'^\mu(x) = \Lambda^\mu{}_\nu(x) V^\nu(x) \quad . \tag{3}$$

The Lorentz transformation $\Lambda(x)$ corresponds in this case to the gauge transformation $\exp\left(-\mathrm{i}g\bar{\theta}(x)\right)$. A well-known technical problem in the theory of relativity is the definition of a vector (in general a tensor) derivative. The evaluation of a quotient of differences

$$\frac{V^\mu(x+h) - V^\mu(x)}{h} \tag{4}$$

yields additional terms owing to the dependence of the metric tensor $g_{\mu\nu}(x)$ on the position vector x.

This problem leads to the definition of the covariant derivative

$$V^\mu_\nu := \partial_\nu V^\mu + \Gamma^\mu{}_{\alpha\nu} V^\alpha \tag{5}$$

with so-called Christoffel symbols $\Gamma^\mu{}_{\alpha\nu}$,

$$\Gamma^\mu{}_{\alpha\nu} = -\frac{1}{2} g^{\mu\sigma}(\partial_\nu g_{\alpha\sigma} + \partial_\alpha g_{\sigma\nu} - \partial_\sigma g_{\nu\alpha}) \quad , \tag{6}$$

representing the position dependence of the metric, that is, the local coordinate systems. The covariant derivative (5) constructed in this way is invariant under Lorentz transformations.

In a completely analogous manner we can write down (5) for the spinor fields Ψ:

$$\partial_\mu \Psi(x) \to \left(\partial_\mu + \hat{\Gamma}_\mu(x)\right)\Psi(x) =: D_\mu \Psi(x) \quad , \tag{7}$$

where $\hat{\Gamma}_\mu(x)$ is a position-dependent matrix with respect to the vectors Ψ. $\hat{\Gamma}_\mu(x)$ is determined by the requirement that the covariant derivative $D_\mu\Psi(x)$ has to be invariant under gauge transformations $\exp\left(-\mathrm{i}g\bar{\theta}\right)$ that correspond to the Lorentz transformations. Writing $-\mathrm{i}g\bar{A}_\mu$ instead of $\hat{\Gamma}_\mu$ we have

$$\mathrm{i}D_\mu\Psi(x) = \left(\hat{p}_\mu - g\bar{A}_\mu\right)\Psi(x) \quad , \tag{8}$$

$$\Psi(x) \to \mathrm{e}^{-\mathrm{i}g\bar{\theta}(x)}\Psi(x) \quad , \tag{9}$$

$$\bar{A}_\mu \to \mathrm{e}^{\mathrm{i}g\bar{\theta}(x)}\left(\bar{A}_\mu(x) + \frac{\mathrm{i}}{g}\partial_\mu\right)\mathrm{e}^{-\mathrm{i}g\bar{\theta}(x)} = \bar{A}'_\mu \quad . \tag{10}$$

Interpreted geometrically (5) is an effect of parallel transport. To evaluate (4), the vector V^μ has to be transported from the position $x+h$ to the point x. But in a position dependent coordinate system this transport changes the vector's coordinate reprentation (think, for example, of the coordinates r, θ, and φ of a fixed vector which moves on the surface of a sphere). $\Gamma^\mu_{\alpha\nu}V^\alpha$ describes this change, which has to be subtracted from $\partial_\nu V^\mu$ in order to reveal the physical, coordinate-independent change of the vector V^μ. Correspondingly $\bar{A}'_\mu - \bar{A}_\mu$ represents the change of Ψ due to the position dependence of the gauge (see Fig. 4.1). Since the covariant derivatives are gauge independent and invariant under Lorentz transformations, this is also valid for their commutator:

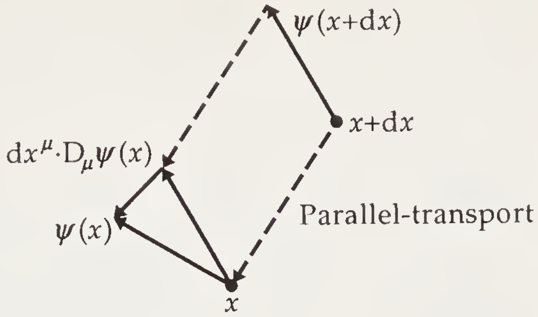


Fig. 4.1. The interpretation of $\overline{A}$ as a description of parallel transport

$$\begin{aligned}[D_\mu, D_\nu] &= \left[(\partial_\mu + \mathrm{i}g\overline{A}_\mu), (\partial_\nu + \mathrm{i}g\overline{A}_\nu)\right] \\ &= \mathrm{i}g\partial_\mu\overline{A}_\nu - \mathrm{i}g\partial_\nu\overline{A}_\mu - g^2\left[\overline{A}_\mu, \overline{A}_\nu\right] \\ &= \mathrm{i}g\Big(\partial_\mu\overline{A}_\nu - \partial_\nu\overline{A}_\mu + \mathrm{i}g\left[\overline{A}_\mu, \overline{A}_\nu\right]\Big) \\ &= \mathrm{i}g\overline{F}_{\mu\nu} \quad . \end{aligned} \tag{11}$$

The $F_{\mu\nu}$ represent the physical part of the gauge fields, i.e., the part that cannot be changed by simply choosing another gauge. Correspondingly in the theory of relativity we obtain the curvature tensor $R^\mu{}_{\nu\sigma\rho}$, describing the physical part of space curvature, i.e., the part that is not due only to the chosen coordinate system. Also $R^\mu{}_{\nu\sigma\rho}$ is simply the commutator of the covariant derivatives (5), with additional terms occurring for the covariant indices. For example, for a tensor it holds that

$$T^\beta{}_{\alpha;\tau} = T^\beta{}_{\alpha,\tau} + \Gamma^\beta_{\mu\tau} T^\mu{}_\alpha - \Gamma^\mu_{\alpha\tau} T^\beta{}_\mu \quad . \tag{12}$$

Because of the outlined similarity one could also focus on gauge symmetries and interpret the general theory of relativity as a special gauge theory. This kind of consideration would shift the whole discussion to the formal definition of a gauge group. Terms like parallel transport or curvature would not play any role. Covariant derivatives and field tensors $\overline{F}_{\mu\nu}$ would be defined by their invariance properties. Only at the very end of such a treatment would one try to connect the obtained structures with physics. Consequently one is free to choose between understanding general relativity as a gauge theory and interpreting gauge theories geometrically.

To formulate the standard model we need an additional concept, namely the idea of spontaneous symmetry breaking. Again the basic idea can be explained quite easily. If scalar fields are considered, interaction terms of the form ϕ^4 and ϕ^2 can be renormalized. Therefore the Lagrangian

$$L = -\Phi^+ \Box \Phi - \frac{\lambda}{4}\left(\Phi^+\Phi - v^2\right)^2 \tag{4.9}$$

sets up a well-defined field theory. (Such scalar quantum field theories are in fact somewhat problematic. It is possible that consistent renormalization requires the coupling constant to be zero. This possibility is called "triviality" and implies that (4.9) makes sense only if the scalar Higgs particle, for example, has some internal structure on the 1–10 TeV scale. For its phenomenological properties at the energies we consider this is irrelevant.) Within the meaning of the Lagrange formalism the second term has to be interpreted as a contribution to the potential energy:

$$V = \frac{\lambda}{4}\left(\Phi^{+}\Phi - v^{2}\right)^{2} \quad . \tag{4.10}$$

Clearly the energy is minimal for $|\phi| = v = \text{const}$. Hence this is the physical ground state. On the other hand a problem occurs if the theory is quantized in the standard way by means of commutator relations between field operators. Since $\langle 0|\hat{\phi}|0\rangle \equiv 0$, the state with $v = |\phi| = \langle \text{vac}|\hat{\phi}|\text{vac}\rangle$ cannot in principle be constructed with one-particle excitations on the usual ground state. Therefore the physical vacuum state $|\text{vac}\rangle$ is not identical with the field theoretical vacuum state, $|\text{vac}\rangle \neq |0\rangle$. The physical excitations are only obtained if fluctuations around $\phi = v$ are considered and if $\hat{\phi}(x)$ is quantized in the usual manner:

$$\Phi(x) \to v + \hat{\Phi}(x) \quad , \quad \langle \text{vac}|\hat{\Phi}|\text{vac}\rangle \quad = \quad 0 \quad . \tag{4.11}$$

There is no deeper physical reason for the fact that the excitations around $\phi = 0$ and $\phi = v$ cannot be expressed by each other. It only means that the canonical quantization scheme is not complete. In fact it is possible to choose another quantization method without this disadvantage, e.g. quantization with functional integrals. We cite here without proof the following essential statement:

If a (gauge) theory is renormalizable for $\Phi(x) = v + \lambda(x)$, i.e., for quantization around $\Phi(x) = v$, then it is also renormalizable for quantization around $\Phi(x) = 0$ and vice versa.

This makes the following trick possible. All particle masses M_i are replaced by

$$M_i \to \frac{\Phi(x)}{v} M_i \quad . \tag{4.12}$$

For physical states one has to set $\phi(x) = v + \chi(x)$. This yields just M_i plus an additional interaction term, whose meaning will be explained later. So long as this term does not cause significant physical effects, the original theory has practically not been changed. In this case the theory is then exactly renormalizable if it is also renormalizable for a quantization around $\phi(x) = 0$, i.e., for vanishing mass terms. As it is possible to prove that certain classes of massless gauge theories are renormalizable, this is also true for the resulting theories with masses generated by (4.12), i.e., by spontaneous symmetry breaking. The crucial point is that spontaneous symmetry breaking preserves gauge invariance, which is essential for the proof of renormalizability.

In general $\phi(x)$ is not a scalar with respect to the current gauge symmetry (i.e. not a singlet) but rather a multiplet. The vacuum expectation value v then defines a particular direction within this multiplet. Because of this one speaks of *spontaneous symmetry breaking*. This expression is quite misleading since the symmetry is really preserved. The choice of v should instead be compared with the choice of a specific direction of quantization in a spherically symmetric problem. Now we want to discuss spontaneous symmetry breaking for the special case of the standard model.

The standard model describes the electromagnetic and weak interactions. The latter was first understood as a four-fermion interaction between, for example, a neutron, proton, electron, and antineutrino. But it has been clear for a long time that this can only be an effective description for a microscopic process, e.g., the exchange of charged so-called W bosons (see Fig. 4.2). Since the W fields are vectors and mediate an interaction (just like the photon), it is natural to regard them as gauge fields of a gauge group. To choose possible candidates for this gauge group does not cause any difficulties, if the following points are considered:

1. The leptons exclusively couple to the corresponding neutrinos, for example the electron only to the electron neutrino. Therefore we have to choose a group with an irreducible two-dimensional representation if we want to avoid additional artificial assumptions.
2. Except for W^+ and W^- only a minimum number of additional gauge fields should be postulated. Hence the number of generators should be slightly bigger than two.

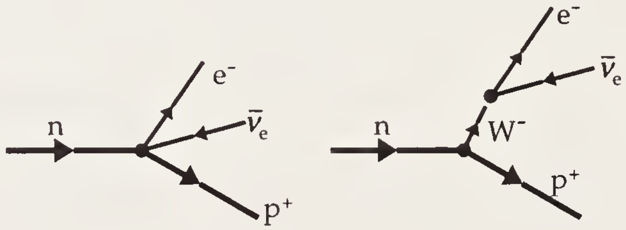


Fig. 4.2 The Fermi interaction and its interpretation as the exchange of charged W bosons

These requirements lead directly to the group SU(2). But the SU(2) contains three generators. One could therefore try to identify the additional third gauge field with the photon. However, it is not possible to carry out this idea consistently. The construction principles, which have been outlined so far and which will be applied in the following to our example, demand a new neutral particle, the Z^0, to be postulated. This involves the existence of a new interaction: so-called neutral currents. The experimental verification of this interaction and of the W and Z particles is one of the greatest triumphs of particle physics.

Hence SU(2) × U(1) is the gauge group of the GSW model. The most frequently used representation of the matrices λ_i is

$$\lambda_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \lambda_4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{4.13}$$

i.e., nothing more than the Pauli matrices for SU(2) and the unit matrix for U(1) (remember that the generators act on the doublet states). If we denote the corresponding gauge fields by W_1^μ, W_2^μ, W_3^μ, and B^μ, then $\bar{F}_{\mu\nu}$ from (4.6) is $(i, j, k = 1, 2, 3)$

$$\overline{F}_{\mu\nu} = \partial_\mu(\lambda^j W^j_\nu + \mathbb{1} \cdot B_\nu) - \partial_\nu(\lambda^j W^j_\mu + \mathbb{1} \cdot B_\mu) - 2g\varepsilon_{ijk}\lambda^k W^i_\mu W^j_\nu \quad , \tag{4.14}$$

since $\mathbb{1}$ commutes with all matrices. Because

$$\mathrm{tr}\{\lambda_i \lambda_j\} = 2\delta_{ij} \quad , \quad \mathrm{tr}\{\lambda_i\} = 0 \quad , \tag{4.15}$$

L_0 is divided into two contributions, where we replace W^i_μ and B_μ by $\frac{1}{2}W^j_\mu$ and $\frac{1}{2}B_\mu$, respectively, in order to obtain the usual factor (usually $t_i = \frac{1}{2}\lambda_i$ rather than λ_i are chosen for the matrices):

$$L_0 = -\frac{1}{4}W^j_{\mu\nu}W^{j\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} \quad , \tag{4.16}$$

$$W^j_{\mu\nu} = \partial_\mu W^j_\nu - \partial_\nu W^j_\mu - g\varepsilon_{ijk}W^i_\mu W^k_\nu \quad , \tag{4.17}$$

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \quad . \tag{4.18}$$

Also (4.7) can be written down directly for the gauge group SU(2) × U(1). The only real novelty is the appearance of two coupling constants g and g' for the two gauge groups SU(2) and U(1):

$$L_{\text{int}} = \bar{\Psi}\left(p_\mu - g\,W^j_\mu\frac{\lambda^j}{2} - g'B_\mu\right)\gamma^\mu\Psi \quad . \tag{4.19}$$

Clearly the total Lagrangian $L_0 + L_{\text{int}}$ is then invariant under the transformations

$$\lambda^j W^j_\mu \to \exp\left(\mathrm{i}g\theta^j\lambda^j\right)\left(\lambda^j W^j_\mu - \frac{\mathrm{i}}{g}\partial_\mu\right)\exp\left(-\mathrm{i}g\theta^j\lambda^j\right) \quad , \tag{4.20}$$

$$B_\mu \to \exp\left(\mathrm{i}g'\theta'(x)\right)\left(B_\mu - \frac{\mathrm{i}}{g'}\partial_\mu\right)\exp\left(-\mathrm{i}g'\theta'(x)\right) = B_\mu + \partial_\mu\theta'(x) \quad , \tag{4.21}$$

$$\Psi(x) \to \exp\left(\mathrm{i}g\theta^j(x)\lambda^j\right)\exp\left(\mathrm{i}g'\theta'(x)\right)\Psi(x) \quad . \tag{4.22}$$

Since the U(1) transformations commute, (4.21) has a very simple form (just as in electrodynamics). Because of this simple feature one can allow different fermions to transform differently under U(1). If we replace (4.19) and (4.22) by

$$L_{\text{int}} = \sum_j \bar{\Psi}_{\text{f}}\left(p_\mu - gW^j_\mu\frac{\lambda^j}{2} - g'\frac{y_f}{2}B_\mu\right)\gamma^\mu\Psi_{\text{f}} \quad , \tag{4.23}$$

$$\Psi(x) \to \exp\left(\mathrm{i}g\theta^j(x)\lambda^j\right)\exp\left(\mathrm{i}g'\frac{y_f}{2}\theta'(x)\right)\Psi_{\text{f}} \tag{4.24}$$

with arbitrary numbers y_f, then the $SU(2) \times U(1)$ symmetry still remains valid.

As mentioned earlier, the spinor doublet is identified with, for example, the electron and the electron-neutrino field; usually ν_{e} is written as the upper component:

$$\Psi_{\text{f}} = \begin{pmatrix}\nu_{\text{e}}\\ \text{e}\end{pmatrix}_{\text{L}} \quad . \tag{4.25}$$

Here we also took the experimenal observation into account that only left-handed leptons and quarks interact weakly:

$$e_{\text{L}} = \frac{1-\gamma_5}{2}e \quad . \tag{4.26}$$

Equation (4.25) then yields the structure of the remaining fermion doublets:

$$\Psi_{\text{f}} : \begin{pmatrix}\nu_\mu\\ \mu\end{pmatrix}_{\text{L}} , \begin{pmatrix}\nu_\tau\\ \tau\end{pmatrix}_{\text{L}} , \begin{pmatrix}\text{u}\\ \text{d}'\end{pmatrix}_{\text{L}} , \begin{pmatrix}\text{c}\\ \text{s}'\end{pmatrix}_{\text{L}} , \begin{pmatrix}\text{t}\\ \text{b}'\end{pmatrix}_{\text{L}} , \cdots . \tag{4.27}$$

Here the quark fields d′, s′ and b′ are orthogonal superpositions of the mass eigenstates d, s, and b. The fact that the mass eigenstates differ from the eigenstates with respect to the weak interaction is one of the most fascinating features of the weak interaction. It is related to exotic effects such as kaon oscillations and could be responsible for CP violation.[1]

Lastly we still have to define the Higgs sector. For Higgs particles we choose also the lowest SU(2) representation, namely the doublet:

$$\phi = \begin{pmatrix} \phi_1 + \mathrm{i}\phi_2 \\ \phi_3 + \mathrm{i}\phi_4 \end{pmatrix} \quad , \quad \phi_1, \phi_2, \phi_3, \text{ and } \phi_4 \text{ are real.} \tag{4.28}$$

To maintain the gauge invariance of the theory, for the ϕ field also all derivatives have to be replaced by covariant derivatives (see Example 4.1). This yields the Higgs Lagrangian

$$L_\mathrm{H} = \left| \left(\partial_\mu - \mathrm{i}g \, \frac{\lambda^j}{2} \, W^j_\mu - \mathrm{i}g' \, \frac{y_\mathrm{H}}{2} \, B_\mu \right) \phi \right|^2 - \frac{\lambda}{4} \left(\phi^+ \phi - v^2 \right)^2 \quad . \tag{4.29}$$

If now ϕ in L_H is replaced by means of (4.11) by

$$\phi \to \begin{pmatrix} 0 \\ v \end{pmatrix} + \chi \quad , \tag{4.30}$$

for example, then (4.29) generates the gauge-field mass terms

$$\begin{aligned} L_\mathrm{H} \to & \frac{1}{4} \left| \begin{pmatrix} g(W^1_\mu - \mathrm{i}W^2_\mu) \\ g' y_\mathrm{H} B_\mu - g W^3_\mu \end{pmatrix} \right|^2 v^2 \\ = & \; \frac{g^2 v^2}{4} (W_1^2 + W_2^2) + \frac{v^2}{4} \, (g' y_\mathrm{H} B_\mu - g W^3_\mu)^2 \quad . \end{aligned} \tag{4.31}$$

Now we expand B_μ and W^3_μ into two new fields A_μ and Z_μ:

$$\begin{aligned} B_\mu &= \frac{1}{\sqrt{g^2 + g'^2 y_\mathrm{H}^2}} (g' y_\mathrm{H} Z_\mu - g A_\mu) \quad , \\ W^3_\mu &= \frac{1}{\sqrt{g^2 + g'^2 y_\mathrm{H}^2}} (-g Z_\mu - g' y_\mathrm{H} A_\mu) \end{aligned} \tag{4.32}$$

and obtain

$$L_\mathrm{H} \to \frac{g^2 v^2}{4} (W_1^2 + W_2^2) + \frac{v^2}{4} (g^2 + g'^2 y_\mathrm{H}^2) Z^2 \quad . \tag{4.33}$$

The A_μ field remains massless and is consequently identified with the photon. One can also absorb y_H in the definition of g' and with these identifications all other y values follow. y, the so-called "weak hypercharge", has to be chosen in such a way that all particles get their correct electric charges. This leads to the well-known Gell-Mann–Nishijima formula

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[1] See Theoretical Physics, Vol. 5 by W. Greiner and B. Müller: *Gauge Theory of Weak Interactions* (Springer, Berlin, Heidelberg, New York 1993)

$$y = 2(Q - t_3) \quad , \tag{4.34}$$

where Q and t_3 denote the electric charge and the weak isospin of each particle, respectively. Finally it is common practice also to generate the fermion masses by spontaneous symmetry breaking. This can be achieved by replacing all mass terms by corresponding couplings to the Higgs field:

$$m_\mathrm{e}\,\bar{\mathrm{e}}\mathrm{e} \longrightarrow \frac{m_\mathrm{e}}{v}\begin{pmatrix}\nu_\mathrm{e}\\ \mathrm{e}\end{pmatrix}_\mathrm{L}^{+}\gamma_0\,\phi\,\mathrm{e}_\mathrm{R} + \bar{\mathrm{e}}_\mathrm{R}\,\phi^{+}\begin{pmatrix}\nu_\mathrm{e}\\ \mathrm{e}\end{pmatrix}_\mathrm{L} \quad . \tag{4.35}$$

With these remarks we conclude our brief sketch of the standard model, which has been experimentally verified to a remarkable degree. At this point we turn to the question of what an analogous theory for quark–quark interactions has to look like.

4.2 The Gauge Theory of Quark–Quark Interactions

In order to construct a gauge theory of quark–quark interactions, one has first of all to determine the number of "charge states", i.e., the number of different kinds of quarks with respect to the new interaction. Following our discussion in Sect. 1.1 the minimum number of quarks needed is three. Now we want to give arguments indicating that there are exactly three quark states, i.e., three colors.

The results of high-energy $\mathrm{e}^+\mathrm{e}^-$ reactions provide the first argument. Here hadrons are created by pair annihilation followed by the creation of quark pairs. Suppose that the interaction in the final channel does not influence the cross section (this assumption is confirmed by deep inelastic scattering experiments (see Chap. 3) and follows from asymptotic freedom). Then the cross section is given by

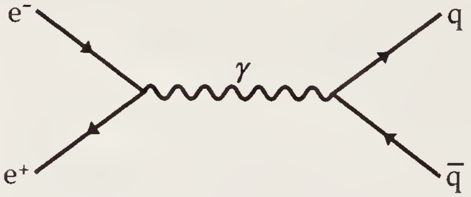


Fig. 4.3. The elementary interaction graph of the reaction $\mathrm{e}^+\mathrm{e}^- \longrightarrow$ hadrons

The graph in Fig. 4.3 looks exactly like the corresponding graph for $\mu^+\mu^-$ creation. Therefore the ratio of the cross sections is simply given by the charges of the particles (provided the particle masses are negligible compared to the energy of the $\mathrm{e}^+\mathrm{e}^-$ pair):

$$R = \frac{\sigma(\mathrm{e}^+\mathrm{e}^- \longrightarrow \bar{q}q)}{\sigma(\mathrm{e}^+\mathrm{c}^- \longrightarrow \mu^+\mu^-)} = \left[\left(\frac{2}{3}\right)^2 + \left(\frac{-1}{3}\right)^2 + \left(\frac{-1}{3}\right)^2\right] N_c = \frac{2}{3}N_c$$

$$\text{for } E_\mathrm{cm} > 2\ \mathrm{GeV} \gg 2m_\mathrm{u},\ 2m_\mathrm{d},\ 2m_\mathrm{s},$$

$$R = \left[\left(\frac{2}{3}\right)^2 + \left(\frac{-1}{3}\right)^2 + \left(\frac{-1}{3}\right)^2 + \left(\frac{2}{3}\right)^2\right] N_c = \frac{10}{9}N_c \tag{4.36}$$

$$\text{for } E_\mathrm{cm} > 3\ \mathrm{GeV} > 2m_\mathrm{c},$$

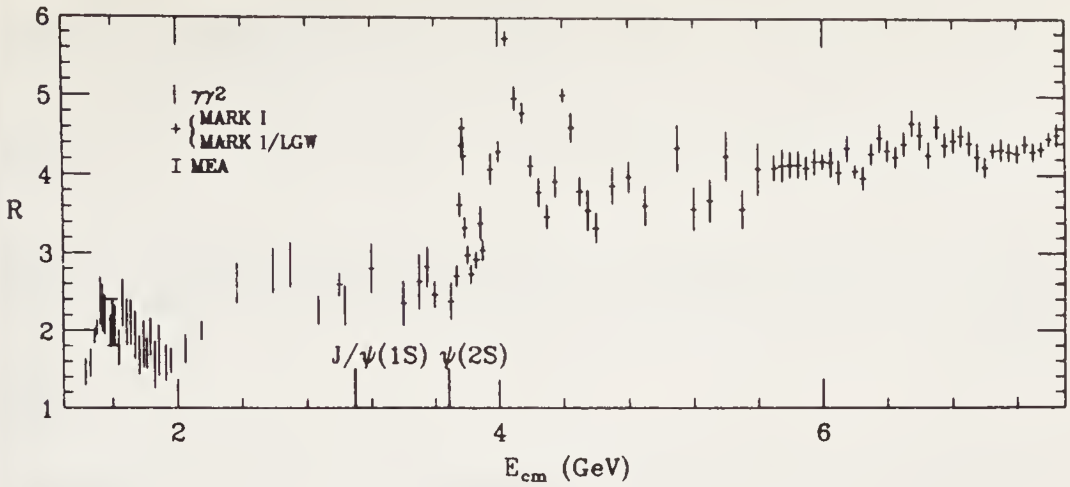


Fig. 4.4. The ratio of the cross sections for hadronic and muonic reactions in e^+e^- pair annihilation, $R = \sigma(e^+e^- \rightarrow \text{hadrons})/\sigma(e^+e^- \rightarrow \mu^+\mu^-)$. (From *Review of Particle Properties:* Phys. Rev. **D45** (1992)). A comparison with theory including radiative corrections is presented in Fig. 7.21

$$R = \left[\left(\frac{2}{3}\right)^2 + \left(\frac{-1}{3}\right)^2 + \left(\frac{-1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{-1}{3}\right)^2\right] N_c = \frac{11}{9} N_c$$

for $E_{\text{cm}} > 10\ \text{GeV} > 2m_{\text{b}}$.

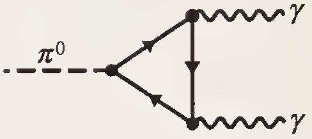


Fig. 4.5. Graph for the $\pi^0 \rightarrow 2\gamma$ decay

If we compare this prediction to the measured values (Fig. 4.4), it becomes clear that the assumptions made above lead to the value $N_c = 3$. The second argument is quite similar. We consider the two-photon decay of the π^0 meson shown in Fig. 4.5. Owing to the relatively small pion mass only the u and d quarks contribute to this graph. The coupling to these quarks is well understood from studies of the pion–baryon interaction and the decay width

$$\Gamma = 7.63\ \text{eV} \left(\frac{N_c}{3}\right)^2 \tag{4.37}$$

is obtained. The experimental value

$$\Gamma_{\text{exp}} = 7.57 \pm 0.32\ \text{eV} \tag{4.38}$$

obviously again favours $N_c = 3$. As some theoretical assumptions enter the derivation of (4.37), this argument is somewhat less direct than the first one.

The last argument we want to discuss is of a purely theoretical nature. From a strictly empirical point of view it is therefore the least well founded. From the theoretical point of view it is, on the contrary, the most fascinating, since it states that the standard model is only internally consistent for three colors. This implies that there must be a fundamental symmetry linking the electroweak sector of the standard model to the QCD sector and thus motivates the search for a Grand Unified Theory.

In the last section we claimed that all gauge theories with or without spontaneous symmetry breaking are renormalizable. However, this statement is not valid in full generality, since there are graphs in the standard model, for example, that are divergent and not subject to the general proof of renormalizability. They can all be traced back to the *triangle anomaly* depicted in Fig. 4.6. If the loop momenta

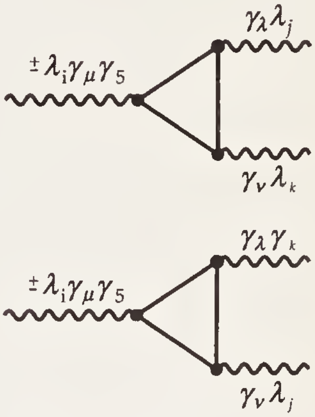

$\gamma_\lambda\lambda_j$
$\pm\lambda_i\gamma_\mu\gamma_5$
$\gamma_\nu\lambda_k$
$\gamma_\lambda\gamma_k$
$\pm\lambda_i\gamma_\mu\gamma_5$
$\gamma_\nu\lambda_j$

Fig. 4.6. Potentially divergent graphs of the standard model. The plus holds for right-handed fermions and the minus sign for left-handed fermions

approach infinity (only this limit is important for renormalizability), the fermion masses can be neglected and the triangle graphs become proportional to

$$\sum_{\mathrm{f}} \pm \left[\langle \mathrm{f} | (\hat{\lambda}_i \hat{\lambda}_j \hat{\lambda}_k) | \mathrm{f} \rangle + \langle \mathrm{f} | (\hat{\lambda}_i \hat{\lambda}_k \hat{\lambda}_j) | \mathrm{f} \rangle \right] \qquad i,j,k = 1,2,3,4 \tag{4.39}$$

with + for left-handed particles and − for right-handed ones. Here the $\hat{\lambda}_j$ denote the SU(2)× U(1) matrices from (4.13), taking into account the hypercharge defined in (4.23):

$$\hat{\lambda}_4 = \hat{y} \cdot \mathbb{1} \quad . \tag{4.40}$$

The contributions of both members of a doublet lead to traces of the $\hat{\lambda}$ matrices:

$$\sum_{\mathrm{fd}} \langle \mathrm{fd} | \mathrm{tr}[\hat{\lambda}_i \{\hat{\lambda}_j, \hat{\lambda}_k\}_+] | \mathrm{fd} \rangle - \sum_{\mathrm{fs}} \langle \mathrm{fs} | 2(\hat{\lambda}_4)^3 | \mathrm{fs} \rangle \delta_{i,4} \delta_{j,4} \delta_{k,4} \quad , \tag{4.41}$$

where fd = fermion doublet, and fs = fermion singlets. In (4.41) we also used the fact that all doublets are left-handed and all singlets are right-handed. For $j, k \neq 4$ it holds that

$$\mathrm{tr}\left(\hat{\lambda}_i \{\hat{\lambda}_j, \hat{\lambda}_k\}_+\right) = 2\, \delta_{jk}\, \mathrm{tr}\left(\hat{\lambda}_i\right) = 4 \delta_{jk} \delta_{i4} \hat{y} \quad , \tag{4.42}$$

for $j = 4$, $k \neq 4$ that

$$\mathrm{tr}\left(\hat{\lambda}_i \{\hat{\lambda}_j, \hat{\lambda}_k\}_+\right) = 2\, \hat{y}\, \mathrm{tr}\left(\hat{\lambda}_i \hat{\lambda}_k\right) = 4 \hat{y} \delta_{ik} \quad , \tag{4.43}$$

and for $j = k = 4$ that

$$\mathrm{tr}\left(\hat{\lambda}_i \{\hat{\lambda}_j, \hat{\lambda}_k\}_+\right) = 2\, \hat{y}^2\, \mathrm{tr}\left(\hat{\lambda}_i\right) = 4 \hat{y}^3 \delta_{i4} \quad . \tag{4.44}$$

Consequently Fig. 4.6 leads to terms that are all proportional either to

$$I_1 = \sum_{\mathrm{fd}} \langle \mathrm{fd} | \hat{y} | \mathrm{fd} \rangle \tag{4.45}$$

or to

$$I_2 = 2 \sum_{\mathrm{fd}} \langle \mathrm{fd} | \hat{y}^3 | \mathrm{fd} \rangle - \sum_{\mathrm{fs}} \langle \mathrm{fs} | \hat{y}^3 | \mathrm{fs} \rangle \tag{4.46}$$

The charges, weak isospins, and weak hypercharges belonging to the quarks and leptons of the first family are listed in the following table.

Let us remember that the quark mass eigenstates d, s, and b are not the eigenstates of weak interaction. This quark mixing is indicated by the subscript C, which stands for “Cabbibo”. With (4.34) it follows for the pair e, ν_{e}, i.e., for the doublet $\nu_{\mathrm{e_L}}$, $\mathrm{e_L}$ and the singlet $\mathrm{e_R}$, that

Table 4.1 The first fermion family of the standard model

	T	T_3	$\hat{y}$	Q
ν_e	$+\frac{1}{2}$	$+\frac{1}{2}$	-1	$+0$
e_L	$+\frac{1}{2}$	$-\frac{1}{2}$	-1	-1
e_R	$+0$	$+0$	-2	-1
u_L	$-\frac{1}{2}$	$+\frac{1}{2}$	$+\frac{1}{3}$	$+\frac{2}{3}$
$(d_C)_L$	$+\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{3}$	$-\frac{1}{3}$
u_R	$+0$	$+0$	$+\frac{4}{3}$	$+\frac{2}{3}$
$(d_C)_R$	$+0$	$+0$	$-\frac{2}{3}$	$-\frac{1}{3}$

$$I_1 = -1 \quad , \quad I_2 = 2 \times (-1)^3 - (-2)^3 = 6 \tag{4.47}$$

and correspondingly for a quark pair, e.g. u_L and d_L, u_R and d_R that

$$I_1 = \frac{1}{3} \quad , \quad I_2 = 2 \times \left(\frac{1}{3}\right)^3 - \left(\frac{4}{3}\right)^3 - \left(-\frac{2}{3}\right)^3 = -2 \quad . \tag{4.48}$$

If there are exactly three times as many quarks as leptons, then I_1 and I_2 vanish, i.e., the large loop momenta do not contribute to the graphs in Fig. 4.6, which then are no longer divergent. Assuming in addition that there are just as many kinds of leptons as quark flavors, we conclude that there must exist three colors.

This argument might have a problem, since the graphs of Fig. 4.6 might be renormalizable in the context of an improved field theory or in a theory going far beyond field theory. We know, however, that they cannot be renormalized on the basis of the present understanding and with any technique developed so far. The role of the anomaly is so fascinating just because it seems to point to something beyond the standard model. It has many most interesting consequences; for example, owing to the anomaly, the standard model should violate separate lepton and baryon number conservation, conserving only B–L, which again shows that it is intimately connected with any unifying model.

Altogether the arguments discussed are sufficiently convincing to demand that every theory of quark–quark interactions has to start with three color states. In analogy to the standard model it is quite natural to regard them as the fundamental representation of a SU(3) color group. It is actually relatively easy to see that SU(3) is the only possible compact simple Lie group. These Lie groups are completely classified and one can look up their irreducible representations in, for example, R. Slansky, Phys. Rep. **79**, p. 1–128. SU(3) is the only group which has complex irreducible triplets. The only alternative to standard QCD are therefore groups that are spontaneously broken to a SU(3) symmetry group plus some high mass residues of the original, larger group. Consequently the basic equations are just (4.6) and (4.7), where we have changed the sign of g and $\overline{\theta}(x)$ in order to agree with common conventions:

$$\begin{aligned}\bar{A}_\nu(x) &= \sum_{a=1}^{8} \frac{1}{2}\hat{\lambda}_a A_\nu^a(x) \qquad (\hat{\lambda}_a = \mathrm{SU(3)\ matrices}) \quad , \\ \bar{\theta}(x) &= \sum_{a=1}^{8} \frac{1}{2}\hat{\lambda}_a \theta^a(x) \quad ,\end{aligned} \tag{4.49}$$

and

$$\begin{aligned}\bar{A}_\nu(x) &\to \mathrm{e}^{\mathrm{i}g\bar{\theta}(x)}\left(\bar{A}_\nu(x) + \frac{\mathrm{i}}{g}\partial_\nu\right)\mathrm{e}^{-\mathrm{i}g\bar{\theta}(x)} \quad , \\ \Psi(x) &\to \mathrm{e}^{\mathrm{i}g\bar{\theta}(x)}\Psi(x) \quad .\end{aligned} \tag{4.50}$$

The quark wave function is then a color-SU(3) triplet

$$\Psi(x) = \begin{pmatrix} \Psi_r(x) \\ \Psi_b(x) \\ \Psi_g(x) \end{pmatrix} \tag{4.51}$$

and the Lagrangian is

$$L = -\frac{1}{2}\,\mathrm{tr}(\bar{F}_{\mu\nu}\bar{F}^{\mu\nu}) + \bar{\Psi}(p_\mu + g\bar{A}_\mu)\gamma^\mu\Psi \tag{4.52}$$

with the tensor of the gluon field strength

$$\bar{F}_{\mu\nu} = \partial_\mu\bar{A}_\nu - \partial_\nu\bar{A}_\mu - \mathrm{i}g\left[\bar{A}_\mu, \bar{A}_\nu\right] \quad . \tag{4.53}$$

Equations (4.49) to (4.53) completely define the theory. They lead to the Feynman rules and allow the evaluation of arbitrary graphs. This procedure has been very successful for all field theories discussed so far, but in the case of QCD it has to be applied with care. Here the essential question is under what circumstances does such a perturbative expansion in the coupling constant converge. One finds that the smallness of the coupling constant is an insufficient criterion. It is still not completely understood, for example, whether the perturbative treatment of QED really converges or whether this is only an asymptotic expansion. In the latter case the evaluation of additional classes of graphs would yield improvements only to a certain order in α. The consideration of higher-order terms then would again worsen the agreement with experimental observations. Unfortunately the QCD coupling constant has to be chosen in such a way that the convergence of the resulting perturbation series is not clear. To be more precise the convergence depends on the value of the momenta that occur. In the case of large momenta, for example, in deep inelastic electron–nucleon scattering reactions, we shall show that perturbative calculations agree very well with experimental results. But for small momenta no convergence can be observed. This fact, however, fits quite well into the general picture, since quark confinement, for instance, is supposed to be a nonperturbative effect. To what extent is this behavior characteristic for the SU(3) theory? In fact only the way in which g decreases with increasing momentum is typical for the SU(3) theory. An SU(4) or SU(2) theory would lead to a stronger or weaker dependence, respectively. For a U(1) theory, e.g., QED, even the opposite behavior occurs. Here perturbation theory converges very well at small momenta and breaks down at unphysically high scales (of the order of 10^{20} GeV).

General features are thus not necessarily characteristic for the chosen theory. Only a thorough and detailed analysis of many processes with different momentum transfers can prove the validity of the QCD equations (4.49)–(4.53). The completion of such an analysis is still out of sight, but the evidence in favor of an SU(3) colour gauge theory is meanwhile so strong that now hardly anybody seriously doubts its validity.

Before we turn in the next chapter to perturbative QCD for large momentum transfers, Example 4.2 introduces the Feynman rules for QCD. Finally, Example 4.3 gives more details on the dependence of the coupling constant on momentum transfer.

EXAMPLE

4.2 The Feynman Rules for QCD

First we discuss the vertices, starting with the *quark–gluon vertex*, which can be deduced from the following part of the Lagrangian:

$$\mathcal{L}_{\text{int}} = \bar{\Psi} g \frac{\lambda^a}{2} \gamma_\mu A^{a\mu} \Psi \quad . \tag{1}$$

Throughout this problem latin letters $a, b, c, \dots$ denote the SU(3) index. One has to sum over indices occurring twice ($a = 1, \dots, 8$).

In general Feynman rules are obtained by varying the corresponding action integral in momentum space. Here

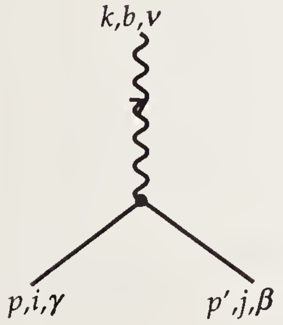
$$\begin{aligned} &\frac{\delta^3}{\delta \Psi^i_\gamma(p) \delta \bar{\Psi}^j_\beta(p') \delta A^{b\nu}(k)} \\ &\quad \times \left[\int \bar{\Psi}(p_1) g \frac{\lambda^a}{2} \gamma_\mu A^{a\mu}(p_2) \Psi(p_3) (2\pi)^4 \delta^4(p_1 + p_2 - p_3)\, \mathrm{d}^4 p_1\, \mathrm{d}^4 p_2\, \mathrm{d}^4 p_3 \right] \end{aligned} \tag{2}$$

has to be evaluted. Using

$$\frac{\delta \Psi(p_1)^i_\gamma}{\delta \Psi(p')^j_\beta} = \delta^4(p_1 - p') \delta_{ij} \delta_{\gamma\beta} \quad \text{etc.} \tag{3}$$

we can simplify the quark–gluon vertex to

$$g \left(\frac{\lambda^b}{2} \right)_{ji} (\gamma_\nu)_{\beta\gamma} (2\pi)^4\, \delta^4(p' + k - p) \quad . \tag{4}$$



Example 4.2.

Since the gauge field is non-Abelian, the gluon Lagrangian also contains second and third powers of the field $\bar{A}_\mu$, which lead to gluon–gluon interactions.

The *three–gluon vertex* is evaluted in an analogous way. From the Lagrangian we obtain with

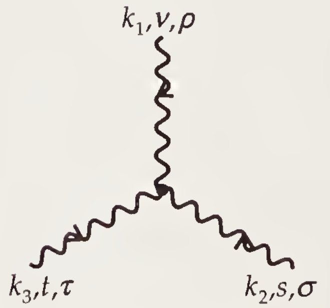
$$\mathrm{tr}\left\{\lambda^a\left[\lambda^b,\lambda^c\right]\right\} = 2\mathrm{i}f_{bcd}\ \mathrm{tr}\left\{\lambda^a\lambda^d\right\} = 4\mathrm{i}f_{abc} \tag{5}$$

the following three-gluon terms:

$$-\frac{1}{2}\mathrm{tr}\left\{\bar{F}_{\mu\nu}F^{\mu\nu}\right\} \to -\frac{g}{2}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right) f_{abc}A^{b\mu}A^{c\nu}. \tag{6}$$

Again this is transformed into momentum space and varied with respect to the fields:

$$\begin{aligned}
&\frac{\delta^3}{\delta A^r_\rho(k_1)\delta A^s_\sigma(k_2)\delta A^t_\tau(k_3)}\\
&\quad\times\left\{-\int\frac{g}{2}\left[\mathrm{i}p_{1,\mu}A^a_\nu(p_1) - \mathrm{i}p_{1,\nu}A^a_\mu(p_1)\right] f_{abc}A^{b\mu}(p_2)\right.\\
&\quad\left.\times A^{c\nu}(p_3)(2\pi)^4\delta^4(p_1+p_2+p_3)\,\mathrm{d}^4p_1\,\mathrm{d}^4p_2\,\mathrm{d}^4p_3\right\}\\
&=\frac{\delta^2}{\delta A^r_\rho(k_1)\delta A^s_\sigma(k_2)}\left\{-\mathrm{i}\frac{g}{2}\int\left[k_{3,\mu}f_{tbc}A^{b\mu}(p_2)A^{c\tau}(p_1)\right.\right.\\
&\quad - k_{3,\nu}f_{tbc}A^{b\tau}(p_1)A^{c\nu}(p_2) + 2f_{tca}A^{c\nu}(p_1)\left(p_2{}^\tau A^a_\nu(p_2)\right.\\
&\quad\left.\left.\left. - p_{2,\nu}A^{a\tau}(p_2)\right)\right](2\pi)^4\delta^4(p_1+p_2+k_3)\,\mathrm{d}^4p_1\,\mathrm{d}^4p_2\right\}\\
&=\frac{\delta}{\delta A^r_\rho(k_1)}(-\mathrm{i}g)\int\left[k_{3,\mu}f_{tbs}A^{b\mu}(p)g^{\sigma\tau} + k_3{}^\sigma f_{tsc}A^{c\tau}(p)\right.\\
&\quad + f_{tsa}\left(p^\tau A^{a\sigma}(p) - p^\sigma A^{a\tau}(p)\right) + k_2{}^\tau f_{tcs}A^{c\sigma}(p)\\
&\quad\left. - k_{2,\nu}f_{tcs}A^{c\nu}(p)g^{\tau\sigma}\right](2\pi)^4\mathrm{d}^4(p+k_2+k_3)\mathrm{d}^4p\\
&= -\mathrm{i}g\left[k_3{}^\rho f_{trs}g^{\sigma\tau} + k_3{}^\sigma f_{tsr}g^{\rho\tau} + f_{tsr}\left(k_1{}^\tau g^{\sigma\rho} - k_1{}^\sigma g^{\tau\rho}\right)\right.\\
&\quad\left. + f_{trs}\left(k_2{}^\tau g^{\sigma\rho} - k_2{}^\rho g^{\tau\sigma}\right)\right](2\pi)^4\,\delta^4(k_1+k_2+k_3)\\
&= \mathrm{i}gf_{rst}\left[\left(k_1{}^\tau - k_2{}^\tau\right)g^{\rho\sigma} + \left(k_2{}^\rho - k_3{}^\rho\right)g^{\sigma\tau}\right.\\
&\quad\left. + \left(k_3{}^\sigma - k_1{}^\sigma\right)g^{\tau\rho}\right](2\pi)^4\,\delta^4(k_1+k_2+k_3)\quad.
\end{aligned} \tag{7}$$



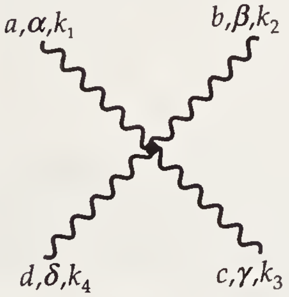
Also the *four–gluon vertex* can be determined in the same manner. Here the relevant part of the Lagrangian is

$$
\begin{aligned}
&\frac{1}{2}\,\mathrm{tr}\left\{\bar{F}_{\mu\nu}\bar{F}^{\mu\nu}\right\}\\
&\to -\frac{1}{2}A^e_\mu A^f_\nu A^{r\mu}A^{s\nu}\frac{-g^2}{16}\,\mathrm{tr}\left\{\left[\lambda^e,\lambda^f\right]\left[\lambda^r,\lambda^s\right]\right\}\\
&= -\frac{1}{2}A^e_\mu A^f_\nu A^{r\mu}A^{s\nu}\frac{g^2}{4}\,\mathrm{tr}\left\{\lambda^g\lambda^t\right\}f_{efg}f_{rst}\\
&= -\frac{g^2}{4}f_{efg}f_{rsg}A^e_\mu A^f_\nu A^{r\mu}A^{s\nu} \quad . \qquad (8)
\end{aligned}
$$

Performing the standard transformation to momentum space simply yields all possible permutations of the four index pairs (a,α), (b,β), (c,γ), and (d,δ), i.e., a total of $4! = 24$ terms:

$$
\begin{aligned}
&-\frac{1}{4}g^2 f_{efg}f_{rsg}\Big[\delta_{ae}\delta_{bf}\delta_{cr}\delta_{ds}g^{\alpha\gamma}g^{\beta\delta} + \delta_{ae}\delta_{bf}\delta_{cs}\delta_{dr}g^{\alpha\delta}g^{\beta\gamma}\\
&+\delta_{ae}\delta_{br}\delta_{cf}\delta_{ds}g^{\alpha\beta}g^{\gamma\delta} + \delta_{ae}\delta_{br}\delta_{cs}\delta_{df}g^{\alpha\beta}g^{\gamma\delta}\\
&+\delta_{ae}\delta_{bs}\delta_{cr}\delta_{df}g^{\alpha\gamma}g^{\beta\delta} + \delta_{ae}\delta_{bs}\delta_{cf}\delta_{dr}g^{\alpha\delta}g^{\beta\gamma}\\
&+(e\to r, f\to s) + (e\to f, r\to s) + (e\to s, f\to r)\Big]\\
&= -\frac{g^2}{4}\Big[f_{gab}f_{gcd}\left(g^{\alpha\gamma}g^{\beta\delta} - g^{\alpha\delta}g^{\beta\gamma}\right)\\
&+f_{gac}f_{gdb}\left(g^{\alpha\delta}g^{\beta\gamma} - g^{\alpha\beta}g^{\gamma\delta}\right) + f_{gad}f_{gbc}\left(g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\gamma}g^{\beta\delta}\right)\Big]4 \quad . \qquad (9)
\end{aligned}
$$

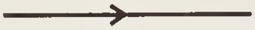
Because of the permutation relations concerning $f_{efg}f_{rsg}$, the three terms denoted by $(e\to r, f\to s)$ etc. contribute the same as the first one. The complete vertex factor is then



$$
\begin{aligned}
&-g^2\Big[f_{eab}f_{ecd}\left(g^{\alpha\gamma}g^{\beta\delta} - g^{\alpha\delta}g^{\beta\gamma}\right)\\
&+f_{eac}f_{edb}\left(g^{\alpha\delta}g^{\beta\gamma} - g^{\alpha\beta}g^{\gamma\delta}\right)\\
&+f_{ead}f_{ebc}\left(g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\gamma}g^{\beta\delta}\right)\Big]\\
&\times(2\pi)^4\,\delta^4(k_1+k_2+k_3+k_4) \quad . \qquad (10)
\end{aligned}
$$

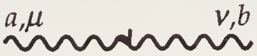
In principle the propagators are identical with those in QED. Hence the quark propagator is

Example 4.2.



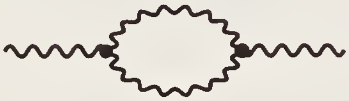
$$\frac{1}{p \cdot \gamma - m + \mathrm{i}\varepsilon} \tag{11}$$

and the gluon propagator could be chosen to be

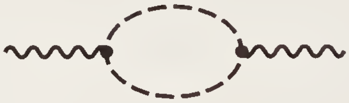


$$-\frac{g_{\mu\nu}\delta_{ab}}{k^2 + \mathrm{i}\varepsilon} \quad . \tag{12}$$

In QED the rule holds that outer lines only propagate real physical degrees of freedom, in particular transverse photons. On the other hand, inner lines (those occurring in loops) represent all degrees of freedom. The latter is possible because the nonphysical components do not contribute to closed loops. However, in QCD additional vertices occur resulting in additional loops that no longer exhibit this feature, e.g.



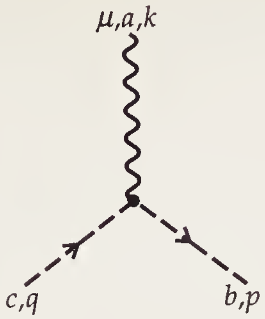
Therefore we have for every inner line to subtract the nonphysical gluonic contributions. This can be achieved by introducing an artificial particle without any physical meaning. The couplings and the propagator of this particle are chosen in such a way that graphs containing this particle cancel the nonphysical gluonic contributions. The corresponding graph is



Here the dashed line denotes the new particle just mentioned. Because of their nonphysical nature these particles are called "ghosts" or "ghost fields". We shall not discuss here, how these fields are exactly defined and how their couplings are determined. We only list the corresponding Feynman rules.

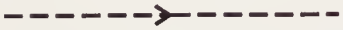
One should be aware of the "i" in the expression for the vertex. Already this different phase compared to (7) shows that certain cancellations of ghost and gluon loops are possible. The Feynman rules discussed above are summarized in (15–21). In order to indicate the different possible gauges, the gluon propagator is given in its most general form, assuming $\partial_\mu A^\mu = 0$ (transverse gauge). From QED we already know the Landau gauge ($\lambda = 1$) and the transverse propagator ($\lambda \longrightarrow 0$). In fact an arbitrary mixing of these two cases can also be chosen. Gauge invariance implies that in the evaluation of a physical process all λ-dependant terms cancel each other or are equal to zero:

Example 4.2.



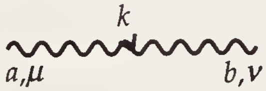
$$-\mathrm{i}gf_{abc}p_\mu(2\pi)^4\,\delta^4(p-k-q) \qquad (13)$$

and

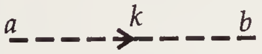


$$\frac{1}{p^2+\mathrm{i}\varepsilon} \qquad (14)$$

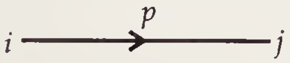
Note that all Feynman graphs are frequently multiplied by an additional factor i because the starting point of a perturbative expansion is really i $\int L(x)\,\mathrm{d}^4x$ and not $\int L(x)\,\mathrm{d}^4x$. However, this only leads to an unimportant overall phase.



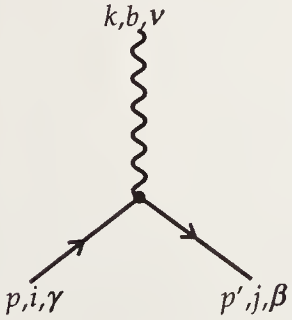
$$-\delta_{ab}\left[\frac{g_{\mu\nu}}{k^2+\mathrm{i}\varepsilon}-(1-\lambda)\frac{k_\mu k_\nu}{(k^2+\mathrm{i}\varepsilon)^2}\right] \quad , \qquad (15)$$



$$\frac{\delta_{ab}}{k^2+\mathrm{i}\varepsilon} \quad , \qquad (16)$$

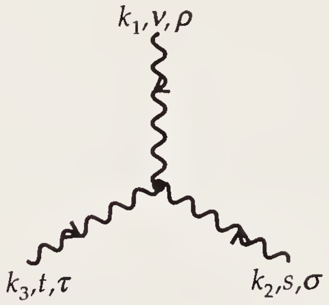


$$\frac{\delta_{ij}}{\not{p}-m+\mathrm{i}\varepsilon} \quad , \qquad (17)$$

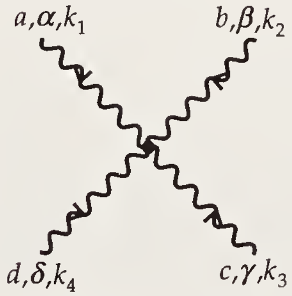


Example 4.2.

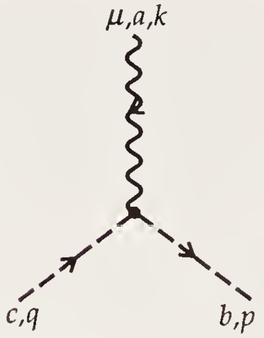
$$g\left(\frac{1}{2}\lambda^b\right)_{ji}(\gamma_\nu)_{\beta\gamma}(2\pi)^4\,\delta^4(p'+k-p)\quad , \tag{18}$$



$$gf_{rst}\left[\left(k_1^\tau-k_2^\tau\right)g^{\rho\sigma}+\left(k_2^\rho-k_3^\rho\right)g^{\sigma\tau}+\left(k_3^\sigma-k_1^\sigma\right)g^{\tau\rho}\right](2\pi)^4\,\delta^4(k_1+k_2+k_3)\quad , \tag{19}$$



$$\begin{aligned}-g^2\Big[&f_{eab}f_{ecd}\left(g^{\alpha\gamma}g^{\beta\delta}-g^{\alpha\delta}g^{\beta\gamma}\right)+f_{eac}f_{edb}\left(g^{\alpha\delta}g^{\beta\gamma}-g^{\alpha\beta}g^{\gamma\delta}\right)\\&+f_{ead}f_{ebc}\left(g^{\alpha\beta}g^{\gamma\delta}-g^{\alpha\gamma}g^{\beta\delta}\right)\Big](2\pi)^4\,\delta^4(k_1+k_2+k_3+k_4)\quad ,\end{aligned} \tag{20}$$

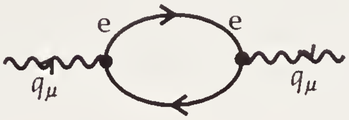


$$-\mathrm{i}gf_{abc}\,p_\mu(2\pi)^4\,\delta^4(p-k-q)\quad , \tag{21}$$

EXAMPLE

4.3 The Running Coupling Constant

We discussed above that in general a perturbation series converges for a given coupling constant only at some momentum transfers. This behavior is easily understood, since the renormalized coupling constant is defined for a specific momentum transfer. In QED, for example, one can choose the incoming and outgoing photons to be on mass shell, and then

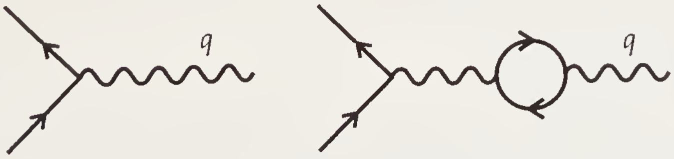


$$e_{\mathrm{R}} = e\left[1 - \frac{\alpha}{2\pi}\log\left(\frac{\Lambda^2}{m^2}\right)\right]^{\frac{1}{2}} \qquad \text{for} \quad q^2 = 0 \quad . \tag{1}$$

For other values of q^2 this graph then yields a finite, q^2-dependent renormalization, i.e., a finite additional contribution to every electromagnetic process. These corrections can be given analytically and taken into account by a redefinition of the electric charge. We obtain in a massless QED

$$e_{\mathrm{R}}^2(q^2) = \frac{e_{\mathrm{R}}^2(q^2 = -M^2)}{1 - Ce_{\mathrm{R}}^2(q^2 = -M^2)\ln\left(q^2/-M^2\right)} \quad . \tag{2}$$

The value of the constant C depends on the number of fermions considered by the theory and on the fermion charges. From (2) it also follows that $q^2 = 0$ is a special value, quite unsuitable for renormalization (here e_{R} is equal to zero). Therefore $q^2 = -M^2$ is usually chosen as the renormalization point. Equation (2) denotes the additional contributions to all massless QED graphs caused by vacuum polarization. $e_{\mathrm{R}}^2(q^2)$ is valid only for a certain q^2 range: if $q^2 < 0, Ce_{\mathrm{R}}^2(-M^2)\ln\left(q^2/-M^2\right) < 1$. In this range it can be interpreted as a physical charge. For $q^2 > 0$, $e_{\mathrm{R}}^2(q^2)$ becomes formally complex and such an interpretation is no longer possible. In the latter case $e_{\mathrm{R}}^2(q^2)$ reflects the fact that the following graphs lead to different phases.



(3)

The divergence of this "charge" for

$$Ce_{\mathrm{R}}^2(-M^2)\ln\left(\frac{q^2}{-M^2}\right) \to 1$$

indicates the breakdown of perturbation theory. For values of $-q^2$ this large the contributions of two-loop corrections become as large as those of one-loop graphs in (1) and so on. If also the higher corrections are taken into account, we obtain instead of (2) the nonconvergent expression.

Example 4.3.

$$e^2(q^2) = \frac{e^2(-M^2)}{1 - Ce^2(-M^2)\ln\left(\frac{q^2}{-M^2}\right) + C'e^4(-M^2)\ln^2\left(\frac{q^2}{-M^2}\right) + \dots} \quad . \tag{4}$$

Since QCD is a scale-free theory, it is possible to define a running coupling constant for it, which takes the finite corrections due to the graphs shown in Fig. 4.7 into account. We get

$$\alpha_s(q^2) = \frac{\alpha_s(-M^2)}{1 + \frac{11 - \frac{2}{3}N_f(q^2)}{4\pi}\alpha_s(-M^2)\ln\left(\frac{q^2}{-M^2}\right)} \quad , \tag{5}$$

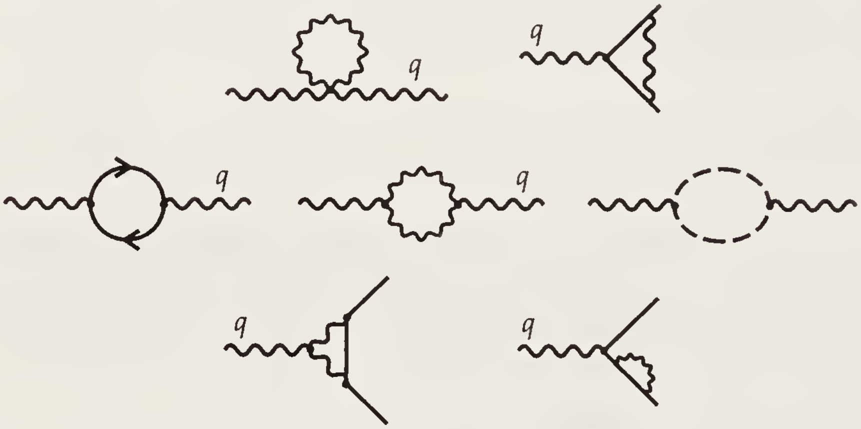


Fig. 4.7. QCD graphs taken into account by $\alpha_s(q^2)$

where N_f denotes the number of quark flavors with a mass much smaller than $\frac{1}{2}\sqrt{-q^2}$. We shall derive (5) in Sect. 4.4 after having introduced the necessary techniques in Sect. 4.3. Here we only want to discuss its phenomenological meaning. Equation (5) includes two parameters M^2 and $\alpha_s(M^2)$, which, however, are not independent of each other. It is possible to introduce a quantity $\Lambda = \Lambda(M)$ such that

$$\frac{11 - \frac{2}{3}N_f(q^2)}{4\pi}\alpha_s(-M^2)\ln(\Lambda^2/M^2) = -1 \quad , \tag{6}$$

leading to

$$\alpha_s(q^2) = \frac{4\pi}{(11 - \frac{2}{3}N_f(q^2))\ln(-q^2/\Lambda^2)} \quad . \tag{7}$$

Hence the running coupling constant is fixed for all momentum transfers by one parameter. Λ can be determined by investigating high energetic e^+e^- pair annihilation and many other processes. The "world average" of these results is

$$\alpha_s(-(34\ \mathrm{GeV})^2) \approx 0.14 \pm 0.02 \quad . \tag{8}$$

Clearly the q^2 dependence of the strong coupling constant is determined by N_f, i.e., the number of quarks with $M^2 < |q^2|$. In Fig. 4.8, (7) is plotted for $N_f = 6$ and $N_f = 12$. Λ has been fixed to obey (8). The plot for $N_f = 6$ increases rapidly for small values of $\sqrt{-q^2}$, i.e., the interaction can certainly not be described by perturbation theory below 500 MeV energy transfer. This conclusion is not valid

Example 4.3.

for $N_f = 12$. Since at these energies the real physical hadrons have a strongly correlated structure rather than appearing as a group of free quarks, it follows immediately that the number of "light"-quarks cannot be much greater than six. In fact only two "light"-quark doublets have been discovered. This example yields

$$\alpha_s(-(100\text{ GeV})^2) = 0.2 \quad \Rightarrow \quad \Lambda = 112\text{ MeV} \quad \text{for} \quad N_f = 6 \quad . \tag{9}$$

The currently discussed values for the scale parameter of QCD, Λ_{QCD} range from 100 MeV to 300 MeV. It should be mentioned that Fig. 4.8 contains an invalid simplification. In contradiction to this figure N_f decreases with decreasing $\sqrt{-q^2}$, because an increasing number of quarks must be considered massive.

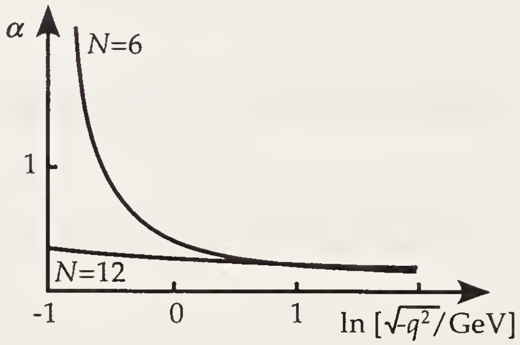


Fig. 4.8. Running coupling constants for the unphysical case of 6 and 12 different massless quarks. Both curves assume the value $\alpha(-100\,\mathrm{GeV}^2) = 0.2$

Equation (7) can be interpreted as antiscreening of the charge unlike the usual screening in QED. The vacuum polarization graph in (1) leads in the case of QED to a polarization of the vacuum, which in the context of field theory is a medium with well-defined, nontrivial features.[2] This behavior is depicted in Fig. 4.9 for an extended charge. In QCD the corresponding graph with a virtual quark loop shows the same effect. Here, however, there is also a gluon contribution. Since the gluons carry a charge, their virtual excitations can also be polarized. Therefore the analogous graph Fig. 4.10 is somewhat more complicated. In particular the total

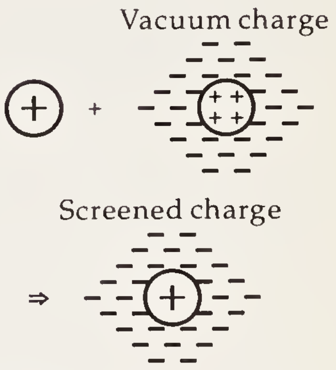


Fig. 4.9. Charge screening in QED

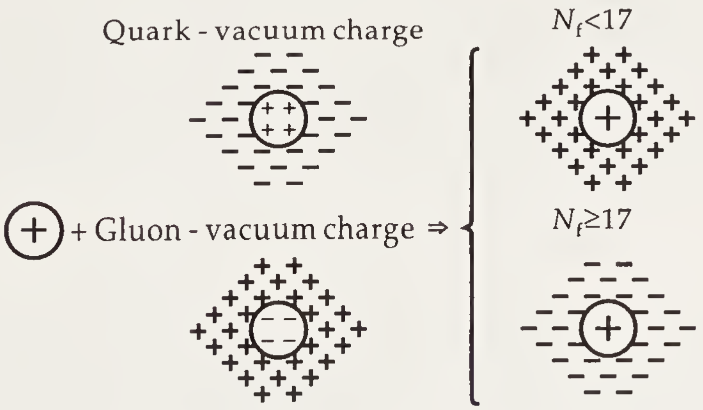


Fig. 4.10. Charge screening in QCD. N_f denotes the number of (massless) quarks

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[2] See W. Greiner / J. Reinhardt: *Quantum Electrodynamics* 2nd, corr. ed. (Springer, Berlin, Heidelberg 1994)

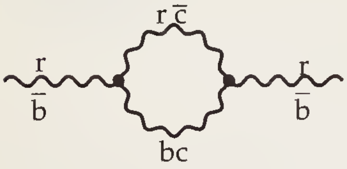


Fig. 4.11. Gluon vacuum polarization for QCD

charge distribution depends on the dominance of the quark or gluon part. While the latter is defined by the gauge group, the first part increases with every additional quark until it dominates the theory and QCD shows the same behavior as QED. From this argument it also follows that the antiscreening in SU(N) gauge theories in principle increases with N. Even the explicit form of that dependence can be understood. The gluon contribution is simply proportional to the number of possible permutations for the loop gluons in Fig. 4.11.

For QCD, i.e. SU(3), c can assume three values; for SU(N) there are correspondingly N values. Therefore (7) becomes, in the case of an SU(N) gauge theory

$$\alpha_N(q^2) = \frac{4\pi}{\left(11 \times \frac{1}{3}N - \frac{2}{3}N_\mathrm{f}\right) \ln\left(-q^2/\Lambda_N^2\right)} \quad . \tag{10}$$

4.3 Dimensional Regularization

Currently so-called dimensional regularization is considered to be the standard procedure for regularizing quantum field theories. Only in special cases, where some specific disadvantages of this method show up, are other procedures, such as Pauli–Villars regularization or the momentum-cutoff method, employed. For lattice gauge theories, however, regularization is automatically provided by the lattice constant, which necessarily yields a rather special regularization scheme. In the following we only discuss dimensional regularization.

The basic observation that motivated dimensional regularization is that only logarithmic divergences in standard quantum field theories are encountered and these vanish for any dimensionality smaller than 4. Thus if it is possible to define a generalized integration for noninteger dimensions "d" the usual divergences will show up as poles in "d-4" which should be relatively easy to isolate. To realize this idea it is advisable first to simplify the divergences that occur by a trick known as "Wick rotation", namely by continuing it to imaginary energies. As mentioned above, only logarithmic divergences are encountered, and so a typical divergent integral has the form

$$\mathrm{i} \int \mathrm{d}^4 k \frac{1}{(k^2 - m_1^2 + \mathrm{i}\varepsilon)} \frac{1}{\left[(p+k)^2 - m_2^2 + \mathrm{i}\varepsilon\right]} \tag{4.54}$$

One awkward property of this integral is that its behavior for large k_μ is not uniform, since $k^2 = (k^0)^2 - (\boldsymbol{k})^2$ can stay small even when k^0 and $|\boldsymbol{k}|$ both become large. This problem can be circumvented by continuing k^0 to $\mathrm{i}k_E^0$ $(k \rightarrow k_E)$, implying that

$$k^2 \rightarrow -(k_E^0)^2 - (\boldsymbol{k}_E)^2 = -k_E^2 \quad ,$$

with $k_E^2 = (k_E^0)^2 + (\boldsymbol{k}_E)^2$. $(k_E)_\mu$ is obviously a Cartesian vector that becomes uniformly large if $(k_E)_\mu \to \infty$, so it is much easier to analyze the ultraviolet divergences for $(k_E)_\mu$ than for k_μ.

To perform the Wick rotation one has to deform the integration path of k^0 in the complex plane. Figures 4.12 and 4.13 show the positions of the poles. We now use

$$\int_{-\infty}^{\infty} \mathrm{d}k^0 \ldots + \int_{+\mathrm{i}\infty}^{-\mathrm{i}\infty} \mathrm{d}k^0 \ldots + \int_{C_1} \mathrm{d}k^0 \ldots + \int_{C_2} \mathrm{d}k^0 \ldots$$

$$= \begin{cases} 0 \text{ for Fig. 4.12} \\ \text{residue for } k_0 = \sqrt{(\boldsymbol{p}+\boldsymbol{k})^2 - m_2^2} - p_0 \quad \text{for Fig. 4.13} \end{cases} . \tag{4.55}$$

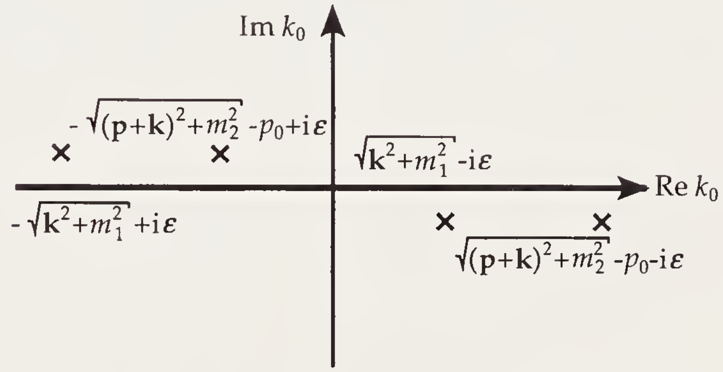

Im k_0
$-\sqrt{(\mathbf{p}+\mathbf{k})^2+m_2^2}-p_0+\mathrm{i}\varepsilon$
$\sqrt{\mathbf{k}^2+m_1^2}-\mathrm{i}\varepsilon$
Re k_0
$-\sqrt{\mathbf{k}^2+m_1^2}+\mathrm{i}\varepsilon$
$\sqrt{(\mathbf{p}+\mathbf{k})^2+m_2^2}-p_0-\mathrm{i}\varepsilon$

Fig. 4.12. Poles of the integrand in (4.54) for $p_0 < \sqrt{(\boldsymbol{p}+\boldsymbol{k})^2 + m_2^2}$

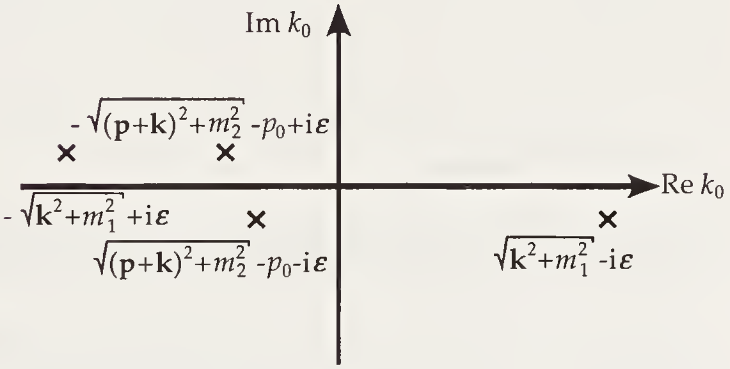

Im k_0
$-\sqrt{(\mathbf{p}+\mathbf{k})^2+m_2^2}-p_0+\mathrm{i}\varepsilon$
Re k_0
$-\sqrt{\mathbf{k}^2+m_1^2}+\mathrm{i}\varepsilon$
$\sqrt{(\mathbf{p}+\mathbf{k})^2+m_2^2}-p_0-\mathrm{i}\varepsilon$
$\sqrt{\mathbf{k}^2+m_1^2}-\mathrm{i}\varepsilon$

Fig. 4.13. Poles of the integrand in (4.54) for $p_0 > \sqrt{(\boldsymbol{p}+\boldsymbol{k})^2 + m_2^2}$

The ϵ prescription in (4.54) guarantees that the integrals over the arcs at infinity $\int_{C_1} \mathrm{d}k^0$ and $\int_{C_2} \mathrm{d}k^0$ vanish. This can easily be seen if we use

$$\frac{1}{(k^2 - m_1^2 + \mathrm{i}\varepsilon)} \frac{1}{(p+k)^2 - m_2^2 + \mathrm{i}\varepsilon}$$

$$= -\mathrm{i} \int_0^\infty \mathrm{d}\alpha\, \mathrm{e}^{\mathrm{i}\alpha(k^2 - m_1^2 + \mathrm{i}\varepsilon)} (-\mathrm{i}) \int_0^\infty \mathrm{d}\beta\, \mathrm{e}^{\mathrm{i}\beta\left[(p+k)^2 - m_2^2 + \mathrm{i}\varepsilon\right]} . \tag{4.56}$$

From $k^0 = R\,\mathrm{e}^{\mathrm{i}\varphi}$, $\quad 0 \le \varphi \le \frac{\pi}{2}$ and $\pi \le \varphi \le \frac{3}{2}\pi$ with $R \to \infty$, we get

$$\mathrm{Im}(k^2) = R^2 \sin(2\mathrm{i}\varphi) + O(R) > 0 \tag{4.57}$$

such that all terms vanish exponentially. The result of the Wick rotation is therefore

$$\begin{aligned} & \mathrm{i}\int\limits_{-\infty}^{\infty} \mathrm{d}^4k \, \frac{1}{k^2 - m_1^2 + \mathrm{i}\varepsilon} \, \frac{1}{(p+k)^2 - m_2^2 + \mathrm{i}\varepsilon} \\ & \qquad = -\int_{-\infty}^{\infty} \mathrm{d}^4k_E \, \frac{1}{k_E^2 + m_1^2 - \mathrm{i}\varepsilon} \, \frac{1}{(p_E + k_E)^2 + m_2^2 - \mathrm{i}\varepsilon} \\ & \qquad + \text{possibly a finite residuum} \quad . \end{aligned} \tag{4.58}$$

The residuum that possibly appears is finite. As we intend to isolate and substract the divergent part of the integral (4.54) we do not have to bother about this finite contribution. We define the renormalized integral as (4.54) minus the divergent part of (4.58); this definition is not affected by the finite residuum.

Furthermore we omit the index "E" assuming all momenta to be Euclidian. Next we proceed to define an abstract mathematical operation which we want to call d-dimensional integral:

$$\int \mathrm{d}^d k \, f(k)$$

First we impose the following conditions:

1. Linearity

$$\int \mathrm{d}^d k [a f(k_\mu) + b g(k_\mu)] = a \int \mathrm{d}^d k \, f(k_\mu) + b \int \mathrm{d}^d k \; g(k_\mu) \quad . \tag{4.59}$$

2. Invariance of the integral under finite shifts (p_μ finite)

$$\int \mathrm{d}^d k f(k_\mu) = \int \mathrm{d}^d k f(k_\mu + p_\mu) \quad . \tag{4.60}$$

3. A scaling property

$$\int \mathrm{d}^d k f(\lambda k_\mu) = \lambda^{-d} \int \mathrm{d}^d k f(k_\mu) \quad . \tag{4.61}$$

From these definitions it is not clear what is meant by a vector in k_μ dimensions. The point is that it is always sufficient to treat scalar integrals. Any vectorlike integral is completely specified by its values if contracted with few linearly independent vectors and those scalar integrals can be analytically continued. Bearing this in mind it is, however, helpful to use vectors like k_μ as a shorthand for a collection of suitably defined scalar products.

The d-dimensional integral is now defined by its action on a set of basis-functions

$$\lambda(A, P_\mu) = \mathrm{e}^{-A(k+P)^2} \quad . \tag{4.62}$$

Properties (4.59) and (4.60) guarantee that all functions decomposed in this basis can be integrated once the integral

$$\int \mathrm{d}^d k \; \mathrm{e}^{-Ak^2} \quad . \tag{4.63}$$

is known. Here the scaling property (4.61) becomes important, since it reduces any such integral to a single one:

$$\int \mathrm{d}^d k \ \mathrm{e}^{-Ak^2} = A^{-d} \int \mathrm{d}^d k \ \mathrm{e}^{-k^2} \quad . \tag{4.64}$$

As we want the d-dimensional integral to coincide with the usual one for integer values of d we impose the condition

$$\int \mathrm{d}^d k \ \mathrm{e}^{-k^2} = \pi^{\frac{d}{2}} \quad . \tag{4.65}$$

This relation is derived for integer d in Exercise 4.4. We note for later use that in Exercise 4.4 we also derive (8)

$$\int\limits_0^\infty \mathrm{d}^d k \, f(k^2) = \frac{1}{(d-2)!!} \left\{ \begin{array}{ll} (2\pi)^{\frac{d}{2}} & d \text{ even} \\ 2\,(2\pi)^{\frac{d-1}{2}} & d \text{ odd} \end{array} \right\} \int\limits_0^\infty \mathrm{d}k \, f(k^2) k^{d-1} \quad . \tag{4.66}$$

With $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ this can be written as

$$\frac{2(\pi)^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} = \left\{ \begin{array}{llll} d \text{ even}: & 2\pi^{\frac{d}{2}} \times \frac{2}{d-2} \times \frac{2}{d-4} \times \cdots \times \frac{2}{2} & = & \frac{(2\pi)^{\frac{d}{2}}}{(d-2)!!} \\ d \text{ odd}: & 2\pi^{\frac{d}{2}} \times \frac{2}{d-2} \times \frac{2}{d-4} \times \cdots \times \frac{2}{1} \times \frac{1}{\sqrt{\pi}} & = & \frac{2(2\pi)^{\frac{d-1}{2}}}{(d-2)!!} \end{array} \right. , \tag{4.67}$$

$$\int\limits_0^\infty \mathrm{d}^d k f(k^2) = \frac{2(\pi)^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} \int\limits_0^\infty \mathrm{d}k k^{d-1} f(k^2) \quad . \tag{4.68}$$

Equation (4.68) will be used constantly during dimensional regularization.

EXERCISE

4.4 The d-Dimensional Gaussian Integral

Problem. Derive (4.65) for integer d.

Solution. For integer d and Euclidian k_μ, $\mu = 0, 1, 2, \ldots, d-1$, we define

$$\begin{aligned} k_0 &= k \cos\vartheta_1 \quad , \\ k_1 &= k \sin\vartheta_1 \cos\vartheta_2 \quad , \\ k_2 &= k \sin\vartheta_1 \sin\vartheta_2 \cos\vartheta_3 \quad , \\ &\vdots \\ k_{d-2} &= k \sin\vartheta_1 \sin\vartheta_2 \ldots \sin\vartheta_{d-2} \cos\vartheta_{d-1} \quad , \\ k_{d-1} &= k \sin\vartheta_1 \sin\vartheta_2 \ldots \sin\vartheta_{d-2} \sin\vartheta_{d-1} \end{aligned} \tag{1}$$

and calculate the Jacobian row by row

Exercise 4.4.

$$J = \frac{\partial(k_0 k_1 k_2 \dots k_{d-1})}{\partial(k, \vartheta_1, \vartheta_2 \dots \vartheta_{d-1})}$$

$$= \begin{vmatrix} \cos\vartheta_1 & -k\sin\vartheta_1 & 0 & \dots \\ \sin\vartheta_1\cos\vartheta_2 & k\cos\vartheta_1\cos\vartheta_2 & -k\sin\vartheta_1\sin\vartheta_2 & \dots \\ \sin\vartheta_1\sin\vartheta_2\cos\vartheta_3 & k\cos\vartheta_1\sin\vartheta_2\cos\vartheta_3 & k\sin\vartheta_1\cos\vartheta_2\cos\vartheta 3 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{vmatrix}$$

$$= k^{d-1}\left\{\cos\vartheta_1\cos\vartheta_1(\sin\vartheta_1)^{d-2} + \sin\vartheta_1\sin\vartheta_1(\sin\vartheta_1)^{d-2}\right\}$$

$$\times \begin{vmatrix} \cos\vartheta_2 & -\sin\vartheta_2 & \dots \\ \sin\vartheta_2\cos\vartheta_3 & \cos\vartheta_2\cos\vartheta_3 & \dots \\ \vdots & \vdots & \ddots \end{vmatrix} = \dots$$

$$= k^{d-1}(\sin\vartheta_1)^{d-2}(\sin\vartheta_2)^{d-3}\dots(\sin\vartheta_{d-3})^2$$

$$\times \begin{vmatrix} \cos\vartheta_{d-2} & -\sin\vartheta_{d-2} & 0 \\ \sin\vartheta_{d-2}\cos\vartheta_{d-1} & \cos\vartheta_{d-2}\cos\vartheta_{d-1} & -\sin\vartheta_{d-2}\sin\vartheta_{d-1} \\ \sin\vartheta_{d-2}\sin\vartheta_{d-1} & \cos\vartheta_{d-2}\sin\vartheta_{d-1} & \sin\vartheta_{d-2}\cos\vartheta_{d-1} \end{vmatrix}$$

$$= k^{d-1}(\sin\vartheta_1)^{d-2}(\sin\vartheta_2)^{d-3}\dots(\sin\vartheta_{d-2})$$

$$\times \begin{vmatrix} \cos\vartheta_{d-1} & -\sin\vartheta_{d-1} \\ \sin\vartheta_{d-1} & \cos\vartheta_{d-1} \end{vmatrix}$$

$$= k^{d-1}(\sin\vartheta_1)^{d-2}(\sin\vartheta_2)^{d-3}\dots(\sin\vartheta_{d-2}) \tag{2}$$

The Gaussian integral thus becomes

$$\int_0^\infty \mathrm{d}^d k\, \mathrm{e}^{-k^2}$$

$$= \int_0^\infty \mathrm{d}k\, k^{d-1}\mathrm{e}^{-k^2} \int_0^\pi (\sin\vartheta_1)^{d-2}\,\mathrm{d}\vartheta_1 \dots \int_0^\pi \sin\vartheta_{d-2}\,\mathrm{d}\vartheta_{d-2} \int_0^\pi \mathrm{d}\vartheta_{d-1} \tag{3}$$

We use partial integration to get

$$\int_0^\pi (\sin\vartheta)^{n-1}\sin\vartheta\,\mathrm{d}\vartheta$$

$$= \left[(\sin\vartheta)^{n-1}(-\cos\vartheta)\right]_0^\pi - (n-1)\int_0^\pi (\sin\vartheta)^{n-2}\cos\vartheta(-\cos\vartheta)\mathrm{d}\vartheta$$

$$= (n-1)\int_0^\pi \left[(\sin\vartheta)^{n-2} - (\sin\vartheta)^n\right]\mathrm{d}\vartheta \tag{4}$$

or

$$\int_0^\pi (\sin\vartheta)^n\mathrm{d}\vartheta = \frac{n-1}{n}\int_0^\pi (\sin\vartheta)^{n-2}\mathrm{d}\vartheta \quad .$$

For even $n \geq 2$ we thus get *Exercise 4.4.*

$$\int_0^\pi (\sin\vartheta)^n \mathrm{d}\vartheta = \frac{(n-1)!!}{n!!} \int_0^\pi \mathrm{d}\vartheta = \frac{(n-1)!!}{n!!}\pi \tag{5}$$

and for $n = 0$ we have

$$\int_0^\pi \mathrm{d}\vartheta = \pi \quad . \tag{6}$$

For odd n we find

$$\begin{aligned}\int_0^\pi (\sin\vartheta)^n \mathrm{d}\vartheta &= \frac{(n-1)!!}{n!!} \int_0^\pi \sin\vartheta\, \mathrm{d}\vartheta \\ &= \frac{(n-1)!!}{n!!} \times 2\end{aligned} \tag{7}$$

Inserting all these equations into (3) we get

$$\begin{aligned}&\int_0^\infty \mathrm{d}^d k\ \mathrm{e}^{-k^2} = \int_0^\infty \mathrm{d}k\ k^{d-1}\mathrm{e}^{-k^2} I_d \frac{1}{(d-2)!!} \quad , \\ &d \text{ even} \Rightarrow I_d = (2\pi)^{\frac{d}{2}} \quad , \\ &d \text{ odd} \Rightarrow I_d = 2 \times (2\pi)^{\frac{d-1}{2}} \quad .\end{aligned} \tag{8}$$

On the other hand partial integration gives

$$\begin{aligned}\int_0^\infty \mathrm{d}k\ k^{d-1}\,\mathrm{e}^{-k^2} &= \frac{1}{2}\int_0^\infty \mathrm{d}k\ k^{d-2} 2k\ \mathrm{e}^{-k^2} \\ &= \frac{d-2}{2}\int_0^\infty \mathrm{d}k\ k^{d-3}\,\mathrm{e}^{-k^2} \\ &= (d-2)!!K_d \quad ;\end{aligned} \tag{9}$$

$$\begin{aligned}d \text{ even} \Rightarrow K_d &= \left(\frac{1}{2}\right)^{\frac{d-2}{2}} \int_0^\infty \mathrm{d}k\ k\mathrm{e}^{-k^2} \\ &= \left(\frac{1}{2}\right)^{\frac{d}{2}} \quad , \\ d \text{ odd} \Rightarrow K_d &= \left(\frac{1}{2}\right)^{\frac{d-1}{2}} \int_0^\infty \mathrm{d}k\ \mathrm{e}^{-k^2} \\ &= \left(\frac{1}{2}\right)^{\frac{d-1}{2}} \frac{\sqrt{\pi}}{2} \quad .\end{aligned} \tag{10}$$

Exercise 4.4.

All together this gives

$$\int_0^\infty \mathrm{d}^d k\, \mathrm{e}^{-k^2} = \frac{1}{(d-2)!!}(d-2)!!\, I_d K_d = \pi^{\frac{d}{2}} \quad , \tag{11}$$

which completes our proof.

With (4.65) all integrals that can be expanded in a sum of Gaussians:

$$f(k) = \sum_n c_n \mathrm{e}^{-A_n(k+p_n)^2} \tag{4.68a}$$

can be integrated d-dimensionally

$$\int \mathrm{d}^d k\, f(k) = \sum_n c_n\, A_n^{-d}\, \pi^{\frac{d}{2}} \quad . \tag{4.68b}$$

However, it is not clear whether this sum converges. If it does for some integer, i.e., if the original integrand converged, then it must also converge since the d-dimensinal integration coincides with the usual one. Thus all finite parts of the momentum integrals are reproduced. The infinite parts, associated, for example, with integrands like $1/(k^2)^\alpha$ or $(k^2)^\alpha$, which we want to subtract, will lead to infinite nonconvergent sums over Gaussians, the resulting term on the right-hand side of (4.68b). These will be subtracted during renormalization. We will see below that these infinite parts are different for d-dimensional integrals than for normal ones. Actually many divergences of usual integrals vanish automatically for d-dimensional integrals, and the important point to keep in mind throughout this discussion is the following:

d-dimensional convergent integrals for $d \to n$ (where n is an integer) coincide with the normal convergent n-dimensional integrals.

n-dimensional divergent integrals are in general different from the corresponding $d \to n$ limit of the d-dimensional integral. The latter is in general less divergent.

It is crucial to understand that our definition of a d-dimensional integral does not reduce to a normal integral in integer d dimensions. Instead it has peculiar properties, for example,

$$\int \mathrm{d}^d k\ \left(k^2\right)^\nu = 0 \qquad \text{for any } \nu \quad , \tag{4.69}$$

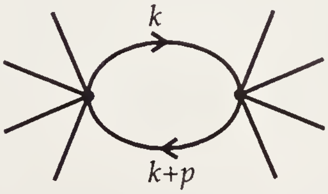
which follows directly from the scaling axiom (4.61). Indeed we have

$$\int_{-\infty}^{\infty} \mathrm{d}^d k\ \left(\lambda^2 k^2\right)^\nu = \lambda^{2\nu} \int_{-\infty}^{\infty} \mathrm{d}^d k\ \left(k^2\right)^\nu = \lambda^{-d} \int_{-\infty}^{\infty} \mathrm{d}^d k\ \left(k^2\right)^\nu \quad . \tag{4.70}$$

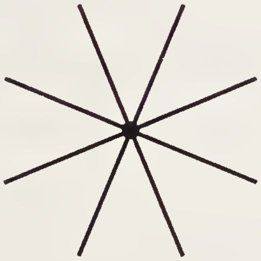
Actually $\nu = d/2$ is the only power for which (4.69) does not imply the vanishing of this integral, but since we make critical use of the fact that our renormalized integrals are analytic functions of all variables involved, like ν in (4.69) we have also to define

$$\int \mathrm{d}^d k \ \left(k^2\right)^{-\frac{d}{2}} = 0 \quad . \tag{4.71}$$

This derivation of (4.69) is actually not without problems, because it assumes that $\int \mathrm{d}^d k\,(k^2)^\nu$ exists, which is only true after a specific definition such as $\lim_{k^2 \to 0} \int \mathrm{d}^d k\,(k^2+m^2)^\nu$. To be on the safe side it is better to regard (4.69) as a definition that is made plausible by (4.70). A peculiar feature of (4.69) seems to be that it leads to finite results even for theories for which arbitrarily high divergences occur. While this is formally true it is of no importance since such theories cannot be renormalized. They would require an infinite number of renormalization constants and counterterms and thus cannot be formulated consistently. As an example let us mention a scalar field theory with a ϕ^6 coupling. This coupling leads to divergent graphs like



which is divergent $\left(\sim \int \mathrm{d}^4 k \left[1/k^2(k+p)^2\right]\right)$ and must be renormalized leading to a counter term of the form



which is a ϕ^8 interaction. Continuing this argument we find that the renormalized theory has infinitely many interaction terms of the general form $\sum_{n=3}^{\infty} c_n \phi^{2n}$, and no consistent theory can be formulated. In fact only those theories can be renormalized which have at most logarithmic divergences, and thus integrals like

$$\int \mathrm{d}^4 k \ \left(k^2\right)^3$$

cannot occur. For renormalizable theories the differences between n-dimensional integrals and d-dimensional integrals (with the limit $d \to n$) is therefore irrelevant.

Equation (4.69) is essential for deriving the following basic formula

$$I_{\nu,\mu} = \int \mathrm{d}^d k \ \frac{\left(k^2\right)^\nu}{\left(k^2+M^2\right)^\mu} = \pi^{\frac{d}{2}} \left(M^2\right)^{\nu-\mu+\frac{d}{2}} \frac{\Gamma\left(\nu+\frac{d}{2}\right)\Gamma\left(\mu-\nu-\frac{d}{2}\right)}{\Gamma\left(\frac{d}{2}\right)\Gamma(\mu)} \quad , \tag{4.72}$$

which we shall prove next. From (4.69) we can rewrite $I_{\nu,\mu}$ in the following manner:

$$\begin{aligned} I_{\nu,\mu} &= \int \mathrm{d}^d k \, \frac{\left(k^2\right)^\nu}{\left(k^2+M^2\right)^\mu} = M^{2\nu-2\mu+d} \int \mathrm{d}^d k \, \frac{\left(k^2\right)^\nu}{\left(k^2+1\right)^\mu} \\ &= M^{2\nu-2\mu+d} \int \mathrm{d}^d k \left[\frac{\left(k^2\right)^\nu}{\left(k^2+1\right)^\mu} - \sum_n \left(k^2\right)^{\nu_n} c_n \right] \end{aligned} \tag{4.73}$$

with arbitrary constant coefficients ν_n and c_n. These coefficients are now chosen such that they cancel all potential divergences, i.e., all high powers of k^2. The integral is then convergent and we simply apply (4.68):

$$I_{\nu,\mu} = M^{2\nu-2\mu+d} \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)} \int\limits_0^\infty \frac{\mathrm{d}k^2}{2} \left(k^2\right)^{\frac{d}{2}-1} \left[\frac{\left(k^2\right)^\nu}{\left(k^2+1\right)^\mu} - \sum_n \left(k^2\right)^{\nu_n} c_n \right] \quad . \tag{4.74}$$

The trick is now to derive by partial integration a recurrence relation for the $I_{\nu,\mu}$:

$$\begin{aligned} I_{\nu,\mu} = {} & \frac{\left(\nu+\frac{d}{2}-1\right)}{\mu-1} I_{\nu-1,\mu-1} + M^{2\nu-2\mu+d} \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)} \frac{1}{2} \\ & \times \left\{ \left(k^2\right)^{\nu+\frac{d}{2}-1} \left[\frac{\left(k^2+1\right)^{1-\mu}}{1-\mu} - \sum_n \left(k^2\right)^{\nu_n+1-\nu} \frac{c_n}{\nu_n+1} \right] \right\}_0^\infty \quad . \end{aligned} \tag{4.75}$$

The second term can obviously be absorbed into the c_n's. In other words c_n can be chosen such that it vanishes. This leaves us with the relation

$$I_{\nu,\mu} = \frac{\nu+\frac{d}{2}-1}{\mu-1} I_{\nu-1,\mu-1} \qquad \text{for all real } \nu, \mu \quad . \tag{4.76}$$

As the relation $\Gamma(z+1) = z\Gamma(z)$ uniquely defines the gamma function, (4.76) implies (d is fixed) that

$$I_{\nu,\mu} = \frac{\Gamma\left(\nu+\frac{d}{2}\right)}{\Gamma(\mu)} K(\nu-\mu) \tag{4.77}$$

with an arbitrary function $K(\nu-\mu)$. To fix $K(\nu-\mu)$ we use the fact that

$$\frac{\left(k^2\right)^\nu}{\left(k^2+1\right)^\mu} = \frac{\left(k^2\right)^\nu}{\left(k^2+1\right)^\mu} \left[\left(k^2+1\right) - k^2\right] \quad , \tag{4.78}$$

$$I_{\nu,\mu} = I_{\nu,\mu-1} - I_{\nu+1,\mu} \quad . \tag{4.79}$$

Inserting (4.67) into (4.79) gives

$$\begin{aligned} \frac{\Gamma\left(\nu+\frac{d}{2}\right)}{\Gamma(\mu)} K(\nu-\mu) &= \left[\frac{\Gamma\left(\nu+\frac{d}{2}\right)}{\Gamma(\mu-1)} - \frac{\Gamma\left(\nu+\frac{d}{2}+1\right)}{\Gamma(\mu)} \right] K(\nu-\mu+1) \\ \Rightarrow K(\nu-\mu) &= \left[(\mu-1) - \left(\nu+\frac{d}{2}\right) \right] K(\nu-\mu+1) \\ &= \left(\mu-\nu-1-\frac{d}{2}\right) K(\nu-\mu+1) \end{aligned} \tag{4.80}$$

$$\Rightarrow K(\nu-\mu) = \Gamma\left(\mu-\nu-\frac{d}{2}\right) \text{const} \quad , \tag{4.81}$$

where we have again used that the relation $\Gamma(x+1) = x\Gamma(x)$ defines the gamma function uniquely, up to a constant. The remaining constant in

$$\int_0^\infty \mathrm{d}^d k \frac{\left(k^2\right)^\nu}{\left(k^2+M^2\right)^\mu} = \frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)} M^{2\nu-2\mu+d} \frac{\Gamma\left(\mu-\nu-\frac{d}{2}\right)\Gamma\left(\nu+\frac{d}{2}\right)}{\Gamma(\mu)} \text{const} \tag{4.82}$$

can easily be fixed. We choose $\nu = 1 - d/2$ and $\mu = 2$. With (4.68) we get

$$\begin{aligned} \int_0^\infty \mathrm{d}^d k \frac{\left(k^2\right)^{1-d/2}}{\left(k^2+M^2\right)^2} &= \frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)} \int_0^\infty \frac{\mathrm{d}k^2}{2} \frac{1}{\left(k^2+M^2\right)^2} \\ &= \frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)} \frac{1}{2} \left[\frac{-1}{k^2+M^2}\right]_0^\infty = \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)} \frac{1}{M^2} \end{aligned} \tag{4.83}$$

and

$$\frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)} M^{2-d-4+d} \frac{\Gamma(2-1)\Gamma(1)}{\Gamma(2)} \text{const} = \frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)} \frac{1}{M^2} \text{const} \quad . \tag{4.84}$$

From this we conclude that const $= 1/2$, or

$$\int_0^\infty \mathrm{d}^d k \frac{\left(k^2\right)^\nu}{\left(k^2+M^2\right)^\mu} = \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)} M^{2\nu-2\mu+d} \frac{\Gamma\left(\mu-\nu-\frac{d}{2}\right)\Gamma\left(\nu+\frac{d}{2}\right)}{\Gamma(\mu)} \quad . \tag{4.85}$$

This is an important formula, since any integral appearing in the calculations of Feynman graphs can be brought into this form. The steps are:

1. Introduce Feynman parameters according to

$$\begin{aligned} &\frac{1}{a_1^{\alpha_1} a_2^{\alpha_2} \ldots a_m^{\alpha_m}} \\ &\quad = \frac{\Gamma(\alpha_1+\alpha_2\ldots\alpha_m)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\ldots\Gamma(\alpha_m)} \int_0^1 \mathrm{d}x_1 \int_0^{x_1} \mathrm{d}x_2 \ldots \int_0^{x_{m-2}} \mathrm{d}x_{m-1} \\ &\qquad \times \frac{x_{m-1}^{\alpha_1-1}(x_{m-2}-x_{m-1})^{\alpha_2-1}\ldots(1-x_1)^{2m-1}}{[a_1 x_{m-1} + a_2(x_{m-2}-x_{m-1}) + \ldots + a_m(1-x_1)]^{\alpha_1+\alpha_2\ldots\alpha_m}} \end{aligned} \tag{4.86}$$

2. Shift the momentum variable of the loop k_μ such that the linear term in the denominator vanishes:

$$\int \mathrm{d}^d k \frac{f(k_\mu)}{\left(k^2+m^2\right)^\beta} \quad . \tag{4.87}$$

3. Use the fact that all k_μ have to be contracted with some other k_λ according to $k_\mu k_\lambda \to k^2/d$. Otherwise the term vanishes if individual components k_μ are inverted, $k_\mu \to -k_\mu$. The resulting expressions are sums of terms of the form (4.85).

For convenience we give a list of expressions which shows the result from different terms following steps 2 and 3. The integrals on the left-hand sides are in Minkowski space; therefore a factor "i" appears on the right owing to the Wick rotation.

$$\int \mathrm{d}^d k \frac{1}{\left(k^2+2k\cdot p+m^2\right)^{\alpha}} = \frac{\mathrm{i}\pi^{d/2}}{\left(m^2-p^2\right)^{\alpha-d/2}} \frac{\Gamma\left(\alpha-d/2\right)}{\Gamma(\alpha)} \quad . \tag{4.88}$$

$$\int \mathrm{d}^d k \frac{k_\mu}{\left(k^2+2k\cdot p+m^2\right)^{\alpha}} = \frac{\mathrm{i}\pi^{d/2}}{\left(m^2-p^2\right)^{\alpha-d/2}} \frac{\Gamma\left(\alpha-d/2\right)}{\Gamma(\alpha)}(-p_\mu) \quad . \tag{4.89}$$

$$\begin{aligned}\int \mathrm{d}^d k \frac{k^2}{\left(k^2+2k\cdot p+m^2\right)^{\alpha}} &= \frac{\mathrm{i}\pi^{d/2}}{\left(m^2-p^2\right)^{\alpha-d/2}} \frac{1}{\Gamma(\alpha)}\left[\Gamma\left(\alpha-\frac{d}{2}\right)p^2\right.\\ &\left.+\Gamma\left(\alpha-1-\frac{d}{2}\right)\frac{d}{2}\left(m^2-p^2\right)\right] \quad .\end{aligned} \tag{4.90}$$

$$\begin{aligned}\int \mathrm{d}^d k \frac{k_\mu k_\nu}{\left(k^2+2k\cdot p+m^2\right)^{\alpha}} &= \frac{\mathrm{i}\pi^{d/2}}{\left(m^2-p^2\right)^{\alpha-d/2}} \frac{1}{\Gamma(\alpha)}\left[\Gamma\left(\alpha-\frac{d}{2}\right)p_\mu p_\nu\right.\\ &\left.+\Gamma\left(\alpha-1-\frac{d}{2}\right)\frac{1}{2}\delta_{\mu\nu}\left(m^2-p^2\right)\right]\end{aligned} \tag{4.91}$$

$$\begin{aligned}\int \mathrm{d}^d k \frac{k_\mu k_\nu k_\lambda}{\left(k^2+2k\cdot p+m^2\right)^{\alpha}} &= \frac{\mathrm{i}\pi^{d/2}}{\left(m^2-p^2\right)^{\alpha-d/2}} \frac{1}{\Gamma(\alpha)}\left[-\Gamma\left(\alpha-\frac{d}{2}\right)p_\mu p_\nu p_\lambda\right.\\ &-\Gamma\left(\alpha-1-\frac{d}{2}\right)\left(m^2-p^2\right)\\ &\left.\times\frac{1}{2}(\delta_{\mu\nu}p_\lambda+\delta_{\mu\lambda}p_\nu+\delta_{\nu\lambda}p_\mu)\right] \quad .\end{aligned} \tag{4.92}$$

To check that (4.85) also reproduces (4.69) we use the following property of the Γ-function:

$$\begin{aligned}\Gamma(\beta+1) = \beta\Gamma(\beta) \Rightarrow \lim_{\beta\to 0}\Gamma(\beta) &= \lim_{\beta\to 0}\frac{1}{\beta}\Gamma(\beta+1)\\ &= \lim_{\beta\to 0}\frac{1}{\beta}\left[1+\beta\Gamma'(1)+O\left(\beta^2\right)\right] = \frac{1}{\beta}+\Gamma'(1) \quad .\end{aligned} \tag{4.93}$$

This implies that

$$\begin{aligned}&\lim_{\mu\to 0}\int \mathrm{d}^d k \frac{\left(k^2\right)^{\nu}}{\left(k^2+M^2\right)^{\mu}}\\ &= \lim_{\mu\to 0}\frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)}M^{2\nu+d}\Gamma\left(-\nu-\frac{d}{2}\right)\Gamma\left(\nu+\frac{d}{2}\right)\mu = 0 \quad .\end{aligned} \tag{4.94}$$

Equation (4.93) also allows us to write down in a uniform manner all the divergences that can occur on the right-hand side of (4.85). Such divergences occur if $\mu-\nu-d/2$ or $\nu+d/2$ are negative integers or zero. The Γ function has no zeros on the positive real axes, so that $1/\Gamma(\mu)$ does not create any divergences. With

$$\lim_{\varepsilon\to 0}\Gamma(-n+\varepsilon) = \lim_{\varepsilon\to 0}\frac{\Gamma(1-n+\varepsilon)}{-n+\varepsilon} = \lim_{\varepsilon\to 0}\frac{\Gamma(2-n+\varepsilon)}{(-n+\varepsilon)(1-n+\varepsilon)} = \dots \tag{4.95}$$

all divergences can be related to $\lim_{\varepsilon\to 0}\Gamma(\varepsilon)$ from (4.93).

Let us discuss as an example

$$\begin{aligned}\lim_{\varepsilon\to 0}\Gamma(-3+\varepsilon) &= \lim_{\varepsilon\to 0}\frac{\Gamma(\varepsilon)}{(-3+\varepsilon)(-2+\varepsilon)(-1+\varepsilon)}\\ &= \lim_{\varepsilon\to 0}\left(-\frac{1}{6}\right)\left(1+\frac{\varepsilon}{3}+\frac{\varepsilon}{2}+\varepsilon\right)\left(\frac{1}{\varepsilon}+\Gamma'(1)\right)\\ &= \lim_{\varepsilon\to 0}\left(\frac{-1}{6\varepsilon}\right)-\frac{1}{6}\left(\Gamma'(1)+\frac{1}{3}+\frac{1}{2}+1\right)\\ &= -\frac{\Gamma'(1)+\frac{11}{6}}{6}-\frac{1}{6}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\quad .\end{aligned}\tag{4.96}$$

All possible divergences (which can be only logarithmic for renormalizable theories) are therefore proportional to $1/\varepsilon$, and (4.85) and (4.96) already determine dimensional regularization.

Dimensional regularization, i.e., the separation of the divergent and finite parts, is realized by identifying the $1/\varepsilon$ terms with the divergent part and the rest with the finite one. The $1/\varepsilon$ terms are then absorbed into the renormalization constants during renormalization.

To end this section let us mention a second valuable formula for dimensional regularization ($d = 4 + 2\varepsilon$):

$$\int \mathrm{d}^d x\,\frac{\mathrm{e}^{\mathrm{i}p\cdot x}}{(-x^2)^\nu} = -\mathrm{i}(-)^\nu\pi^2\frac{\Gamma(2-\nu+\varepsilon)}{\Gamma(\nu)}\left(\frac{-4\pi^2\mu^2}{p^2}\right)^\varepsilon\left(\frac{-p^2}{4}\right)^{\nu-2}\quad ,$$

which allows us to perform d-dimensional Fourier transformations. It is derived in Exercise 4.5.

EXERCISE

4.5 The d-Dimensional Fourier Transform

Problem. Derive the equation

$$\int \mathrm{d}^d x\,\frac{\mathrm{e}^{\mathrm{i}p\cdot x}}{(-x^2)^\nu} = -\mathrm{i}\pi^2\frac{\Gamma(2-\nu+\varepsilon)}{\Gamma(\nu)}\left(\frac{-4\pi^2\mu^2}{p^2}\right)^\varepsilon\left(\frac{-p^2}{4}\right)^{\nu-2}\quad .\tag{1}$$

Solution. In three dimensions plane waves can be expanded into Bessel functions and Legendre polynomials according to

$$\mathrm{e}^{\mathrm{i}p\cdot x} = \mathrm{e}^{\mathrm{i}|p|\,|x|\,\cos\theta} = \dots\quad .\tag{2}$$

To prove (1) we need the generalization of this expansion to arbitrary dimensions. The exponential itself looks the same for any Euclidian dimensional, namely

$$\mathrm{e}^{\mathrm{i}p\cdot x} = \mathrm{e}^{\mathrm{i}|p|\,|x|\,\cos\theta}\quad .\tag{3}$$

Therefore the dimensionality enters only in the orthogonality property. The functions of θ that we call $C_l(\theta)$ have to be orthogonal with the weight $(\sin\theta)^{d-2}$:

Example 4.5.

$$\begin{aligned}
&\int \mathrm{d}^d\Omega\, C_i(\cos\theta)\, C_j(\cos\theta) \\
&\quad \approx \int\limits_0^{\pi} \mathrm{d}\theta (\sin\theta)^{d-2}\, C_i(\cos\theta)\, C_j(\cos\theta) \\
&\quad = \int\limits_{-1}^{1} \mathrm{d}\cos\theta\, (\sin^2\theta)^{(d-3)/2}\, C_i(\cos\theta)\, C_j(\cos\theta) \\
&\quad = \int\limits_{-1}^{1} \mathrm{d}x\, (1-x^2)^{(d-3)/2}\, C_i(x)\, C_j(x) \quad . \qquad (4)
\end{aligned}$$

Orthogonal polynomials with the weight $(1-x^2)^{d-1/2}$ are the "Gegenbauer polynomials" $C_n^{(\alpha)}(x)$.[3] The important properties for us are

$$\mathrm{e}^{\mathrm{i}px\cos\theta} = \Gamma(\nu)\left(\frac{px}{2}\right)^{-\nu} \sum_{k=0}^{\infty} (\nu+k)\, \mathrm{i}^k J_{\nu+k}(px)\, C_k^{(\nu)}(\cos\theta) \qquad (5)$$

with arbitrary ν,

$$\begin{aligned}
&\int_{-1}^{1} \mathrm{d}x\, (1-x^2)^{\alpha-1/2}\, C_n^{(\alpha)}(x) C_{n'}^{(\alpha)}(x) \\
&\quad = \delta_{nn'} \frac{\pi 2^{1-2\alpha}\Gamma(n+2\alpha)}{n!\,(n+\alpha)\,[\Gamma(\alpha)]^2} \quad \left(\alpha > -\frac{1}{2}\right) \quad , \qquad (6)
\end{aligned}$$

and

$$C_0^{(\alpha)}(x) = 1 \quad . \qquad (7)$$

From (6) and (7) we find that

$$\begin{aligned}
&\int\limits_{-1}^{1} \mathrm{d}x\, (1-x^2)^{\alpha-1/2} C_n^{(\alpha)}(x) \\
&\quad = \int\limits_{-1}^{1} \mathrm{d}x\, (1-x^2)^{\alpha-1/2} C_n^{(\alpha)}(x) C_0^{(\alpha)}(x) \\
&\quad = \delta_{n0} \frac{\pi 2^{1-2\alpha}\Gamma(2\alpha)}{\alpha[\Gamma(\alpha)]^2} \\
&\quad = \delta_{n0} \int\limits_{-1}^{1} \mathrm{d}x\, (1-x^2)^{\alpha-1/2} \quad . \qquad (8)
\end{aligned}$$

Comparing (8) with the d-dimensional integral we find that α has to be chosen as $\alpha = \frac{d}{2} - 1$. The angular integral is now easy to perform. We first substitute $x^0 \to -\mathrm{i}x_\mathrm{E}^0$ to go to Euclidian coordinates. We also substitute $p^0 \to \mathrm{i}p_\mathrm{E}^0$.

---

[3] Their properties can be found for example in M. Abramowitz and A. Stegun, *Handbook of Mathematical Functions*, Chap. 22.

Example 4.5.

$$
\begin{aligned}
&-\mathrm{i}\int \mathrm{d}^d x_{\mathrm{E}}\,\frac{\mathrm{e}^{\mathrm{i}p_{\mathrm{E}}x_{\mathrm{E}}\cos\theta}}{(x_{\mathrm{E}}^2)^\nu}\\
&= -\mathrm{i}\int_0^\infty \mathrm{d}x_{\mathrm{E}}\,\frac{x_{\mathrm{E}}^{d-1}}{(x_{\mathrm{E}}^2)^\nu}\,\Gamma(\alpha)\left(\frac{p_{\mathrm{E}}x_{\mathrm{E}}}{2}\right)^{-\alpha}\\
&\quad\times\sum_{k=0}^\infty(\alpha+k)\mathrm{i}^k\,J_{\alpha+k}(p_{\mathrm{E}}x_{\mathrm{E}})\int \mathrm{d}^d\Omega\, C_k^{(\alpha)}(\cos\theta)\\
&= -\mathrm{i}\int_0^\infty \mathrm{d}x_{\mathrm{E}}\,(x_{\mathrm{E}})^{d-1-2\nu}\,\Gamma(\alpha)\left(\frac{p_{\mathrm{E}}x_{\mathrm{E}}}{2}\right)^{-\alpha}\alpha J_\alpha(p_{\mathrm{E}}x_{\mathrm{E}})\int \mathrm{d}^d\Omega\\
&= -\mathrm{i}\frac{2\pi^{d/2}}{\Gamma(d/2)}\Gamma(d/2-1)\left(\frac{p_{\mathrm{E}}}{2}\right)^{1-d/2}\left(\frac{d}{2}-1\right)\\
&\quad\times\int_0^\infty \mathrm{d}x_{\mathrm{E}}\,(x_{\mathrm{E}})^{d-1-2\nu-d/2+1}J_{d/2-1}(p_{\mathrm{E}}x_{\mathrm{E}})\\
&= -\mathrm{i}2\pi^{d/2}\left(\frac{p_{\mathrm{E}}}{2}\right)^{1-d/2}\int_0^\infty \mathrm{d}x_{\mathrm{E}}\,x_{\mathrm{E}}^{d/2-2\nu}J_{d/2-1}(p_{\mathrm{E}}x_{\mathrm{E}})\\
&= -\mathrm{i}2\pi^{d/2}\left(\frac{p_{\mathrm{E}}}{2}\right)^{1-d/2}(p_{\mathrm{E}})^{-1-d/2+2\nu}\int_0^\infty \mathrm{d}y\,y^{d/2-2\nu}J_{d/2-1}(y)\quad. \qquad (9)
\end{aligned}
$$

The remaining integral can be found, in appropriate integral tables.[4]

$$
\int_0^\infty \mathrm{d}y\,y^{d/2-2\nu}J_{d/2-1}(y) = 2^{d/2-2\nu}\frac{\Gamma\left(\frac{d}{4}+\frac{d}{4}-\nu\right)}{\Gamma\left(\frac{d}{4}-\frac{d}{4}+\nu\right)}\quad. \qquad (10)
$$

Putting all this together we get

$$
\int \mathrm{d}^d x\,\frac{\mathrm{e}^{\mathrm{i}p\cdot x}}{(-x^2)^\nu} = -\mathrm{i}\pi^{d/2}(p_{\mathrm{E}})^{2\nu-d}\,2^{d/2}\,2^{d/2-2\nu}\frac{\Gamma(d/2-\nu)}{\Gamma(\nu)}\quad. \qquad (11)
$$

Finally we insert again $d = 4 + 2\varepsilon$ to obtain the result

$$
\int \mathrm{d}^d x\,\frac{\mathrm{e}^{\mathrm{i}p\cdot x}}{(-x^2)^\nu} = -\mathrm{i}\pi^2\left(\frac{p_E^2}{4}\right)^{\nu-2}\left(\frac{4\pi}{p_E^2}\right)^{\varepsilon}\frac{\Gamma(2-\nu+\varepsilon)}{\Gamma(\nu)}\quad, \qquad (12)
$$

which completes our proof.

---

[4] See, for example, I. Gradshtein and I. Ryshik: *Tables of Series, Products and Integrals*, No. 6.151.14 (Harri Deutsch, Frankfurt am Main (1981)).

4.4 The Renormalized Coupling Constant of QCD

We shall now use dimensional regularization to calculate the renormalized QCD coupling constant to lowest order. The divergent graphs contributing to the renormalization of g_s belong to three classes, shown in Fig. 4.14, 4.15, and 4.16.

We start with vacuum polarization, with the fermion graph. Using the QCD Feynman rules it is easy to write down the polarization tensor:

$$\Pi^{(\mathrm{a1})}_{\mu\mu'}(k) = \sum_{i=1}^{N_\mathrm{F}} g^2 \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \times \mathrm{tr}\left\{ \frac{\lambda^a}{2}\frac{\lambda^{a'}}{2} \frac{\not{q}+\not{k}+m}{(q+k)^2-m^2+\mathrm{i}\eta}\gamma_\mu \frac{\not{q}+m}{q^2-m^2+\mathrm{i}\eta}\gamma_{\mu'} \right\} . \tag{4.97}$$

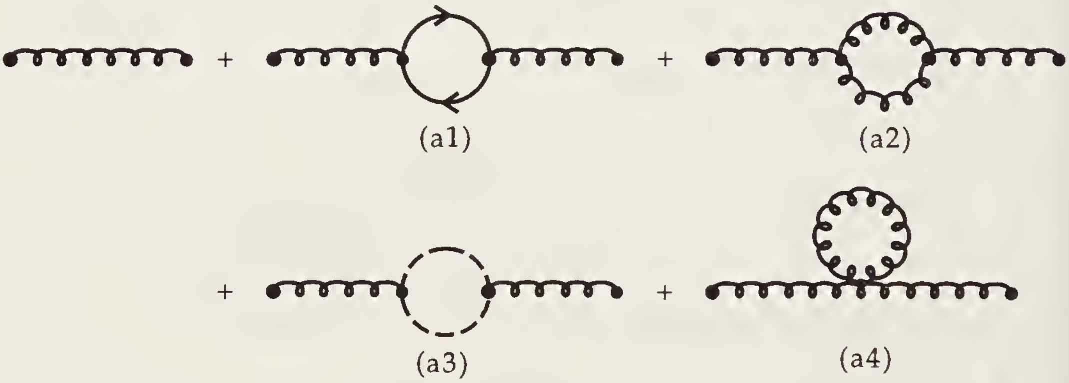

(a1) (a2) (a3) (a4)

Fig. 4.14. The vacuum polarization graphs of QCD

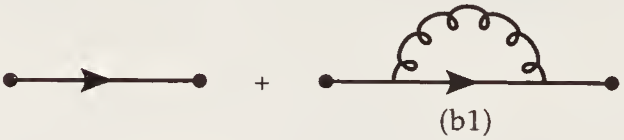

(b1)

Fig. 4.15. The self energy graphs of QCD

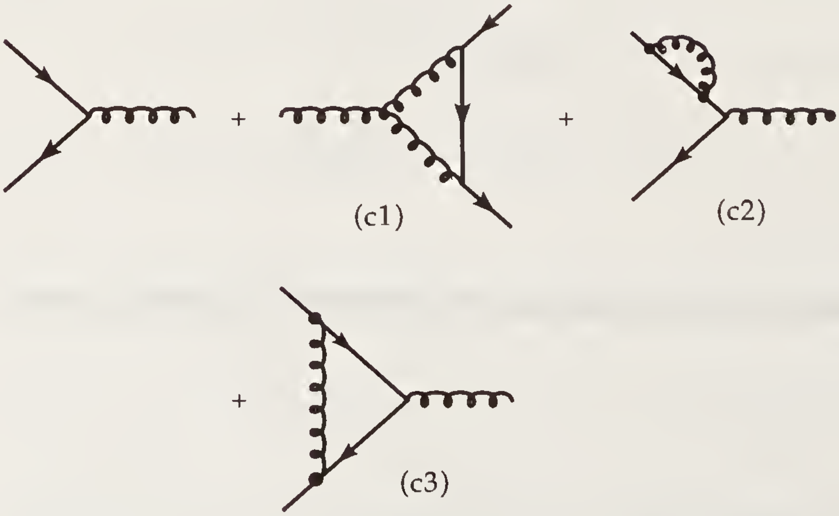

(c1) (c2) (c3)

Fig. 4.16. The vertex correction graphs of QCD

N_F is the number of quarks in the theory, or more precisely the number of quarks with $m^2 \ll |k^2|$. The whole calculation can be carried through for arbitrary m^2, but one finds that the contribution is suppressed for $m^2 > |k^2|$. N_F therefore counts only the light flavors and for simplicity we can set $m = 0$ in what follows whenever this does not lead to infrared divergencies. The trace over the color indices gives simply

$$\operatorname{tr}\left\{\frac{\lambda^a}{2}\frac{\lambda^{a'}}{2}\right\} = \delta_{aa'}\frac{1}{2} \quad . \tag{4.98}$$

As usual we introduce Feynman parameters

$$\begin{aligned}
&\frac{1}{(q+k)^2 - m^2 + \mathrm{i}\eta}\,\frac{1}{q^2 - m^2 + \mathrm{i}\eta} \\
&\quad = \int_0^1 \mathrm{d}z\, \frac{1}{[q^2 + 2kqz + (k^2 - m^2)z - m^2(1-z) + \mathrm{i}\eta]^2} \\
&\quad = \int_0^1 \mathrm{d}z\, \frac{1}{[(q+kz)^2 + k^2 z(1-z) - m^2 + \mathrm{i}\eta]^2} \quad .
\end{aligned} \tag{4.99}$$

To get rid of the linear term in the denominator we substitute $q_\mu \to q_\mu - k_\mu z$:

$$\Pi^{(\mathrm{al})}_{\mu\mu'}(k) = g^2 \frac{N_F}{2} \int_0^1 \mathrm{d}z \int_0^\infty \frac{\mathrm{d}^4 q}{(2\pi)^4}\, \frac{\operatorname{tr}\{[\not{q} + \not{k}(1-z)]\gamma_\mu[\not{q} - \not{k}z]\gamma_{\mu'}\}}{[q^2 + k^2 z(1-z) - m^2 + \mathrm{i}\eta]^2} \quad . \tag{4.100}$$

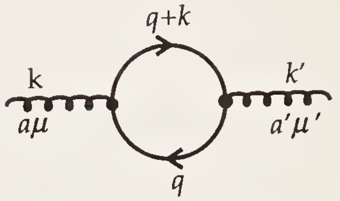


Fig. 4.17. The chosen variables for the quark loop contribution to vacuum polarization

Next we take the trace and neglect all terms proportional to odd powers of q. They are zero as can be seen by substituting q_ν for $-q_\nu$. For the same reason $q_\mu q_{\mu'}$ can be substituted by $g_{\mu\mu'}q^2/4$

$$\begin{aligned}
\Pi^{(\mathrm{al})}_{\mu\mu'}(k) &= g^2 \frac{N_F}{2} 4 \int_0^1 \mathrm{d}z \\
&\quad \times \int_0^\infty \frac{\mathrm{d}^4 q}{(2\pi)^4}\, \frac{2q_\mu q_{\mu'} - q^2 g_{\mu\mu'} - z(1-z)(2k_\mu k_{\mu'} - k^2 g_{\mu\mu'})}{[q^2 + k^2 z(1-z) - m^2 + \mathrm{i}\eta]^2} \\
&= 2g^2 N_F \int_0^1 \mathrm{d}z \\
&\quad \times \int_0^\infty \frac{\mathrm{d}^4 q}{(2\pi)^4}\, \frac{-\frac{1}{2}q^2 g_{\mu\mu'} - z(1-z)(2k_\mu k_{\mu'} - k^2 g_{\mu\mu'})}{[q^2 + k^2 z(1-z) - m^2 + \mathrm{i}\eta]^2}
\end{aligned} \tag{4.101}$$

Now only q^2 appears in the integrand, so it can safely be continued to Euclidian space, $q_\mu \to \tilde{q}_\mu = (\mathrm{i}q^0, \boldsymbol{q})$:

$$\Pi^{(\mathrm{al})}_{\mu\mu'}(k) \Rightarrow 2g^2 N_{\mathrm{F}} \int\limits_0^1 \mathrm{d}z \int\limits_0^\infty \mathrm{i}\, \frac{\mathrm{d}^d\tilde{q}}{(2\pi)^4}\, \mu^{4-d}\, \frac{\frac{1}{2}\tilde{q}^2 g_{\mu\mu'} - z(1-z)(2k_\mu k_{\mu'} - k^2 g_{\mu\mu'})}{[\tilde{q}^2 - k^2 z(1-z) + m^2 - \mathrm{i}\eta]^2} \tag{4.102}$$

We have introduced a dimensional quantity μ to keep the total dimension of the expression unchanged. With (4.85) it follows that

$$\begin{aligned}\Pi^{(\mathrm{al})}_{\mu\mu'}(k) = {} & \frac{2\mathrm{i}g^2 N_{\mathrm{F}}}{(2\pi)^4} \int\limits_0^1 \mathrm{d}z \frac{\pi^{\frac{d}{2}}\mu^{4-d}}{\Gamma\left(\frac{d}{2}\right)} \left[m^2 - k^2 z(1-z) - \mathrm{i}\eta\right]^{\frac{d}{2}-2} \\ & \times \frac{\Gamma\left(2-\frac{d}{2}\right)\Gamma\left(\frac{d}{2}\right)}{\Gamma(2)} [-z(1-z)] \left(2k_\mu k_{\mu'} - k^2 g_{\mu\mu'}\right) \\ & + \frac{2\mathrm{i}g^2 N_{\mathrm{F}}}{(2\pi)^4} \frac{g_{\mu\mu'}}{2} \int\limits_0^1 \mathrm{d}z \frac{\pi^{\frac{d}{2}}\mu^{4-d}}{\Gamma\left(\frac{d}{2}\right)} \left[m^2 - k^2 z(1-z) - \mathrm{i}\eta\right]^{\frac{d}{2}-1} \\ & \times \frac{\Gamma\left(1-\frac{d}{2}\right)\Gamma\left(1+\frac{d}{2}\right)}{\Gamma(2)} \quad .\end{aligned} \tag{4.103}$$

The terms on the right-hand side are divergent. For $d = 4 - 2\varepsilon$ the first is proportional to

$$\Gamma\left(2 - \frac{d}{2}\right) = \Gamma(\varepsilon) = \frac{1}{\varepsilon} + \Gamma'(1) + O(\varepsilon) \quad , \tag{4.104}$$

while the second term is proportinal to

$$\begin{aligned}\frac{\Gamma\left(1-\frac{d}{2}\right)\Gamma\left(1+\frac{d}{2}\right)}{\Gamma\left(\frac{d}{2}\right)} & = \frac{4-2\varepsilon}{2}\Gamma(-1+\varepsilon) \\ & = (2-\varepsilon)\frac{\Gamma(\varepsilon)}{(-1+\varepsilon)} \\ & = (2-\varepsilon)(-1-\varepsilon)\left(\frac{1}{\varepsilon} + \Gamma'(1)\right) + O(\varepsilon) \\ & = -\frac{2}{\varepsilon} - 1 - 2\Gamma'(1) + O(\varepsilon) \quad .\end{aligned} \tag{4.105}$$

We thus have

$$\begin{aligned}\Pi^{(\mathrm{al})}_{\mu\mu'}(k) = {} & \frac{2\mathrm{i}g^2 N_{\mathrm{F}}}{(2\pi)^4}\, \mu^{4-d}\, \pi^{\frac{d}{2}} \left(\frac{1}{\varepsilon} + \Gamma'(1)\right) \\ & \times \int\limits_0^1 \mathrm{d}z\, \frac{-z(1-z)}{[-k^2 z(1-z)]^{\varepsilon}} \left(2k_\mu k_{\mu'} - k^2 g_{\mu\mu'}\right) \\ & - \frac{2\mathrm{i}g^2 N_{\mathrm{F}}}{(2\pi)^4}\, \mu^{4-d}\, \pi^{\frac{d}{2}} \left(\frac{2}{\varepsilon} + 1 + 2\Gamma'(1)\right) \\ & \times \int\limits_0^1 \mathrm{d}z\, [-k^2 z(1-z)]^{1-\varepsilon} \frac{1}{2} g_{\mu\mu'} \quad .\end{aligned} \tag{4.106}$$

It can now be seen that m^2 can safely be neglected. The z integrals are elementary:

$$
\begin{aligned}
\int_0^1 \mathrm{d}z \ \ [z(1-z)]^{1-\varepsilon} &= \frac{\Gamma(2-\varepsilon)\,\Gamma(2-\varepsilon)}{\Gamma(4-2\varepsilon)} \\
&= \frac{[1-\varepsilon\Gamma'(2)]^2}{3!-2\varepsilon\Gamma'(4)} \\
&= \frac{1}{3!} - \frac{\varepsilon}{3}\Gamma'(2) + \frac{\varepsilon}{3}\Gamma'(4) \quad , \qquad (4.107)
\end{aligned}
$$

$$
\begin{aligned}
\Pi^{(\mathrm{a1})}_{\mu\mu'}(k) &= \frac{2\mathrm{i}g^2N_\mathrm{F}}{16\pi^2}\,\pi^\varepsilon\,\mu^{2\varepsilon}\left(-2k_\mu k_{\mu'}+2k^2g_{\mu\mu'}\right)\left(-k^2\right)^{-\varepsilon}\frac{1}{3!}\left(\frac{1}{\varepsilon}+\mathrm{const}\right) \\
&= g^2\mathrm{i}\frac{N_\mathrm{F}}{16\pi^2}\,\frac{2}{3}\left(\frac{\pi\mu^2}{-k^2}\right)^\varepsilon\left(\frac{1}{\varepsilon}+\mathrm{const}\right)\left(g_{\mu\mu'}k^2-k_\mu k_{\mu'}\right)+\ldots \quad (4.108)
\end{aligned}
$$

Finally we expand $y^\varepsilon = 1+\varepsilon\ln(y)$

$$
\Pi^{(\mathrm{a1})}_{\mu\mu'}(k) = g^2\mathrm{i}\frac{N_\mathrm{F}}{16\pi^2}\,\frac{2}{3}\left[\frac{1}{\varepsilon}+\ln\left(\frac{\pi\mu^2}{-k^2}\right)+\mathrm{const}\right]\left(g_{\mu\mu'}k^2-k_\mu k_{\mu'}\right) \quad . \quad (4.109)
$$

Note that every $1/\varepsilon$ term is always accompanied by a term $\ln(-k^2/\mu^2)$.

With $\ln(-k^2/\pi\mu^2) = \ln(-k^2/\mu^2) - \ln(\pi)$ we can write

$$
\Pi^{(\mathrm{a1})}_{\mu\mu'}(k) = g^2\mathrm{i}\ \frac{N_\mathrm{F}}{16\pi^2}\,\frac{2}{3}\left(\frac{1}{\varepsilon}-\ln\left(\frac{-k^2}{\mu^2}\right)+\mathrm{const.}\right)\left(g_{\mu\mu'}k^2-k_\mu k_{\mu'}\right) \ .(4.110)
$$

Next we calculate the gluon loop with two 3-gluon vertices:

$$
\begin{aligned}
\Pi^{(\mathrm{a2})}_{\mu\mu'}(k) = g^2\frac{1}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\Big\{ & f^{abc}\left[g_{\mu\lambda}(k-q)_\nu+g_{\nu\mu}(-q-2k)_\lambda+g_{\lambda\nu}(q+q+k)_\mu\right] \\
& \times f^{a'bc}\left[g_{\mu'\lambda}(-k+q)_\nu+g_{\nu\mu'}(q+2k)_\lambda+g_{\nu\lambda}(-2q-k)_{\mu'}\right]\Big\} \\
& \times\frac{1}{(q+k)^2+\mathrm{i}\eta}\,\frac{1}{q^2} \quad . \qquad (4.111)
\end{aligned}
$$

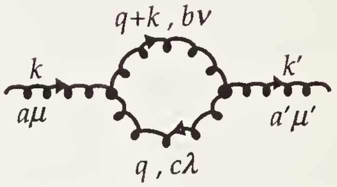


Fig. 4.18. The chosen variables from the first gluon loop contribution to vacuum polarization

The factor 1/2 in front follows from combinatorics. Such combinatoric factors are fairly easy to derive, as we shall now demonstrate for all the graphs relevant for our calculation.

Let us start with the gluon loop. The 3-gluon vertex contains 6 terms, correspondingly to the 3! orientations of the 3-gluon vertex. The total symmetry factor is therefore $(3!)^2 \cdot \frac{1}{2!}$, see Fig. 4.19.

As the two factors $6^2 = (3!)^2$ are absorbed in the 2×6 terms of the two 3-gluon vertices we are left with a factor 1/2. For the other graphs the corresponding factors are derived in Fig. 4.20.

Following this scheme it is easy to obtain (see Figs. 4.21–4.24) any symmetry factor of interest. The whole expression for $\Pi^{(\mathrm{a2})}_{\mu\mu'}$ is obviously very similar to (4.97) except for the different nominator, which we simplify first using

$$
f^{abc}f^{a'bc} = \delta_{aa'}C_2 = \delta_{aa'}N \quad \text{for} \quad \mathrm{SU}(N) \quad . \qquad (4.112)
$$

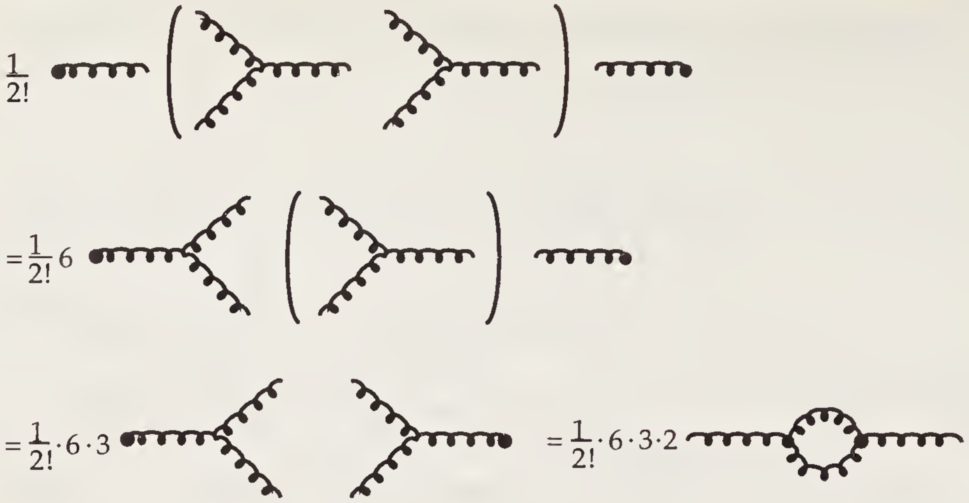


Fig. 4.19. The symmetry factor for the graph in Fig. 4.18. The factor 1/2! is due to the fact that we are considering second order in perturbation theory

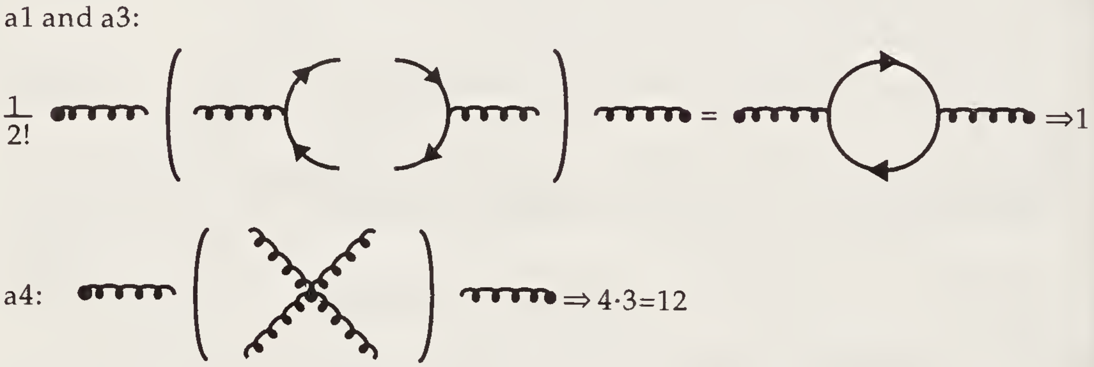


Fig. 4.20. The symmetry factor for the other graphs in Fig. 4.14

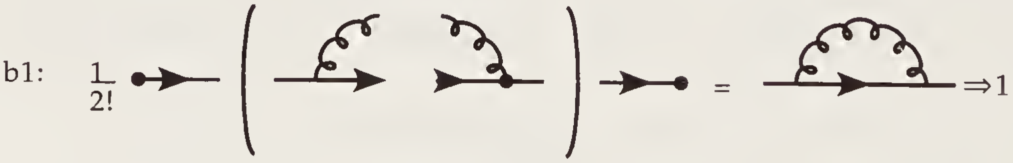


Fig. 4.21. The symmetry factor for the second graph in Fig. 4.15

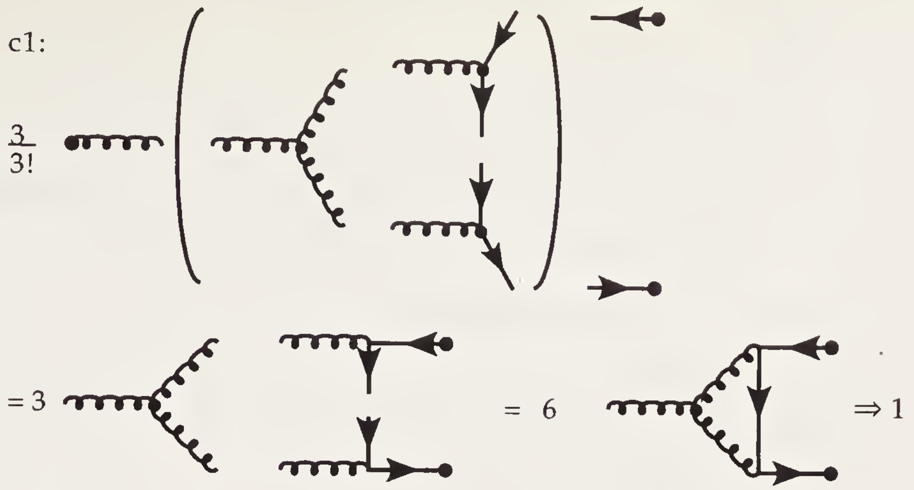


Fig. 4.22. The symmetry factor for the second graph in Fig. 4.16 (3rd-order perturbation theory implies 1/3! binomial for the combination of one 3-gluon vertex and two quark gluon vertices: $\binom{2}{3} = \frac{3!}{2!\,1!} = 3$)

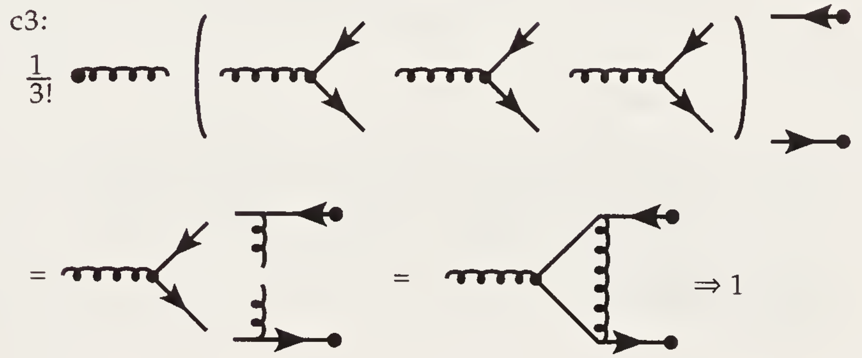


Fig. 4.23. The symmetry factor for the fourth graph in Fig. 4.16

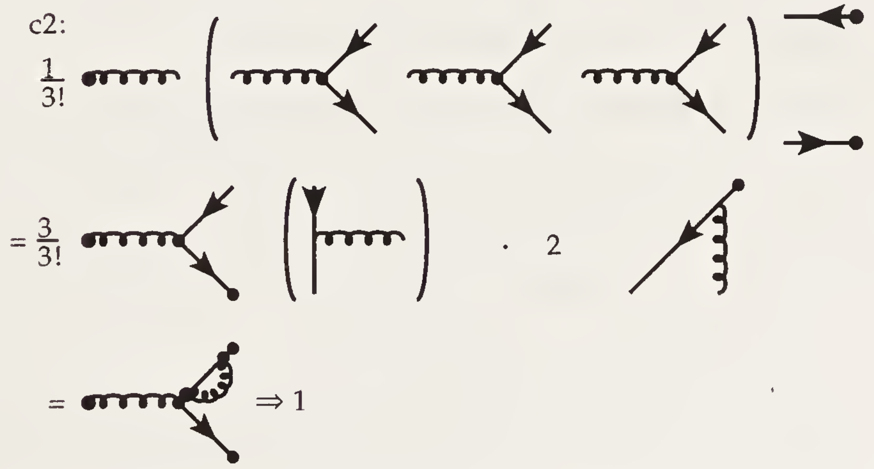


Fig. 4.24. The symmetry factor for the third graph in Fig. 4.16

We get

$$
\begin{aligned}
\{\ldots\} = 3\delta_{aa'} \Big[& -g_{\mu\mu'}(k-q)^2 + (k-q)_{\mu'}(q+2k)_\mu - (k-q)_\mu(k+2q)_{\mu'} \\
& -(q-k)_\mu(2k+q)_{\mu'} - g_{\mu\mu'}(q+2k)^2 + (2q+k)_{\mu'}(q+2k)_\mu \\
& +(q-k)_{\mu'}(2q+k)_\mu + (q+2k)_{\mu'}(2q+k)_\mu \\
& -4(2q+k)_\mu(2q+k)_{\mu'} \Big] \\
= 3\delta_{aa'} \Big[& -g_{\mu\mu'}\left(k^2 - 2k\cdot q + q^2 + q^2 + 4k\cdot q + 4k^2\right) \\
& + k_\mu k_{\mu'}(2-1+2+2-1+2-4) \\
& + k_\mu q_{\mu'}(-2-2+1+4+1+1-8) \\
& + q_\mu k_{\mu'}(1+1-2+1-2+4-8) \\
& + q_\mu q_{\mu'}(-1+2-1+2+2+2-16) \Big] \\
= 3g^2\delta_{aa'} \big[& -g_{\mu\mu'}\left(2q^2 + 2k\cdot q + 5k^2\right) + 2k_\mu k_{\mu'} \\
& -5k_\mu q_{\mu'} - 5k_{\mu'}q_\mu - 10q_\mu q_{\mu'} \big] \quad . \qquad (4.113)
\end{aligned}
$$

We again substitute

$$
q_\mu \longrightarrow q_\mu - k_\mu z \qquad (4.114)
$$

and keep only the terms with even powers in q_μ

$$
\begin{aligned}
\{\ldots\} \Rightarrow 3\delta_{aa'} \big[& -g_{\mu\mu'}\left(2q^2 + 2k^2z^2 - 2k^2z + 5k^2\right) \\
& +2k_\mu k_{\mu'} + 5k_\mu k_{\mu'}z + 5k_\mu k_{\mu'}z - 10k_\mu k_{\mu'}z^2 - 10q_\mu q_{\mu'} \big] \\
\Rightarrow 3\delta_{aa'} \Big[& -g_{\mu\mu'}\left(-2k^2z(1-z) + 5k^2\right) \\
& +2k_\mu k_{\mu'} + 10k_\mu k_{\mu'}z(1-z) - 2g_{\mu\mu'}q^2 - \frac{5}{2}g_{\mu\mu'}q^2 \Big] \\
=: 3\delta_{aa'} & \left(A_{\mu\mu'} + B_{\mu\mu'}q^2\right) \quad , \qquad (4.115)
\end{aligned}
$$

defining $A_{\mu\mu'}$ and $B_{\mu\mu'}$ to abbreviate the corresponding terms. The analog to (4.102) is now

$$
\begin{aligned}
\Pi^{(a2)}_{\mu\mu'}(k) &= \frac{3}{2}g^2\delta_{aa'}\mathrm{i}\int\limits_0^1 \mathrm{d}z \int\limits_0^\infty \frac{\mathrm{d}^d\tilde{q}}{(2\pi)^4}\,\mu^{4-d}\,\frac{-B_{\mu\mu'}\tilde{q}^2 + A_{\mu\mu'}}{[\tilde{q}^2 - k^2z(1-z) - \mathrm{i}\eta]^2} \\
&= \frac{3}{2}g^2\delta_{aa'}\mathrm{i}\int\limits_0^1 \mathrm{d}z\, \frac{\mu^{4-d}\pi^{\frac{d}{2}}}{(2\pi)^4}\,\frac{1}{\varepsilon}\,\left[A_{\mu\mu'} + 2B_{\mu\mu'}k^2z(1-z)\right] \\
&\quad \times \left(-k^2z(1-z)\right)^{-\varepsilon} + O(\varepsilon) \quad . \qquad (4.116)
\end{aligned}
$$

In the last step we have directly inserted (4.106). The z integral now becomes

$$
\begin{aligned}
&g_{\mu\mu'}\left(2k^2+\frac{9}{2}k^2 2\right)\int\limits_0^1 \mathrm{d}z\ [z(1-z)]^{1-\varepsilon} - 5g_{\mu\mu'}k^2\int\limits_0^1 \mathrm{d}z\ [z(1-z)]^{-\varepsilon} \\
&\quad + 2k_\mu k_{\mu'}\int\limits_0^1 \mathrm{d}z\ [z(1-z)]^{-\varepsilon} + 10k_\mu k_{\mu'}\int\limits_0^1 \mathrm{d}z\ [z(1-z)]^{1-\varepsilon} \\
&\quad = \frac{11}{6}g_{\mu\mu'}k^2 - 5g_{\mu\mu'}k^2 + 2k_\mu k_{\mu'} + 10k_\mu k_{\mu'}\frac{1}{6} + O(\varepsilon) \\
&\quad = -\frac{19}{6}g_{\mu\mu'}k^2 + \frac{22}{6}k_\mu k_{\mu'} + O(\varepsilon) \quad , \qquad (4.117)
\end{aligned}
$$

and all together we get

$$
\begin{aligned}
\Pi^{(\mathrm{a2})}_{\mu\mu'}(k) = {}& \frac{3\mathrm{i}g^2\delta_{aa'}}{(16\pi)^2}\frac{1}{12}\left[\frac{1}{\varepsilon} - \ln\left(\frac{-k^2}{\mu^2}\right) + \mathrm{const}\right] \\
& \left(-19g_{\mu\mu'}k^2 + 22k_\mu k_{\mu'}\right) \quad . \qquad (4.118)
\end{aligned}
$$

This expression is not gauge invariant ($k^\mu \Pi^{(\mathrm{a2})}_{\mu\mu'} \neq 0$), which is quite reasonable, since gauge invariance is only restored by including the ghost terms. More precisely the ghost fields are constructed such that they cancel the contribution from the unphysical degrees of freedom of the gluon field. These unphysical degrees of freedom violate the gauge symmetry. To see how this happens we next calculate the ghost-contribution,

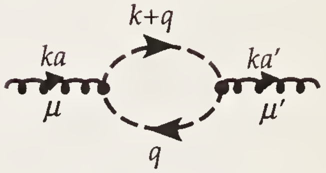


Fig. 4.25. The chosen variables for the ghost loop contribution to vacuum polarization

$$
\Pi^{(\mathrm{a3})}_{\mu\mu'}(k) = -g^2 f^{abc}f^{a'cb}\int \frac{\mathrm{d}^4q}{(2\pi)^4}\frac{q_{\mu'}(k+q)_\mu}{[(q+k)^2+\mathrm{i}\eta](q^2+\mathrm{i}\eta)} \qquad (4.119)
$$

Using (4.112) and repeating the introduction of a Feynman parameter and shifting according to $q_\mu \rightarrow q_\mu - k_\mu z$ we obtain

$$
\begin{aligned}
\Pi^{(\mathrm{a3})}_{\mu\mu'}(k) &= 3g^2\delta_{aa'}\int\limits_0^1 \mathrm{d}z\int\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{\frac{1}{4}g_{\mu\mu'}q^2 - k_\mu k_{\mu'}z(1-z)}{\left(q^2+k^2z(1-z)+\mathrm{i}\eta\right)^2} \qquad (4.120) \\
&= \frac{3g^2\delta_{aa'}\mathrm{i}}{16\pi^2}\left(\frac{\pi\mu^2}{-k^2}\right)^\varepsilon\left(\frac{1}{\varepsilon}+\mathrm{const}\right)\int\limits_0^1 \mathrm{d}z\left(-\frac{2}{4}g_{\mu\mu'}k^2 - k_\mu k_{\mu'}\right)(1-z)z \\
&= \frac{3g^2\delta_{aa'}\mathrm{i}}{16\pi^2}\left[\frac{1}{\varepsilon} - \ln\left(\frac{-k^2}{\mu^2}\right) + \mathrm{const}\right]\frac{1}{12}\left(-g_{\mu\mu'}k^2 - 2k_\mu k_{\mu'}\right) \quad . \qquad (4.121)
\end{aligned}
$$

Together this gives

$$
\begin{aligned}
&\Pi^{(\mathrm{a2})}_{\mu\mu'}(k) + \Pi^{(\mathrm{a3})}_{\mu\mu'}(k) \\
&\quad = \frac{3g^2\delta_{aa'}\mathrm{i}}{16\pi^2}\left[\frac{1}{\varepsilon} - \ln\left(\frac{-k^2}{\mu^2}\right) + \mathrm{const}\right]\frac{20}{12}\left(k_\mu k_{\mu'} - g_{\mu\mu'}k^2\right) \quad . \qquad (4.122)
\end{aligned}
$$

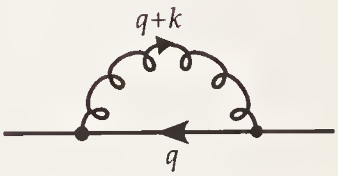


Fig. 4.26. The chosen variables for the self-energy graph

The sum of the gluon-plus-ghost vacuum graph is thus again gauge invariant. This illustrates nicely the role of the ghost fields, namely to substract out the unphysical components of the A_μ field, which are fixed by the chosen gauge.

Next we calculate the self-energy graph

$$\Sigma^{\mathrm{b}}(k) = g^2 \int \frac{\mathrm{d}^4 q}{(2\pi)^4}\gamma_\mu \frac{\not{q}}{q^2+\mathrm{i}\eta}\gamma^\mu \frac{1}{(q+k)^2+\mathrm{i}\eta}\,\mathrm{tr}\left(\frac{\lambda_a}{2}\frac{\lambda_a}{2}\right)\frac{1}{3} \quad . \tag{4.123}$$

In this case we have to average over colors which leads to the factor 1/3:

$$\Sigma^{\mathrm{b}}(k) = \frac{-2}{3}4g^2 \int \frac{\mathrm{d}^4 q}{(2\pi)^4}\frac{\not{q}}{[(q+k)^2+\mathrm{i}\eta][q^2+\mathrm{i}\eta]} \quad . \tag{4.124}$$

Following the same steps as before gives

$$\begin{aligned} &-\frac{8}{3}g^2\frac{\mathrm{i}}{16\pi^2}\left(\frac{1}{\varepsilon}-\ln\frac{-k^2}{\mu^2}+\mathrm{const}\right)(-\not{k})\int_0^1 \mathrm{d}z\, z \\ &= \frac{4}{3}g^2\frac{\mathrm{i}}{16\pi^2}\left(\frac{1}{\varepsilon}-\ln\frac{-k^2}{\mu^2}+\mathrm{const}\right)\not{k} \quad . \end{aligned} \tag{4.125}$$

Finally we turn to the vertex corrections:

$$\begin{aligned} \Gamma_\mu^{(\mathrm{c1})} = g^3 \int \frac{\mathrm{d}^4 q}{(2\pi)^4} f^{abc}\left[g_{\mu\lambda}(k-q)_\nu + g_{\nu\mu}(-q-2k)_\lambda + g_{\lambda\nu}(2q+k)_\mu\right] \\ \times \frac{\lambda^b}{2}\frac{\lambda^c}{2}\frac{\gamma^\nu \not{q}\gamma^\lambda}{(q^2+\mathrm{i}\eta)^2\left[(q+k)^2+\mathrm{i}\eta\right]} \quad . \end{aligned} \tag{4.126}$$

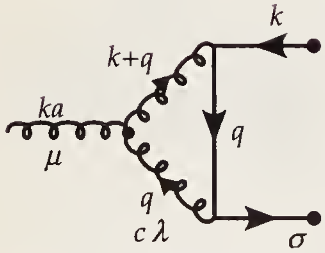


Fig. 4.27. The chosen variables for the first vertex correction graph

To make things easy we choose a very special kinematics: we choose the outgoing quark to have zero momentum. Or in other words we calculate in a specific Lorentz frame where this is the case, relying on the fact that the divergencies and thus the renormalization constants for coupling constants, masses, and wave functions are Lorentz invariant. The divergent term is not affected by this; only the finite ones are, and this makes things much easier.

$$\begin{aligned} \Gamma_\mu^{(\mathrm{c1})} &= \frac{1}{2}C_2\frac{\lambda^a}{2}g^3\int\frac{\mathrm{d}^4 q}{(2\pi)^4}\int_0^1 \mathrm{d}z\,\frac{\Gamma(3)}{\Gamma(2)\Gamma(1)}(1-z) \\ &\quad\times\frac{(\not{k}\not{q}-q^2)\gamma_\mu-\gamma_\mu(q^2+2\not{k}\not{q})-2\not{q}(2q+k)_\mu}{(q^2+2qkz+k^2z+i\eta)^3} \\ &= \frac{1}{2}C_2\frac{\lambda^a}{2}2g^3\int_0^1 \mathrm{d}z\,(1-z)\int\frac{\mathrm{d}^4 q}{(2\pi)^4}\frac{-2q^2\gamma_\mu-q^2\gamma_\mu+\dots}{(q^2+k^2z(1-z)+\mathrm{i}\eta)^3} \\ &= \mathrm{i}C_2\frac{\lambda^a}{2}g^3(+3)\int_0^1 \mathrm{d}z\,(1-z)\frac{\mu^{4-d}}{16\pi^2}\gamma_\mu\pi^\varepsilon\left(-k^2z(1-z)\right)^{-\varepsilon} \\ &\quad\times\frac{\Gamma(3)\Gamma(\varepsilon)}{\Gamma(2)\Gamma(3)}+\dots \\ &= \mathrm{i}C_2\frac{\lambda^a}{2}3g^3\frac{1}{2}\frac{\gamma_\mu}{16\pi^2}\left(\frac{1}{\varepsilon}-\ln\frac{-k^2}{\mu^2}+\dots\right) \quad . \end{aligned} \tag{4.127}$$

The self-energy insertion can be directly read off from (4.125). The factor $\not k$ in (4.125) just cancels the additional quark propagator and one gets the same contribution for the self-energy twice. For the specific kinematics we have chosen the result seems, however, to be undefined because for the outgoing antiquark one formally gets log(0). Therefore in this case we choose the kinematics defined in Fig. 4.28 leading to

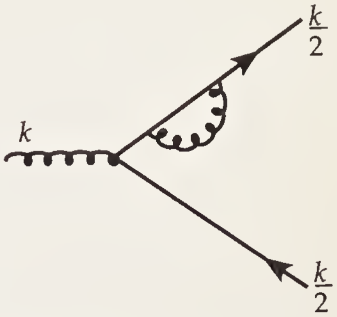


Fig. 4.28. The chosen variables for the second vertex correction graph

$$\Gamma_\mu^{(c2)} = -2 \times \frac{4}{3}\frac{1}{16\pi^2}\left(\frac{1}{\varepsilon} - \ln\frac{-k^2}{\mu^2} + \dots\right) \mathrm{i}\frac{\lambda^a}{2}\gamma_\mu g^3 \quad . \tag{4.128}$$

Again, owing to Lorentz invariance the divergences have to be the same. And the final vertex graph is

$$\begin{aligned}\Gamma_\mu^{(c_3)} &= g^3 \int \frac{\mathrm{d}^4q}{(2\pi)^4}\frac{\lambda^b}{2}\gamma^\lambda\frac{\lambda^a}{2}(\not q + \not k)\gamma_\mu \not q \frac{\lambda^b}{2}\gamma_\lambda \frac{1}{(q^2+\mathrm{i}\eta)^2\left((q+k)^2+\mathrm{i}\eta\right)} \\ &= g^3\left(\frac{4}{3} - \frac{1}{2}C_2\right)\frac{\lambda^a}{2}(-2)\int \frac{\mathrm{d}^4q}{(2\pi)^4}\frac{\frac{1}{4}(-2)\gamma_\mu q^2}{(q^2+\mathrm{i}\eta)^2\left((q+k)^2+\mathrm{i}\eta\right)} \\ &= g^3\left(\frac{4}{3} - \frac{C_2}{2}\right)\frac{\mathrm{i}\lambda^a}{2}\frac{\gamma_\mu}{16\pi^2}\left(\frac{1}{\varepsilon} - \ln\frac{-k^2}{\mu^2} + \dots\right) \quad .\end{aligned}$$

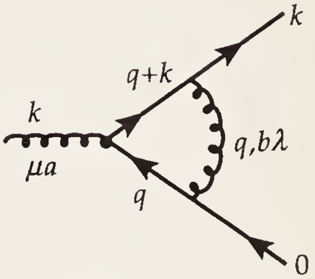
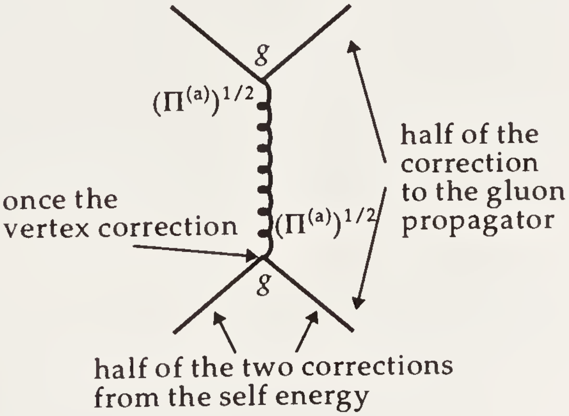


Fig. 4.29. The chosen variables for the third vertex correction graph

All together this adds up to

$$\begin{aligned}\Gamma_\mu^{(c)} &= g^3\mathrm{i}\frac{\lambda^a}{2}\frac{\gamma_\mu}{16\pi^2}\left(\frac{1}{\varepsilon} - \ln\frac{-k^2}{\mu^2} + \dots\right)\left(-\frac{8}{3} + \frac{4}{3} + \frac{3}{2}C_2 - \frac{C_2}{2}\right) \\ &= g^3 i\frac{\lambda^a}{2}\frac{\gamma_\mu}{16\pi^2}\left(\frac{1}{\varepsilon} - \ln\frac{-k^2}{\mu^2} + \dots\right)\left(C_2 - \frac{4}{3}\right) \quad .\end{aligned} \tag{4.129}$$

Now we can renormalize the coupling constant. The way in which the various contributions combine is shown in Fig. 4.30. The renormalization procedure is in fact rather easy. The square root of the vacuum polarization factor and the self-energy factor multiply each of the respective lines. Thus the quark–gluon vertex acquires the factors $(\Sigma^{(b)})^{\frac{1}{2}}\,(\Sigma^{(b)})^{\frac{1}{2}}\,(\Pi^{(a)})^{\frac{1}{2}}\,\Gamma^{(c)}$:



g
$(\Pi^{(a)})^{1/2}$
half of the correction to the gluon propagator
once the vertex correction
$(\Pi^{(a)})^{1/2}$
g
half of the two corrections from the self energy

Fig. 4.30. The combination of radiative corrections contributing to the renormalization of the coupling constant g

$$\begin{aligned}\Gamma_\mu^{\mathrm{a(total)}} &= -\mathrm{i}\frac{\lambda^a}{2}g_0\gamma_\mu\left\{1-\frac{g^2}{16\pi^2}\left[\frac{1}{2}\left(\frac{20}{12}C_2-\frac{2}{3}N_\mathrm{F}\right)\right.\right.\\ &\qquad\left.\left.-\frac{1}{2}\times 2\times\frac{4}{3}+\frac{4}{3}-\frac{C_2}{2}+\frac{3}{2}C_2\right]\left[\frac{1}{\epsilon}-\ln\left(\frac{-k^2}{\mu^2}\right)+\dots\right]\right\}\\ &= -\mathrm{i}\frac{\lambda^a}{2}g_0\gamma_\mu\left\{1-\frac{g^2}{16\pi^2}\left(\frac{22}{12}C_2-\frac{1}{3}N_\mathrm{F}\right)\left[\frac{1}{\epsilon}-\ln\left(\frac{-k^2}{\mu^2}\right)+\dots\right]\right\}\quad .\end{aligned} \tag{4.130}$$

The renormalized coupling constant is obtained from (4.130) by subtracting the value of the correction at some renormalization scale M^2:

$$\begin{aligned}g_\mathrm{R} &= g_0-\frac{g_0^3}{16\pi^2}\left(\frac{11}{6}C_2-\frac{1}{3}N_\mathrm{F}\right)\left[\frac{1}{\epsilon}-\ln\left(\frac{-k^2}{\mu^2}\right)-\frac{1}{\epsilon}+\ln\left(\frac{-M^2}{\mu^2}\right)\right]\\ &= g_0+\frac{g_0^3}{16\pi^2}\left(\frac{11}{6}C_2-\frac{1}{3}N_\mathrm{F}\right)\ln\left(\frac{-k^2}{M^2}\right)\quad .\end{aligned} \tag{4.131}$$

The β function can now be easily calculated

$$\begin{aligned}\beta = M\frac{\partial g_R}{\partial M} &= -\frac{g_0^3}{16\pi^2}\left(\frac{11}{6}C_2-\frac{1}{3}N_\mathrm{F}\right)\left(\frac{2M}{M^2}\right)M\\ &= -\frac{g_0^3}{16\pi^2}\left(\frac{11}{3}C_2-\frac{2}{3}N_\mathrm{F}\right)+O(g_0^5)\quad .\end{aligned} \tag{4.132}$$

This is the first term in the perturbative expansion for the β function. For QCD we have to insert $C_2 = 3$.

It is instructive to compare our results with those obtained for a different gauge, namely the Landau gauge, in which the gluon propagator is

$$\frac{-g_{\mu\nu}+\frac{k_\mu k_\nu}{k^2}}{k^2+\mathrm{i}\eta}\quad . \tag{4.133}$$

In this gauge instead of (4.122) we get

$$\begin{aligned}\Pi^{(\mathrm{a}_2)}_{\mu\mu'}+\Pi^{(\mathrm{a}_3)}_{\mu\mu'}\big|_\mathrm{Landau} &= \mathrm{i}\frac{3\delta_{aa'}}{16\pi^2}\left[\frac{1}{\varepsilon}-\ln\left(\frac{-k^2}{\mu^2}\right)+\dots\right]\\ &\quad\times\frac{26}{12}(k_\mu k_{\mu'}-g_{\mu\mu'}k^2)\quad ,\end{aligned} \tag{4.134}$$

instead of (4.123)

$$\Pi^\mathrm{b}(k) = 0\quad , \tag{4.135}$$

and instead of (4.129)

$$\Gamma_\mu^{(\mathrm{c})} = \mathrm{i}\frac{\lambda^a}{2}\frac{\gamma_\mu}{16\pi^2}\left[\frac{1}{\varepsilon}-\ln\left(\frac{-k^2}{\mu^2}\right)+\dots\right]\frac{3}{4}C_2\quad . \tag{4.136}$$

Together this gives

$$
\begin{aligned}
g_{\mathrm{R}} &= g_0 - \frac{g_0^3}{16\pi^2}\left(\frac{13}{12}C_2 + \frac{3}{4}C_2 - \frac{1}{3}N_{\mathrm{F}}\right)\left[\frac{1}{\varepsilon} - \ln\left(\frac{-k^2}{\mu^2}\right) + \dots\right] \\
&= g_0 - \frac{g_0^3}{16\pi^2}\left(\frac{11}{6}C_2 - \frac{1}{3}N_{\mathrm{F}}\right)\left[\frac{1}{\varepsilon} - \ln\left(\frac{-k^2}{\mu^2}\right) + \dots\right] \quad . \qquad (4.137)
\end{aligned}
$$

Thus the β function as an observable is gauge independent, as it must be, but the renormalization of the wave function owing to the self-energy graph and the vacuum polarization graphs is gauge dependent. For completeness let us also cite the result for the beta function to third order:[5]

$$
\begin{aligned}
\beta(g) = &-\frac{g^3}{(4\pi)^2}\left(\frac{11}{3}C_2 - \frac{3}{2}N_{\mathrm{F}}\right) - \frac{g^5}{(4\pi)^4}\left(\frac{34}{3}C_2^2 - 2C_1N_{\mathrm{F}} - \frac{10}{3}C_2N_{\mathrm{F}}\right) \\
&- \frac{g^7}{(4\pi)^6}\left(\frac{2857}{54}C_2^3 + C_1^2N_{\mathrm{F}} - \frac{205}{18}C_1C_2N_{\mathrm{F}}\right. \\
&\left. - \frac{1415}{54}C_2^2N_{\mathrm{F}} + \frac{11}{9}C_1N_{\mathrm{F}}^2 + \frac{79}{54}C_2N_{\mathrm{F}}^2\right) \qquad (4.138)
\end{aligned}
$$

with $C_2 = N$ for SU(N) and $C_1 = 4/3$ for quarks in the fundamental representation.

For QCD this simplifies to

$$
\begin{aligned}
\beta(g) = &-\frac{g^3}{(4\pi)^2}\left(11 - \frac{2}{3}N_{\mathrm{F}}\right) - \frac{g^5}{(4\pi)^4}\left(102 - \frac{38}{3}N_{\mathrm{F}}\right) \\
&- \frac{g^7}{(4\pi)^6}\left(\frac{2857}{2} - \frac{5033}{18}N_{\mathrm{F}} + \frac{325}{54}N_{\mathrm{F}}^2\right) \quad . \qquad (4.139)
\end{aligned}
$$

Up to second order this can be translated into an explicit solution for α_{s}

$$
\alpha_{\mathrm{s}}(\mu) = \frac{12\pi}{(33 - 2N_{\mathrm{F}})\ln\left(\frac{\mu^2}{\Lambda^2}\right)}\left[1 - \frac{6(153 - 19N_{\mathrm{F}})}{(33 - 2N_{\mathrm{F}})^2}\,\frac{\ln\left[\ln\left(\frac{\mu^2}{\Lambda^2}\right)\right]}{\ln\left(\frac{\mu^2}{\Lambda^2}\right)}\right] \quad , \qquad (4.140)
$$

where we have introduced the famous Λ_{QCD}-parameter. To third order, such an explicit expansion cannot be given. This is because that one actually expands simultaneously in $\ln M^2/\Lambda^2$ and $\ln\left[\ln\left(\frac{M^2}{\Lambda^2}\right)\right]$. Such a double expansion becomes ambiguous at higher orders.

Let us finally note that the higher terms of the β function, i.e., the cofficient of g_0^3 and g_0^5 depend on the regularization scheme used, leading to different Λ_{QCD} parameters for different renormalization schemes, which are correlated with different expressions in the large bracket on the right-hand side of (4.140). This will be explicitly discussed for the lattice gauge regularization in Sect. 7.1.

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[5] See Tarasov, Vladimirov, and Zhankov: Phys. Lett. **B93**, 429 (1980); Larin and Vermaseren: Phys. Lett. **B303**, 334 (1993).

5. Perturbative QCD I: Deep Inelastic Scattering

In the last chapter we discussed how the QCD coupling constant depends on the transferred momenta. Because of this specific dependence, QCD can only be treated perturbatively in the case of large momentum transfers. For practical purposes, however, it is indispensable to know at what momentum values the transition between perturbative and non-perturbative effects takes place. To this end we again take a look at Fig. 1.1. The quark confinement problem will be considered later together with different models for its solution. A common feature of all these models is that they postulate nonperturbative effects. Since, for example the masses of the N resonances in Fig. 1.1 reach a value of 2.5 GeV and since on the other hand the rest masses of the constituent up and down quarks are negligible, we can conclude that for momentum transfers less than 1 GeV QCD is certainly still in the nonperturbative region. On the other hand Fig. 4.4 shows that even at momentum transfers of a few GeV, quarks inside nucleons behave almost like free particles. Hence the transition from the nonperturbative to the perturbative region must take place quite rapidly, i.e., between $\sqrt{Q^2} \approx 1$ GeV and $\sqrt{Q^2} \approx 3$ GeV. According to Fig. 4.10 QCD can only yield such an immediate transition if the number of quark flavors with masses less than 1-2 GeV is not much larger than six. In fact there are only three or four such quarks: up ($m_\mathrm{u} = 5.6 \pm 1.1$ MeV), down ($m_\mathrm{d} = 9.9 \pm 1.1$ MeV), strange ($m_\mathrm{s} = 199 \pm 33$ MeV), and charm ($m_\mathrm{c} = 1.35 \pm 0.05$ GeV). The bottom quark is with $m_\mathrm{b} \approx 5$ GeV too heavy, and so is the top quark, which is claimed to have $m_\mathrm{t} = 174 \pm 17$ GeV. As a further consequence of this sudden transition it is almost certain that for somewhat larger momentum transfers $\sqrt{Q^2}$ all processes can be evaluated by means of the usual perturbation theory, i.e., the QCD Feynman rules. It therefore seems quite obvious to calculate QCD corrections to the parton model of deep inlastic lepton–nucleon scattering. The next section treats these questions in some detail.

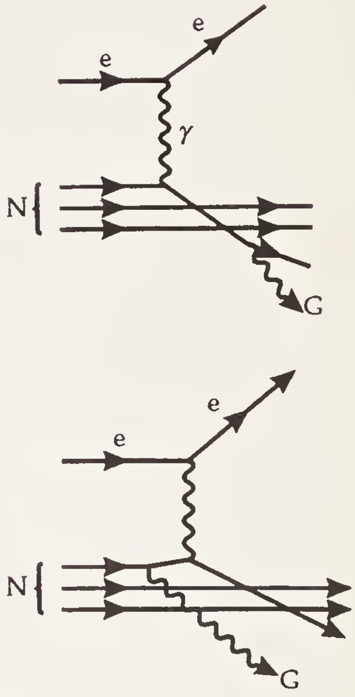


Fig. 5.1. Two correction graphs for deep-inelastic electron-nucleon scattering

5.1 The Gribov–Lipatov–Altarelli–Parisi Equations

The Gribov–Lipatov–Altarelli–Parisi equations (GLAP) describe the influence of the perturbative QCD corrections on the distribution functions that enter the parton model of deep inelastic scattering processes. At this point we investigate their structure and the functions that occur only for the two corrections graphs in Fig. 5.1. In line with the parton-model assumptions, both the scattering parton and the "emitted" gluon can be treated here as free particles. Having determined the GLAP

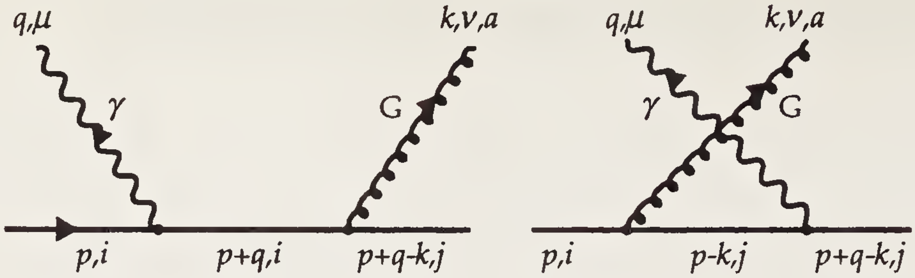


Fig. 5.2. Definition of the employed quantities

equations for the two graphs mentioned above it is quite easy to extrapolate to their general form. We shall use the notation defined in Fig. 5.2.

Obviously these graphs are similar to those for Compton scattering. Hence their contribution to the scattering tensor $W_{\mu\nu}$ can be evaluated in analogy to the corresponding QED graphs. The first step towards this end is the determination of the correct normalization factor. $W_{\mu\nu}$ is then just a factor entering the cross section (see (3.22)); the photon progagator and the normalization factor of the incoming photon have been separated. Bearing all these facts in mind one is led to the correct result, which we want, however, to derive in a slightly different way. We start with (3.33):

$$W_{\mu\mu'} = \frac{1}{2\pi}\int \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}qx}\frac{1}{2}\sum_{\text{pol.}}\langle N|J_\mu(x)J_{\mu'}(0)|N\rangle \quad . \tag{5.1}$$

In order to obtain the contributions due to Fig. 5.2 we clearly have to insert

$$J_\mu(x) = \int \bar{\Psi}(y) g G^a_\rho(y)\gamma^\rho\frac{\lambda^a}{2}S(y-x)Q_f\gamma_\mu\Psi(x)\,\mathrm{d}^4y$$

for the transition current $J_\mu(x)$ and

$$J_\mu(x) = \int \bar{\Psi}(x)Q_f\gamma_\mu S(x-y)\frac{\lambda^a}{2}G^a_\rho\gamma^\rho\Psi(y)\,\mathrm{d}^4y \tag{5.2}$$

as the exchange term. Ψ and G^a_ρ denote the wave functions of a quark with flavor f and a gluon, respectively. Transforming this into momentum space we obtain the Feynman graphs of Fig. 5.3. Here it must be taken into account that the outgoing (supposedly massless) parton and the outgoing gluon are on the mass shell, i.e.,

$$(p+q-K)^2 = K^2 = 0 \quad . \tag{5.3}$$

The usual propagators therefore have to be substituted by δ functions if these particles occur as "inner lines" in the $W_{\mu\mu'}$ graphs:

$$\begin{aligned}\frac{1}{K^2+\mathrm{i}\varepsilon} &= P(1/K^2) - \pi\mathrm{i}\delta(K^2) \rightarrow -2\pi\mathrm{i}\delta(K^2)\Theta(K_0)\\ \frac{\gamma_\mu(p^\mu+q^\mu-K^\mu)}{(p+q-K)^2+\mathrm{i}\varepsilon} &\rightarrow -2\pi\mathrm{i}\gamma_\mu(p^\mu+q^\mu-K^\mu)\delta[(p+q-K)^2]\\ &\quad\times\Theta(p_0+q_0-K_0) \quad .\end{aligned} \tag{5.4}$$

Using the Feynman rules of Exercise 4.2, together with these modifications yields

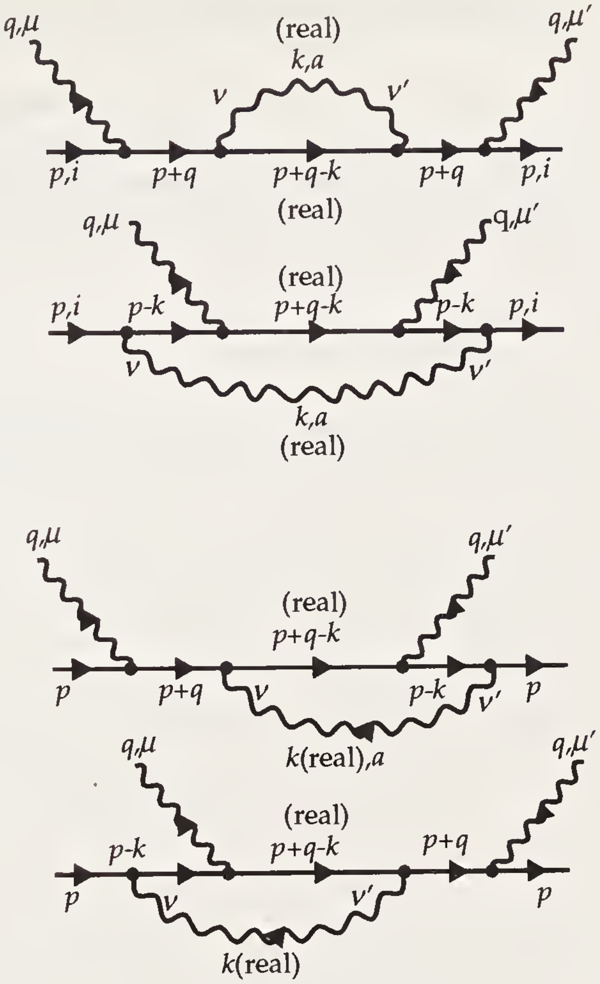


Fig. 5.3. The Feynman representation of the corrections to $W_{\mu\mu'}$ shown in Fig. 5.1

$$
\begin{aligned}
W_{\mu\mu'} = \int & \left[g^2 \frac{1}{3} \sum_{a,i,j} \left(\frac{\lambda^a}{2}\right)_{ji} \left(\frac{\lambda^a}{2}\right)_{ij} \right] \frac{1}{2} \left(\sum_{\varepsilon} \varepsilon^{*\nu} \varepsilon^{\nu'} \right) Q_f^2 \\
& \times \frac{1}{2} \sum_s \{ \overline{u}(p,s)(\gamma_{\nu'}(\not{p} - \not{K})^{-1}\gamma_{\mu'} + \gamma_{\mu'}(\not{p} + \not{q})^{-1}\gamma_{\nu'}) \\
& \times (\not{p} + \not{q} - \not{K}) \Theta(p_0 + q_0 - K_0) \delta\left[(p+q-K)^2\right] \Theta(K_0) \delta(K^2) \\
& \times \left[\gamma_\mu(\not{p} - \not{K})^{-1}\gamma_\nu + \gamma_\mu(\not{p} + \not{q})^{-1}\gamma_\nu\right] \cdot u(p,s)\} \\
& \times (-4\pi^2) \frac{\mathrm{d}^4 K}{(2\pi)^4} \frac{1}{2\pi} \quad .
\end{aligned}
\tag{5.5}
$$

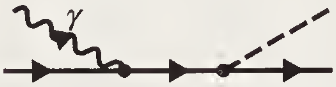
Here we have averaged over spin s, quark color i, and photon polarization ε of the initial state. Note that the factor e^2 (charge squared) has been separated from $W_{\mu\nu}$. Q_f denotes the electric charge of the flavor f, and the additional factor $1/2\pi$ is due to (5.1). In the Feynman gauge we have

$$
\sum_{\varepsilon} \varepsilon^{*\nu} \varepsilon^{\nu'} = -g^{\nu\nu'} \quad .
\tag{5.6}
$$

This is simply the Lorentz-invariant extension of the corresponding QED relation, where one has for $\boldsymbol{K} = K\boldsymbol{e}_3$:

$$\sum_{\varepsilon} \varepsilon^{*\nu} \varepsilon^{\nu'} = -g^{\nu\nu'}(\delta_{\nu 1} + \delta_{\nu 2}) \quad . \tag{5.7}$$

The degrees of freedom additionally introduced are compensated by the ghost fields, as discussed earlier. However, ghost fields do not have to be taken into account in our calculation, since we are considering physical gluons. Formally this is due to the lack of quark–ghost coupling, implying that no graph of the kind



exists. The expressions that have to be substituted for $\sum_{\varepsilon} \varepsilon^{*\nu} \varepsilon^{\nu'}$ in various gauges are discussed in Example 5.1 in more detail. Because

$$\frac{1}{3} \sum_a \mathrm{tr} \left(\frac{\lambda^a}{2} \right)^2 = \frac{1}{3} \sum_a \frac{1}{2} = \frac{4}{3} \quad , \tag{5.8}$$

we obtain

$$\begin{aligned} W_{\mu\mu'} = & \frac{4g^2}{3} \frac{Q_f^2}{8\pi^3} \frac{1}{4} \int \mathrm{d}^4K \ \mathrm{tr} \left\{ \not{p} \left(\gamma_\nu \frac{(\not{p} - \not{K})}{(p-K)^2} \gamma_{\mu'} + \gamma_{\mu'} \frac{(\not{p} + \not{q})}{(p+q)^2} \gamma_\nu \right) \right. \\ & \times \left. \left\{ (\not{p} + \not{q} - \not{K}) \left(\gamma_\mu \frac{(\not{p} - \not{K})}{(p-K)^2} \gamma^\nu + \gamma^\nu \frac{(\not{p} + \not{q})}{(p+q)^2} \gamma_\mu \right) \right\} \right. \\ & \times \Theta(K_0) \Theta(p_0 + q_0 - K_0) \delta\left(K^2\right) \delta\left[(p + q - K)^2\right] \quad . \end{aligned} \tag{5.9}$$

Fortunately we do not have to evaluate the trace completely, since we already know that all the information provided by $W_{\mu\mu'}$ is contained in the structure functions W_1 and W_2. Therefore it is sufficient to determine $p^\mu p^{\mu'} W_{\mu\mu'}$ and $W_\mu{}^\mu$. We abbreviate the trace by $S_{\mu\mu'}$ and delegate the evaluation of $p^\mu p^{\mu'} S_{\mu\mu'}$ and $S_\mu{}^\mu$ to Exercise 5.2. The results are

$$\begin{aligned} & p^\mu p^{\mu'} S_{\mu\mu'} = 4u \quad , \\ & S_\mu{}^\mu = -8 \left(\frac{s}{t} + \frac{t}{s} - \frac{2Q^2 u}{st} \right) \end{aligned} \tag{5.10}$$

with the Mandelstam variables

$$\begin{aligned} & t = (p - K)^2 = -2p \cdot K \quad , \\ & s = (q + p)^2 = 2\nu - Q^2 \quad , \quad \nu = p \cdot q \quad , \\ & u = (q - K)^2 = (q + p - K - p)^2 = -2p \cdot (q + p - K) = -2\nu - t \quad . \end{aligned} \tag{5.11}$$

Note the change in our notation. The ν in (5.11) is equal to the ν of Chap. 3 multiplied by M_{N}.

EXAMPLE

5.1 Photon and Gluon Polarization Vectors

Real photons and gluons have two physical degrees of freedom. Thus the simplest choice would be ($\boldsymbol{k}$ is the photon or gluon momentum) that ε_1 and ε_2 are defined such that

$$(\varepsilon_1)^2 = (\varepsilon_2)^2 \,, \; (\varepsilon_1)^0 = (\varepsilon_2)^0 = 0$$
$$\varepsilon_{1,2} \cdot \boldsymbol{k} = 0 \quad , \quad \text{and} \quad \varepsilon_1 \cdot \varepsilon_2 = 0 \quad . \tag{1}$$

The sum over the vector particle polarization is then

$$\sum_{\varepsilon} \varepsilon^{\mu} \varepsilon^{*\nu} = \begin{cases} 0 & \text{if } \mu = 0 \text{ or } \nu = 0 \\ \sum_{\varepsilon=\varepsilon_1,\varepsilon_2} \varepsilon^{\mu}\varepsilon^{*\nu} & \text{otherwise} \end{cases} \quad . \tag{2}$$

This, however, is not explicitly Lorentz invariant. Therefore it is advantageous to choose different vectors. The following is a rather general covariant form:

$$\Sigma^{\mu\nu} = \sum_{\varepsilon} \varepsilon^{\mu} \varepsilon^{*\nu} = -g^{\mu\nu} + \frac{n^{\mu}k^{\nu} + n^{\nu}k^{\mu}}{n \cdot k} - \frac{n^2 k^{\mu} k^{\nu}}{(n \cdot k)^2} \quad . \tag{3}$$

This form has the property

$$k_{\mu}\Sigma^{\mu\nu} = \frac{n^{\nu}}{n \cdot k} k^2 - \frac{n^2 k^2}{(n \cdot k)^2} k^{\nu} \quad , \tag{4}$$

which vanishes for real photons as it should. Furthermore it fullfills

$$n_{\mu}\Sigma^{\mu\nu} = -n^{\nu} + \frac{n^2 k^{\nu}}{n \cdot k} + n^{\nu} + \frac{n^2}{n \cdot k} k^{\nu} = 0 \quad , \tag{5}$$

implying the gauge

$$n_{\mu}A^{\mu} = 0 \quad . \tag{6}$$

From (6) the specific form of n_{μ} for the gauge one is interested in can be read off. We give two standard examples.

1. The Lorentz gauge:

$$\begin{aligned} \Sigma^{\mu\nu}(1) &= -g^{\mu\nu} + \frac{2k^{\mu}k^{\nu}}{k^2} - \frac{k^{\mu}k^{\nu}}{k^2} \\ &= -\left[g^{\mu\nu} - \frac{k^{\mu}k^{\nu}}{k^2} \right] \quad . \end{aligned} \tag{7}$$

2. The axial gauge:

$$\Sigma^{\mu\nu}(2) = -g^{\mu\nu} + \frac{n^{\mu}k^{\nu} + n^{\nu}k^{\mu}}{n \cdot k} \quad , \tag{8}$$

where n^{ν} = fixed four-vector with $n^2 = 0$. However, form (3) also allows us to choose n_{μ} according to the problem we are treating. It might be very advantageous to choose, for example,

Example 5.1.

$$n_\mu = k_\mu + \alpha p_\mu \quad , \tag{9}$$

where p is one of the momenta of the problem. This might be useful, for example, if $k^2 = 0$ and $p^2 = 0$, which describe outgoing partons. Then (9) leads to

$$\begin{aligned} p_\mu \Sigma^{\mu\nu} &= -p^\nu + \frac{k \cdot p\, k^\nu + (k^\nu + \alpha p^\nu) p \cdot k}{\alpha p \cdot k} - \frac{2\alpha\, k \cdot p\; k \cdot p\; k^\nu}{(\alpha p \cdot k)^2} \\ &= -p^\nu + \frac{2k^\nu + \alpha p^\nu - 2k^\nu}{\alpha} = 0 \quad , \end{aligned} \tag{10}$$

$$k_\mu \Sigma^{\mu\nu} = 0 \quad , \tag{11}$$

which allows us to set all vectors p_μ, k_μ contracted with $\Sigma^{\mu\nu}$ equal to zero.

EXERCISE

5.2 More on the Derivation of QCD Corrections to Electron–Nucleon Scattering

Problem. Evaluate $S_\mu{}^\mu$ and $p^\mu p^{\mu'} S_{\mu\mu'}$ for the trace in (5.9).

Solution. First we employ

$$\begin{aligned} \gamma_\mu \not{a} \gamma^\mu &= -2\not{a} \quad , \\ \gamma_\mu \not{a}\not{b} \gamma^\mu &= 4a \cdot b \quad , \\ \gamma_\mu \not{a}\not{b}\not{c} \gamma^\mu &= -2\not{c}\not{b}\not{a} \quad , \end{aligned} \tag{1}$$

in order to simplify $S_{\mu\mu'}$:

$$\begin{aligned} S_{\mu\mu'} = {} & \frac{-2}{(p-K)^4} \operatorname{tr}\left[\not{p}(\not{p} - \not{K})\gamma_{\mu'}(\not{p} + \not{q} - \not{K})\gamma_\mu(\not{p} - \not{K})\right] \\ & + \frac{-2}{(p-K)^2(p+q)^2} \operatorname{tr}\left[\not{p}\gamma_\mu(\not{p} + \not{q})(\not{p} - \not{K})\gamma_{\mu'}(\not{p} + \not{q} - \not{K})\right] \\ & + \frac{-2}{(p-K)^2(p+q)^2} \operatorname{tr}\left[\not{p}\gamma_{\mu'}(\not{p} + \not{q})(\not{p} - \not{K})\gamma_\mu(\not{p} + \not{q} - \not{K})\right] \\ & + \frac{-2}{(p-q)^4} \operatorname{tr}\left[\not{p}\gamma_{\mu'}(\not{p} + \not{q})(\not{p} + \not{q} - \not{K})(\not{p} + \not{q})\gamma_\mu\right] \quad . \end{aligned} \tag{2}$$

Then $S_\mu{}^\mu$ is brought into the form

$$\begin{aligned} S_\mu{}^\mu = {} & \frac{4}{(p-K)^4} \operatorname{tr}\left[\not{p}(\not{p} - \not{K})(\not{p} - \not{K} + \not{q})(\not{p} - \not{K})\right] \\ & - \frac{16}{(p-K)^2(p-q)^2} (p+q) \cdot (p-K) \operatorname{tr}\left[\not{p}(\not{p} - \not{K} + \not{q})\right] \\ & + \frac{4}{(p+q)^4} \operatorname{tr}\left[\not{p}(\not{p} + \not{q})(\not{p} - \not{K} + \not{q})(\not{p} + \not{q})\right] \quad . \end{aligned} \tag{3}$$

Exercise 5.2.

For massless quarks and real gluons,

$$\begin{aligned}
\not{p}^2 &= p^2 = 0 \quad , \\
(\not{p} - \not{K} + \not{q})^2 &= (p - K + q)^2 = 0 \quad , \\
(p - K)^2 &= -2p \cdot K \quad , \\
(p + q)^2 &= (p + q - K + K)^2 \\
&= 2K \cdot (p + q - K) = 2K \cdot (p + q) \quad ,
\end{aligned} \tag{4}$$

which simplifies (3) to

$$\begin{aligned}
S_\mu{}^\mu = &\frac{1}{(p \cdot K)^2} \mathrm{tr}\left[\not{p}\not{K}(\not{p} - \not{K} + \not{q})\not{K}\right] \\
&+ \frac{4}{p \cdot K\, K \cdot (p + q)} (p + q) \cdot (p - K) 4p \cdot (p - K) \\
&+ \frac{1}{[K \cdot (p + q)]^2} \mathrm{tr}\left[\not{p}\not{K}(\not{p} - \not{K} + \not{q})\not{K}\right] \\
= &\left(\frac{4}{(p \cdot K)^2} + \frac{4}{[K \cdot (p + q)]^2}\right) \left[2p \cdot K (p + q) \cdot K\right] \\
&+ \frac{16}{p \cdot K\, K \cdot (p + q)} (p + q) \cdot (p - K) p \cdot (q - K) \quad .
\end{aligned} \tag{5}$$

Next we introduce the Mandelstam variables

$$\begin{aligned}
t &= (p - K)^2 = -2p \cdot K \quad , \\
s &= (p + q)^2 = 2K \cdot (p + q) \quad , \\
u &= (q - K)^2 \\
&= (q - K + p)^2 - 2p \cdot (q - K + p) + p^2 \\
&= -2p \cdot (q - K) \quad .
\end{aligned} \tag{6}$$

Taking into account

$$\begin{aligned}
Q^2 &= -q^2 = -(q + p - p)^2 \\
&= -2K \cdot (p + q) + 2p \cdot (q + p) \\
&= 2(q + p) \cdot (p - K) \quad ,
\end{aligned} \tag{7}$$

we finally obtain

$$\begin{aligned}
S_\mu{}^\mu &= -8\frac{s}{t} - 8\frac{t}{s} + 16\frac{Q^2 u}{st} \\
&= -8\left(\frac{s}{t} + \frac{t}{s} - 2\frac{Q^2 u}{st}\right) \quad .
\end{aligned} \tag{8}$$

Now we evaluate $p^\mu p^{\mu'} S_{\mu\mu'}$ in a completely analogous manner:

$$p^\mu p^{\mu'} S_{\mu\mu'} = \frac{-2}{(p - K)^4} \mathrm{tr}\left[\not{p}(\not{p} - \not{K})\not{p}(\not{p} - \not{K} + \not{q})\not{p}(\not{p} - \not{K})\right] . \tag{9}$$

All other terms vanish because $\not{p}^2 = p^2 = 0$.

Exercise 5.2.

$$p^{\mu}p^{\mu'}S_{\mu\mu'} = \frac{-2}{(p-K)^4}\,\mathrm{tr}\left[\not{p}\not{K}\not{p}(-\not{K}+\not{q})\not{p}\not{K}\right] \quad . \tag{10}$$

The final simplifications are achieved by exchanging the first two factors in the trace:

$$p^{\mu}p^{\mu'}S_{\mu\mu'} = \frac{-4}{(p-K)^4}p\cdot K\,\mathrm{tr}\left[\not{p}(-\not{K}+\not{q})\not{p}\not{K}\right] + 0 \tag{11}$$

and then the last two factors:

$$\begin{aligned} p^{\mu}p^{\mu'}S_{\mu\mu'} &= -2\mathrm{tr}\left[(-\not{K}+\not{q})\not{p}\right] \\ &= -8p\cdot(q-K) = 4u \quad . \end{aligned} \tag{12}$$

Since the traces in (5.10) are Lorentz invariant, the K integration can be performed in an arbitrary reference system. We choose the Breit system (see Chap. 3), i.e.,

$$p_\mu = (p,0,0,-p) \quad ; \quad q_\mu = (0,0,0,\sqrt{Q^2}) \quad . \tag{5.12}$$

The corresponding kinematics is depicted in Fig. 5.4.

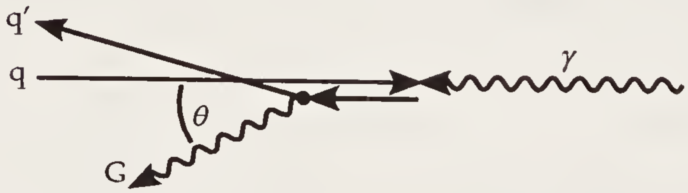


Fig. 5.4 Kinematics of the graphs depicted in Fig. 5.2 in the Breit system

Since the process considered is clearly cylindrically symmetric, the K_0 and φ integrations can be performed immediately:

$$\begin{aligned} I &= \int \mathrm{d}^4K\,f(Q^2,t,\nu)\Theta(K_0)\Theta(p_0+q_0-K_0)\delta(K^2)\delta\left[(p+q-K)^2\right] \\ &= 2\pi\int\limits_0^\infty \mathrm{d}K\int\limits_{-1}^{1}\mathrm{d}\cos\theta\,f(Q^2,t,\nu)\frac{K^2}{2K}\Theta(p_0+q_0-K)\delta\left[(p+q-K)^2\right] \quad . \end{aligned} \tag{5.13}$$

Next we rewrite the second δ function by inserting the explicit form of K_μ:

$$K_\mu = (K,0,K\sin\theta,K\cos\theta) \quad , \tag{5.14}$$

$$\begin{aligned} (p_\mu+q_\mu-K_\mu)^2 &= \left(p-K,0,-K\sin\theta,-p+\sqrt{Q^2}-K\cos\theta\right)^2 \quad , \\ &= p^2+K^2-2pK \\ &\quad -\left(K^2+p^2+Q^2+2pK\cos\theta-2K\sqrt{Q^2}\cos\theta-2p\sqrt{Q^2}\right) \\ &= -2pK(1+\cos\theta)-Q^2+2\nu+2K\sqrt{Q^2}\cos\theta \quad . \end{aligned} \tag{5.15}$$

Now $\cos\theta$ is replaced by t:

$$t = (p_\mu - K_\mu)^2 = -2pK(1+\cos\theta) \quad , \tag{5.16}$$

$$(p_\mu + q_\mu - K_\mu)^2 = t - Q^2 + 2\nu - 2K\sqrt{Q^2} + 2K\frac{Q^2}{p\sqrt{Q^2}}p(1+\cos\theta) \, , \tag{5.17}$$

$$= t(1 - Q^2/\nu) - Q^2 + 2\nu - 2K\sqrt{Q^2} \quad . \tag{5.18}$$

Finally we replace $\mathrm{d}\cos\theta$ by

$$\mathrm{d}\cos\theta \to -\frac{1}{2pK}\mathrm{d}t \tag{5.19}$$

and obtain

$$\begin{aligned} I &= 2\pi \int\limits_0^\infty \mathrm{d}K \int \mathrm{d}t \left(\frac{-1}{2pK}\right) \frac{K}{2} \Theta(p_0 + q_0 - K) \\ &\quad \times \delta\left[t\left(1 - \frac{Q^2}{\nu}\right) - Q^2 + 2\nu - 2K\sqrt{Q^2}\right] f(Q^2, t, \nu) \\ &= -\int\limits_{t_1}^{t_2} \frac{\pi}{4p\sqrt{Q^2}} \mathrm{d}t\, f(Q^2, t, \nu) = \frac{\pi}{4\nu} \int\limits_{t_2}^{t_1} \mathrm{d}t\, f(Q^2, t, \nu) \quad . \end{aligned} \tag{5.20}$$

The boundaries of the t integration are not yet determined. This is achieved by substituting K from (5.16) into the argument of the δ function:

$$K = \frac{-t}{2p(1+\cos\theta)} \quad , \tag{5.21}$$

$$0 = t\left(1 - \frac{Q^2}{\nu}\right) - Q^2 + 2\nu + \frac{\sqrt{Q^2}t}{p(1+\cos\theta)} \quad . \tag{5.22}$$

The second equation yields for t $\left(\nu = \sqrt{Q^2}\cdot p\right)$

$$t = \frac{Q^2 - 2\nu}{1 - \frac{Q^2}{\nu} + \frac{Q^2}{\nu(1+\cos\theta)}} = \frac{Q^2 - 2\nu}{1 - \frac{Q^2}{2\nu} + \frac{Q^2}{\nu}\left(\frac{1}{1+\cos\theta} - \frac{1}{2}\right)} \quad . \tag{5.23}$$

Since $x = Q^2/2\nu$ is less than or equal to one, the denominator is bigger than or equal to zero. The numerator is always less than or equal to zero and consequently t is a monotonically decreasing, negative function of $\cos\theta$. The boundaries t_1 and t_2 are simply

$$t_1 = t(\cos\theta = -1) = 0 \quad , \quad t_2 = t(\cos\theta = 1) = -2\nu \quad . \tag{5.24}$$

Now we have to investigate whether the Θ function leads to further restrictions. With (5.21), (5.23), and (5.12)

$$p_0 + q_0 - K_0 = p - K = p - \frac{2\nu - Q^2}{2\sqrt{Q^2} + 2p(1+\cos\theta)\left(1 - \frac{\sqrt{Q^2}}{p}\right)}$$

$$= p - \frac{2\nu - Q^2}{-2\sqrt{Q^2}\cos\theta + 2p(1+\cos\theta)}$$

$$= p - \frac{2\nu - Q^2}{\left(4p - 2\sqrt{Q^2}\right)\cos\theta + 2p(1-\cos\theta)} \quad . \tag{5.25}$$

Because $x = \sqrt{Q^2}/2p \leq 1 \Rightarrow \sqrt{Q^2} \leq 2p$, the denominator is always positive. It assumes its smallest value at $\cos\theta = -1$ for $p > \sqrt{Q^2}$ and at $\cos\theta = 1$ for $p < \sqrt{Q^2}$.

$p > \sqrt{Q^2}$:

$$\frac{2\nu - Q^2}{\left(4p - 2\sqrt{Q^2}\right)\cos\theta + 2p(1-\cos\theta)} \leq \frac{2\nu - Q^2}{2\sqrt{Q^2}} = p - \frac{\sqrt{Q^2}}{2} \leq p \, , \tag{5.26}$$

$p < \sqrt{Q^2}$:

$$\frac{2\nu - Q^2}{\left(4p - 2\sqrt{Q^2}\right)\cos\theta + 2p(1-\cos\theta)} \leq \frac{2\nu - Q^2}{4p - 2\sqrt{Q^2}} = \frac{\sqrt{Q^2}}{2} \leq p \, , \tag{5.27}$$

Hence the argument of the Θ function is positive definite:

$$p_0 + q_0 - k_0 \geq p - p = 0 \quad . \tag{5.28}$$

We thus obtain the result

$$W_{\mu\mu'} = -\frac{4}{3} g^2 \frac{Q_f^2}{16\pi^3} \frac{\pi}{8\nu} \int\limits_{-2\nu}^{0} \mathrm{d}t \, S_{\mu\mu'} \quad , \tag{5.29}$$

$$p^\mu p^{\mu'} S_{\mu\mu'} = 4u = -4(2\nu + t) \quad , \tag{5.30}$$

$$S_\mu{}^\mu = -8\left(\frac{2\nu - Q^2}{t} + \frac{t}{2\nu - Q^2} + \frac{2Q^2(2\nu + t)}{t(2\nu - Q^2)}\right) \quad . \tag{5.31}$$

The t integration yields a logarithmic singularity for $S_\mu{}^\mu$, which is the same kind of infrared singularity that occurs in the evaluation of the QED bremsstrahlung cross section. In fact the graphs in Fig. 5.1 can also be interpreted as bremsstrahlung processes. Before we discuss this singularity further, however, we should like to finish our calculation. First the upper integration boundary is replaced by $-\lambda^2$:

$$p^\mu p^{\mu'} W_{\mu\mu'} = \frac{4}{3} g^2 \frac{Q_f^2}{16\pi^2} \frac{1}{2\nu} 2\nu^2 = \frac{4}{3}\alpha_s \frac{Q_f^2}{8\pi} 2\nu \quad , \tag{5.32}$$

$$W_\mu{}^\mu = \frac{4}{3}\alpha_s \frac{Q_f^2}{4\pi} \frac{1}{\nu} \left(\frac{(2\nu - Q^2)^2 + 4\nu Q^2}{(2\nu - Q^2)} \int\limits_{-2\nu}^{-\lambda^2} \frac{\mathrm{d}t}{t} + \frac{-2\nu^2}{2\nu - Q^2} + \frac{2Q^2 2\nu}{2\nu - Q^2} \right)$$

$$= \frac{4}{3}\alpha_s \frac{Q_f^2}{4\pi} \left(2\frac{1+x^2}{1-x} \ln\frac{\lambda^2 x}{Q^2} - \frac{1}{1-x} + 4\frac{x}{1-x} \right) \quad . \tag{5.33}$$

How are the structure functions F_1 and F_2 connected with the expressions in (5.32) and (5.33)? Equations (3.18) and (3.23) yield the relation

$$\begin{aligned}W_{\mu\mu'} &= \left(-g_{\mu\mu'} + \frac{q_\mu q_{\mu'}}{q^2}\right)\frac{F_1}{M_\mathrm{N}} + \left(P_\mu - q_\mu \frac{q\cdot P}{q^2}\right)\left(P_{\mu'} - q_{\mu'}\frac{q\cdot P}{q^2}\right)\frac{F_2}{M_\mathrm{N}\nu}\\ &= \frac{1}{M_\mathrm{N}}\left[\left(-g_{\mu\mu'} + \frac{q_\mu q_{\mu'}}{q^2}\right)F_1\right.\\ &\quad \left. + \left(P_\mu - q_\mu \frac{q\cdot P}{q^2}\right)\left(P_{\mu'} - q_{\mu'}\frac{q\cdot P}{q^2}\right)\frac{F_2}{\nu}\right] \quad . \qquad (5.34)\end{aligned}$$

The factor $1/M_\mathrm{N}$ is due to different normalization factors for nucleons and massless quarks. For quarks, (5.34) therefore becomes

$$W_{\mu\mu'} = \left(-g_{\mu\mu'} + \frac{q_\mu q_{\mu'}}{q^2}\right)F_1^{Qu} + \frac{1}{\nu}\left(p_\mu - q_\mu\frac{q\cdot p}{q^2}\right)\left(p_{\mu'} - q_{\mu'}\frac{q\cdot p}{q^2}\right)F_2^{Qu} \,. \qquad (5.35)$$

It follows that

$$\begin{aligned}W^\mu_\mu &= (-4+1)F_1^{Qu} + \left(-2\frac{(q\cdot p)^2}{q^2} + \frac{(q\cdot p)^2}{q^2}\right)\frac{1}{\nu}F_2^{Qu}\\ &= -3F_1^{Qu} + \frac{\nu}{Q^2}F_2^{Qu}\\ &= -3F_1^{Qu} + \frac{F_2^{Qu}}{2x} \qquad (5.36)\end{aligned}$$

and

$$\begin{aligned}p^\mu p^{\mu'} W_{\mu\mu'} &= -\frac{\nu^2}{Q^2}F_1^{Qu} + \frac{\nu^3}{Q^4}F_2^{Qu}\\ &= -\frac{Q^2}{4x^2}F_1^{Qu} + \frac{Q^2}{8x^3}F_2^{Qu}\\ &= \frac{Q^2}{8x^3}\left(F_2^{Qu} - 2xF_1^{Qu}\right) \quad . \qquad (5.37)\end{aligned}$$

Obviously $p^\mu p^{\mu'} W_{\mu\mu'}$ is a measure of the violation of the Callan–Gross relation:

$$F_2^{Qu} = 2xF_1^{Qu} \qquad (5.38)$$

by the interaction. In addition to this Q^2-dependent violation we can choose

$$F_2^{Qu}(x, Q^2) = -xW^\mu_\mu + 12\frac{x^3}{Q^2}p^\mu p^{\mu'} W_{\mu\mu'} \qquad (5.39)$$

as a second, linearly independent function. The additional contributions to these functions due to "gluon bremsstrahlung" (see Fig. 5.1) are therefore

$$\Delta\left[F_2^{Qu}(x,Q^2) - 2xF_1^{Qu}(x,Q^2)\right] = \frac{4}{3}\alpha_s Q_f^2 \frac{x^2}{\pi} \quad , \tag{5.40}$$

$$\begin{aligned} \Delta F_2^{Qu}(x,Q^2) &= \frac{4}{3}\alpha_s Q_f^2 \left\{ \frac{3x^2}{2\pi} - \frac{x}{4\pi}\left[2\frac{1+x^2}{1-x} \right.\right. \\ &\quad \left.\left. \times \left(\ln\frac{\lambda^2}{Q^2} + \ln x \right) + \frac{4x-1}{1-x} \right] \right\} \\ &\approx -\frac{4}{3}\alpha_s Q_f^2 \frac{x}{2\pi}\frac{1+x^2}{1-x}\ln\frac{\lambda^2}{Q^2} + \dots \end{aligned} \tag{5.41}$$

Now we investigate the dependence of the nucleon structure functions $F_{1,2}$ on the quark structure functions F_1^{Qu} and F_2^{Qu}. To this end we have to go back to (3.70–79). There F_2 is given by

$$F_2 = \sum_i \int_0^1 \mathrm{d}\xi_i \, f_i(\xi_i) Q_i^2 \xi_i \delta(\xi_i - X) \tag{5.42}$$

with

$$p_\mu = \xi_i P_\mu \quad \text{and} \quad \int_0^1 \mathrm{d}\xi_i \, f(\xi_i) = 1 \quad .$$

X denotes $Q^2/2P_{\mathrm{N}} \cdot q$ while x is equal to $Q^2/2p \cdot q$. ξ_i is the fraction of the ith parton of the total momentum P_μ and $f(\xi_i)$ is the probability of a parton to having that momentum. Q_i denotes the charge of the ith parton. By definition the structure functions are factors in the scattering tensor, which turn the contributions of free pointlike particles into those of extended particles. Correspondingly one has to multiply the right-hand side of (5.42) by the factor $F_2^{\mathrm{Qu}}(x,Q^2)$ and to modify the δ function according to

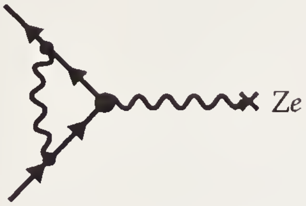
$$x = \frac{Q^2}{2p \cdot q} = \frac{Q^2}{2\xi_i P_{\mathrm{N}} \cdot q} = \frac{X}{\xi_i} \quad , \tag{5.43}$$

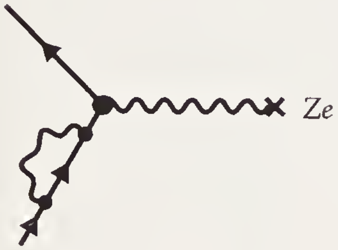
$$F_2 = \sum_i \int_0^1 \mathrm{d}\xi_i \, f_i(\xi_i)\xi_i \int_0^1 \mathrm{d}x \, \delta(X - x\xi_i) F_2^{\mathrm{Qu},i}(x,Q^2) \quad . \tag{5.44}$$

Note that we have integrated over all values of x that contribute to a given X. In addition the factor Q_i^2 has been omitted, since it is contained in $F_2^{\mathrm{Qu},i}$. With the help of (5.41) we finally obtain the following contribution of the graphs in Fig. 5.1 to the nucleon structure function F_2 of deep inelastic electron–nucleon scattering:

$$\begin{aligned} \Delta F_2 &= \Delta F_2^{Qu}(Q_1^2) - \Delta F_2^{Qu}(Q_2^2) \\ &= -\frac{4}{3}\alpha_s \frac{1}{2\pi}\sum_i Q_i^2 \int_X^1 \mathrm{d}\xi_i \, f(\xi_i) \frac{X}{\xi_i}\frac{1+(X/\xi_i)^2}{1-(X/\xi_i)} \ln\frac{Q_2^2}{Q_1^2} \quad . \end{aligned} \tag{5.45}$$

The integral is divergent, because the integrand has a singularity at $x = 1$ or $X = \xi_i$. Since $x = 1$ means elastic scattering, the gluon emitted in this reaction carries neither energy nor momentum. The occurrence of such an infrared divergence is not surprising: one encounters similar behavior in QED bremsstrahlung. There it can be shown that, up to a given order in α, the infrared divergences cancel each other if all scattering processes, including the elastic channel, are taken into account. To lowest order this is achieved by the following additional graphs:





In QCD many more graphs exist and an analogous proof is more sophisticated. The desired compensation can to lowest order be shown by an explicit evaluation. The result can be brought into the following form:

ΔF_2(bremsstrahlung + radiative corrections)

$$\approx -\frac{4}{3}\frac{\alpha_s}{2\pi}\sum_i Q_i^2 \ln\frac{Q_2^2}{Q_1^2}\int\limits_X^1 \mathrm{d}\xi_i\, f_i(\xi_i)\frac{X}{\xi_i}\left[\frac{1+(X/\xi_i)^2}{1-(X/\xi_i)}\right]_+ \quad , \qquad (5.46)$$

where $[(1+z^2)/(1-z)]_+$ is defined by

$$\int\limits_0^1 \mathrm{d}z\, F(z)\left[\frac{1+z^2}{1-z}\right]_+ = \int\limits_0^1 \mathrm{d}z\,\left[F(z)-F(1)\right]\frac{1+z^2}{1-z} \qquad (5.47)$$

for every sufficiently regular function $F(z)$. (From now on we shall again write x for X.) One possible choice for $f(\xi_i)$ is the distribution function derived in the context of the simple parton model (see Fig. 3.11). In this case the result would be an expression for the scale violation of the function F_2 depending on one fit parameter Λ_{QCD}. From Fig. 3.5 it is clear that for $x > 0.5$ scale violations in fact occur, i.e., F_2 becomes a function of x and Q^2. In order to be more precise, F_2 decreases with increasing Q^2 for $x > 0.5$. The very same behavior can also be encountered in other scattering reactions, e.g., in neutrino–nucleon scattering (Fig. 5.5). Here it is seen that for small x the Q^2 dependence is just the opposite, i.e., for $x < 0.1$ F_2 increases with Q^2. In the case of electron–nucleon scattering, respectively muon-nucleon scattering the data in Fig. 5.7 show the same tendency.

The overall behavior (see Fig. 5.6) is readily understood. For increasing momentum transfers QCD processes become more important, and on average more quarks, antiquarks, and gluons occur, between which the total momentum is distributed. Owing to (3.79), F_2 is given by

$$F_2^{\mathrm{eN}}(x) = \sum_i F_i(x)Q_i^2 x \quad , \qquad (5.48)$$

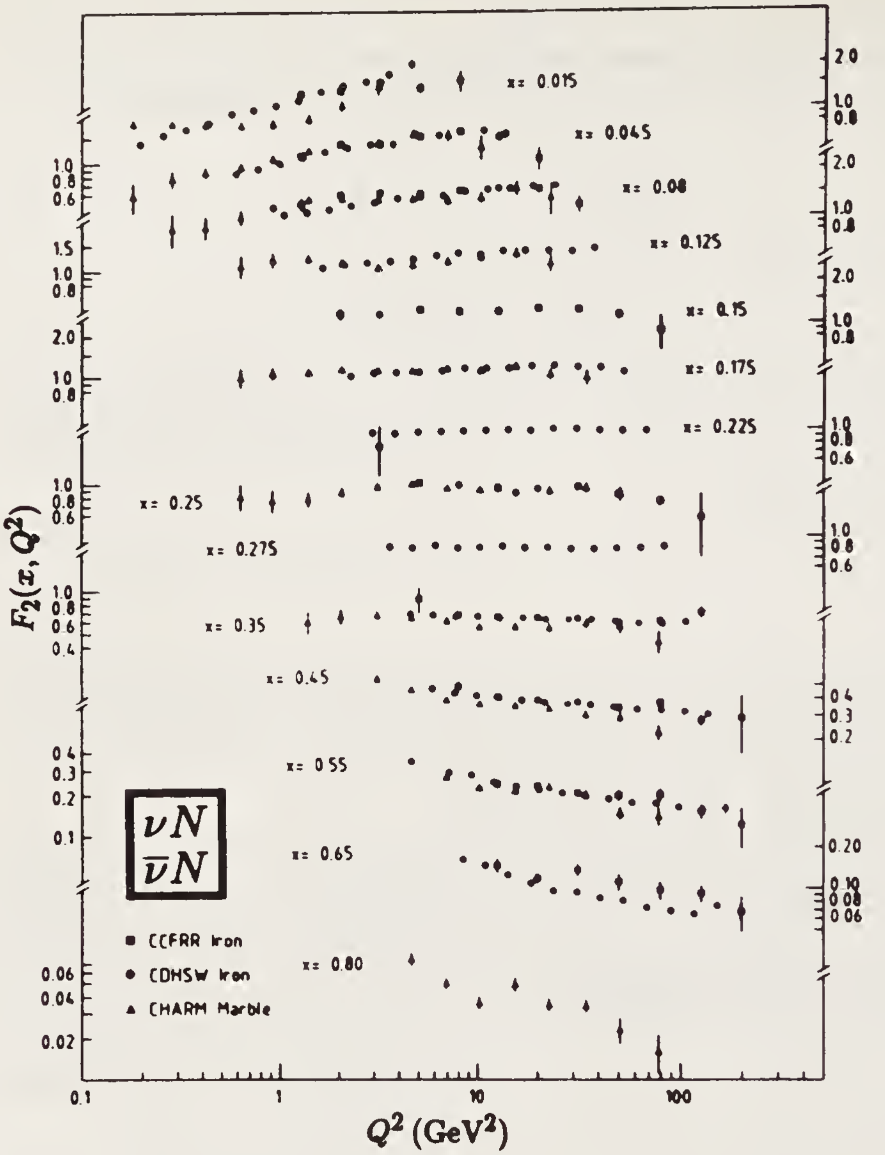


Fig. 5.5. The structure functions of neutrino–nucleon scattering. (From Review of Particle Properties, Physical Review **D 45** (1992))

i.e., by x times the probability that a parton carries the momentum fraction x. Therefore $F_2(x)$ decreases for large x values with increasing Q^2. Since the total momentum is constant

$$\int \mathrm{d}x \sum_i F_i(x)x = 1 \quad , \tag{5.49}$$

an enhancement for small x values should follow. However, this can be prevented by the charge factors, for example, by Q_i^2 in (5.48). In this way the gluons contribute to the sum (5.49), but not to the F_2^{eN} structure functions. As a matter of fact, according to Chap. 3 the charged partons alone only contribute about fifty percent to (5.49).

Structure functions for muon–carbon and muon–iron scattering are depicted in Fig. 5.7. Again the same Q^2 dependance is observed. Furthermore it is remarkable that in the region $0.5 < x < 0.65$ and for very small x the F_2 function for iron is considerably smaller than for μ–p scattering. Part of this discrepancy is due to the

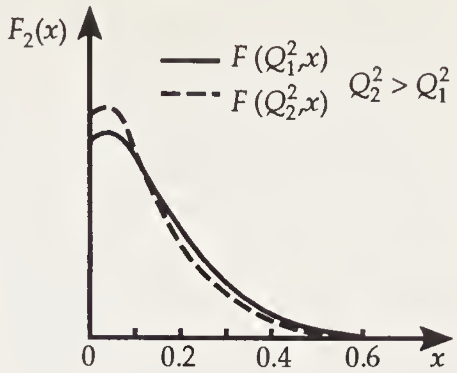


Fig. 5.6. The schematic dependence of the structure functions on Q^2, with $Q_2^2 \gg Q_1^2$

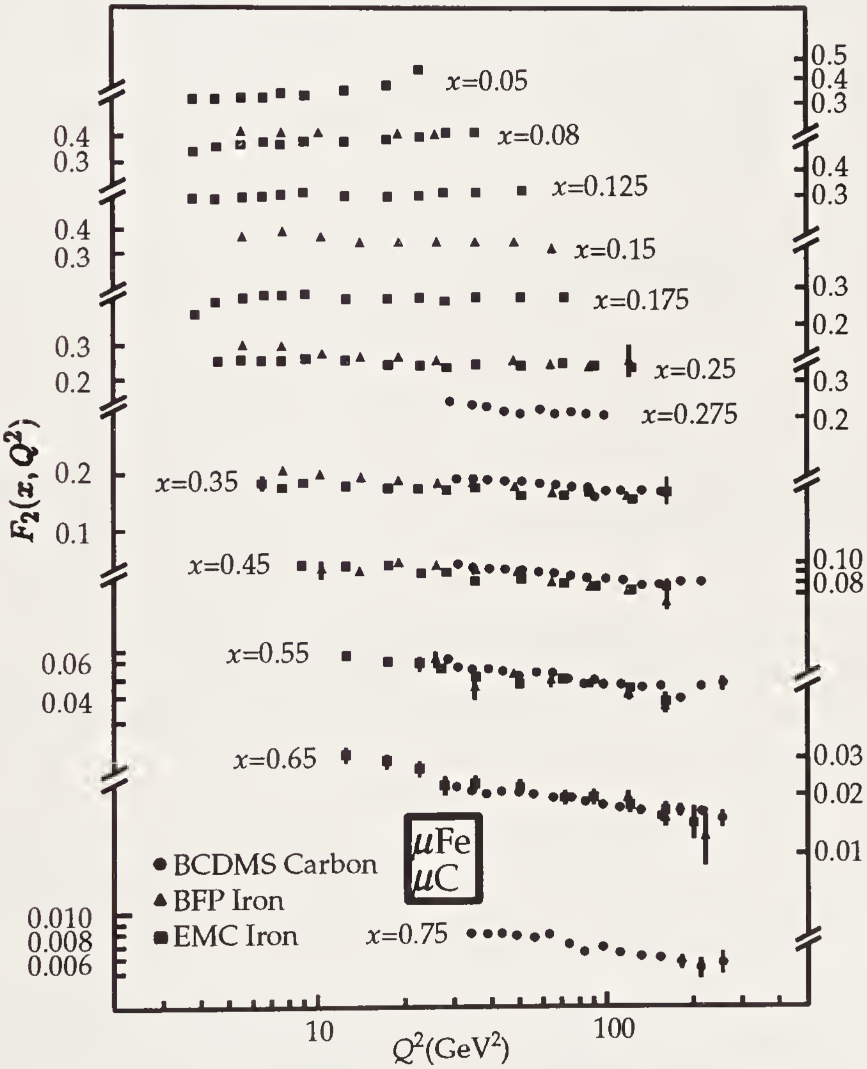


Fig. 5.7. Structure functions of muon–carbon and muon–iron scattering. (From Review of Particle Properties, Physical Review **D 45** (1992))

difference between neutrons and protons (the sums of the squared charges of the quarks have a ratio of 3 to 2). The remainder of the discrepancy is known as the EMC (European muon collaboration) effect, recalling the experimental collaboration that first discovered the effect.

The history of the EMC effect has been quite involved. Eventually it was found that part of the originally observed effect was due to a wrong measurement, since data analysis for small x values is extremely difficult. However, it is now clear that the structure functions depend on the size of the nucleus. Figure 5.8 gives recent experimental results.

A number of theoretical models for the observed phenomena exist, but their physical meaning is still heavily disputed. If we disregard this problem, pertubative QCD yields a reasonable description of the structure functions. To this end further corrections besides the gluon–bremsstrahlung correction have to be taken into account. They could of course be determined graph by graph, but there is a more elegant way to evaluate these corrections. Every calculation again and again employs the same techniques and the specific kinematics of the parton picture. Thus it is useful to formulate a general scheme suited for the special situation of deep inelastic scattering. In this way we are led directly to the Gribov–Lipatov–Altarelli–Parisi–equations, which we derive here in an intuitive fashion. Later on we discuss a more formal derivation.

First we rewrite the result of the bremsstrahlung calculation. Inserting (5.48) into (5.46) yields

$$\begin{aligned} 0 = \sum_i Q_i^2 \Bigg[& x \Delta f_i(x) \\ & + \frac{4}{3}\frac{\alpha_s}{2\pi} \ln\frac{Q_2^2}{Q_1^2} \int_x^1 \mathrm{d}\xi_i\, f_i(\xi_i)\frac{x}{\xi_i} \left(\left[1+\left(\frac{x}{\xi_i}\right)^2\right]\left(1-\frac{x}{\xi_i}\right)^{-1}\right)_+ \Bigg] \end{aligned} \tag{5.50}$$

Since the different flavors are completely independent, every single term in the sum must be equal to zero:

$$\Delta f_i(x) = -\frac{4}{3}\frac{\alpha_s}{2\pi} \ln\frac{Q_2^2}{Q_1^2} \int_x^1 \mathrm{d}\xi_i \frac{f_i(\xi_i)}{\xi_i} \left(\frac{1+\left(\frac{x}{\xi_i}\right)^2}{1-\frac{x}{\xi_i}} \right)_+ \quad . \tag{5.51}$$

We define

$$P_{qq}(z) = \frac{4}{3}\left\{\frac{1+z^2}{1-z}\right\}_+ \quad . \tag{5.52}$$

In addition we omit the index i:

$$\Delta f(x) = -\frac{\alpha_s}{2\pi} \ln\frac{Q_2^2}{Q_1^2} \int_x^1 \frac{\mathrm{d}y}{y} f(y) P_{qq}\left(\frac{x}{y}\right) \quad . \tag{5.53}$$

Furthermore we define the logarithmic momentum variable

$$\begin{aligned} t(Q^2) &= \ln\frac{Q^2}{Q_0^2} \quad , \\ \Delta t &= t(Q_1^2) - t(Q_2^2) = \ln\frac{Q_1^2}{Q_2^2} \quad , \end{aligned} \tag{5.54}$$

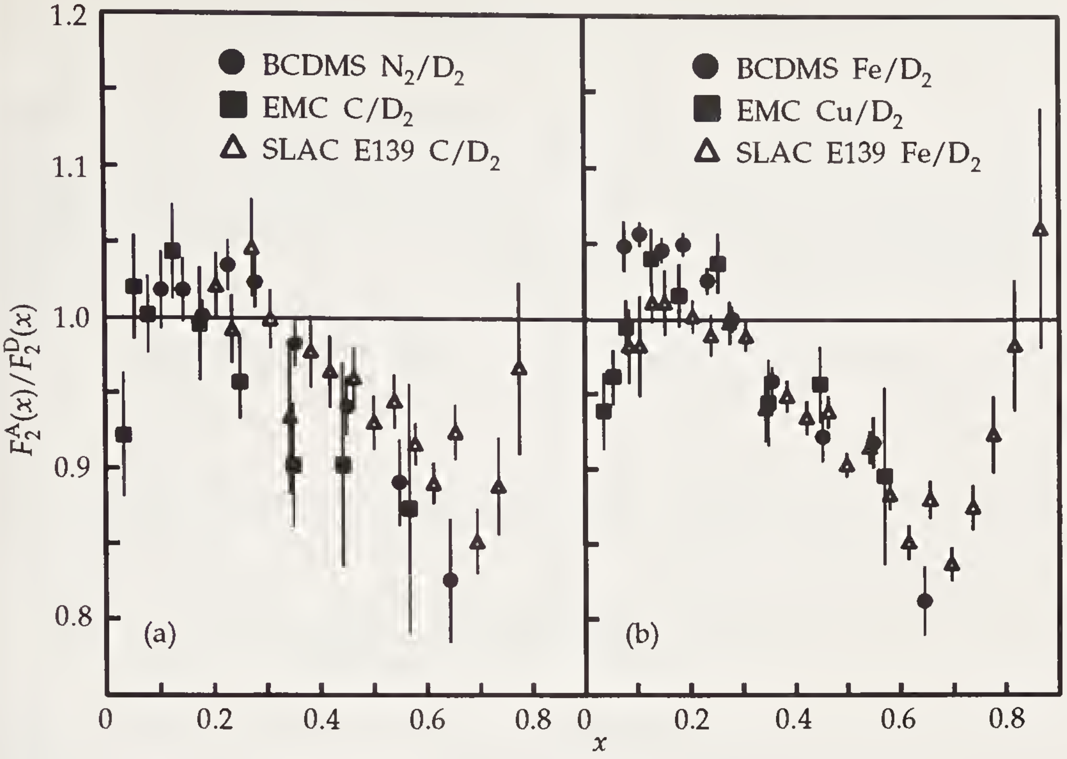


Fig. 5.8. The EMC effect: the ratio of F_2 for carbon, nitrogen, iron and copper to F_2 of deuterium (From Review of Particle Properties, Physical Review **D 45** (1992))

which gives

$$\frac{\Delta f(x)}{\Delta t} = \frac{\alpha_s}{2\pi} \int_x^1 \frac{\mathrm{d}y}{y} f(y) P_{qq}\left(\frac{x}{y}\right) \quad . \tag{5.55}$$

The last equation is only valid for small Δt. For large Δt values it must, for example, be taken into account that α_s also depends on Q^2. The correct equation is therefore:

$$\frac{\mathrm{d}f(x,t)}{\mathrm{d}t} = \frac{\alpha_s(t)}{2\pi} \int_x^1 \frac{\mathrm{d}y}{y} f(y,t) P_{qq}\left(\frac{x}{y}\right) \quad . \tag{5.56}$$

Equation (5.56) is already a Gribov–Lipatov–Altarelli–Parisi (GLAP) equation. All further QCD corrections are described by additional terms on the right-hand side.

We should like to remark that the expression $[(1+z^2)/(1-z)]_+$ can be written in different ways. Two of those used more frequently in the literature are the following:

$$P'_{qq}(z) = \frac{4}{3}\left(\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z)\right) \quad , \tag{5.57}$$

and

$$\begin{aligned} P''_{qq}(z) = &-\frac{4}{3}(1+z) + 2\delta(1-z) \\ &+ \lim_{\eta\to 0} \frac{8}{3}\left(\frac{1}{1-z+\eta} - \delta(1-z)\int_0^1 \mathrm{d}z' \frac{1}{1-z'+\eta}\right) \quad . \end{aligned} \tag{5.58}$$

The equivalence of (5.57) and (5.58) with (5.52) can be proven by applying them to an arbitrary smooth and continous function.

$$\begin{aligned} \int_0^1 P'_{qq}(z)f(z)\mathrm{d}z &= \frac{4}{3}\int_0^1 \mathrm{d}z \left(\frac{1+z^2}{1-z}f(z) - \frac{2}{1-z}f(1)\right) + \frac{4}{3}\cdot\frac{3}{2}f(1) \\ &= \frac{4}{3}\int_0^1 \mathrm{d}z \left(\frac{1+z^2}{1-z}f(z) - \frac{2}{1-z}f(1) + (1+z)f(1)\right) \\ &= \frac{4}{3}\int_0^1 \mathrm{d}z \left(\frac{1+z^2}{1-z}f(z) - \frac{z^2+1}{1-z}f(1)\right) \\ &= \frac{4}{3}\int_0^1 \mathrm{d}z \left\{\frac{1+z^2}{1-z}\right\}_+ f(z) \quad . \end{aligned} \tag{5.59}$$

For $P''_{qq}(z)$ a similar calculation can be performed:

$$\begin{aligned} \int_0^1 P''_{qq}(z)f(z)\mathrm{d}z = &-\frac{4}{3}\int_0^1 \mathrm{d}z \left(f(z)(1+z) + \lim_{\eta\to+0}\frac{8}{3}\frac{1}{1-z+\eta}f(z)\right) \\ &+ 2f(1) - \frac{8}{3}f(1)\lim_{\eta\to+0}\int_0^1 \mathrm{d}z \frac{1}{1-z+\eta} \\ = &\lim_{\eta\to+0}\frac{8}{3}\int \mathrm{d}z \frac{1}{\{1-z+\eta\}_+}f(z) \\ &-\frac{4}{3}\int_0^1 \mathrm{d}z \, [f(z)(1+z) - f(1)(1+z)] \\ = &\frac{8}{3}\int \mathrm{d}z \frac{1}{\{1-z\}_+}f(z) + \frac{4}{3}\int_0^1 \mathrm{d}z \frac{z^2-1}{1-z}[f(z)-f(1)] \\ = &\frac{4}{3}\int \mathrm{d}z \left(\frac{z^2+1}{1-z}\right)_+ f(z) \quad . \end{aligned} \tag{5.60}$$

EXERCISE

5.3 The Bremsstrahlung Part of the GLAP Equation

Problem. The bremsstrahlung process of Fig. 5.1 does not change the number of quarks or antiquarks. Prove that this statement is also contained in the GLAP equation (5.56).

Solution. The number of quarks with a specific flavor is given by

$$N = \int_0^1 \mathrm{d}x\, f(x) \quad . \tag{1}$$

One has to show that N does not depend on Q^2 and consequently not on t. Therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}N = \int_0^1 \mathrm{d}x \frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\alpha_s}{2\pi}\int_0^1 \mathrm{d}x \int_x^1 \frac{\mathrm{d}y}{y} f(y) P_{qq}\left(\frac{x}{y}\right) \tag{2}$$

must vanish. We exchange the x and y integrations:

$$\frac{\mathrm{d}}{\mathrm{d}t}N = \frac{\alpha_s}{2\pi}\int_0^1 \mathrm{d}y \int_0^y \frac{\mathrm{d}x}{y} f(y) P_{qq}\left(\frac{x}{y}\right) \tag{3}$$

and introduce the new variable $z = x/y$

$$\frac{\mathrm{d}}{\mathrm{d}t}N = \frac{\alpha_s}{2\pi}\int_0^1 \mathrm{d}y\, f(y) \int_0^1 \mathrm{d}z P_{qq}(z) = \frac{\alpha_s}{2\pi} N \int_0^1 \mathrm{d}z P_{qq}(z) \quad . \tag{4}$$

By means of (5.47) we have

$$\int_0^1 \mathrm{d}z \frac{4}{3}\left(\frac{1+z^2}{1-z}\right)_+ = \int_0^1 \mathrm{d}z \left(\frac{4}{3}-\frac{4}{3}\right)\frac{1+z^2}{1-z} = 0 \tag{5}$$

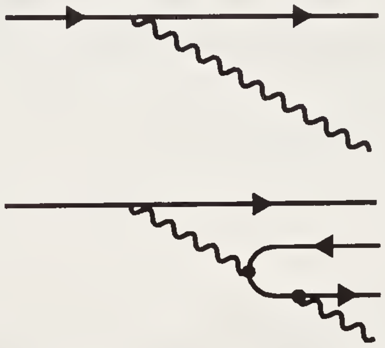
and hence

$$\frac{\mathrm{d}}{\mathrm{d}t}N = 0 \quad ,$$

which was to be shown.

It is no coincidence that the right-hand side of (5.56) integrated over x separates into two factors, namely the integrals over y and z. Since QCD is a dimensionless theory, P_{qq} can only depend on ratios of the momenta that occur, i.e., on x/y. Consequently all contributions to the GLAP equation exhibit the same feature. In the next section we discuss how this mathematical structure can be derived from very general assumptions.

The content of (5.56) can also be interpreted in the following way: owing to the QCD interaction a quark can split up into a couple of particles with the same total charge, for example, into a quark and a gluon or into two quarks, one antiquark, and a gluon.



Now the question arises whether the quark "fine structure" can be resolved by the exchanged photon. Clearly this depends on the photon momentum. For larger values of Q^2 smaller structures which might exist only for shorter times become visible (see Fig. 5.9).

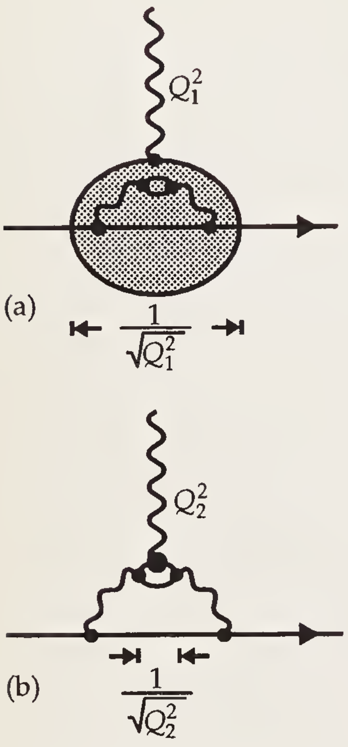
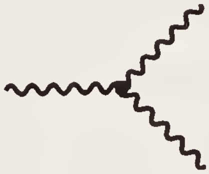
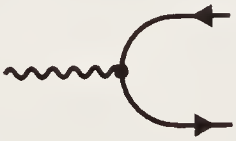


Fig. 5.9a,b. An illustration of the GLAP equation. Larger values of Q^2 ($Q_2^2 \gg Q_1^2$) resolve smaller structures

In this sense P_{qq} is also denoted a "splitting-function" and interpreted as the probability that a quark shows the inner structure quark + gluon at a resolution of $t = \ln(Q^2/Q_0^2)$. Now it is easy to guess in what way one has to generalize (5.56) in order to describe the following processes also:



and



First we introduce new symbols. The single quark distribution functions $q_i(x)$ and $\overline{q}_i(x)$ are replaced by a total quark distribution,

$$\Sigma(x) = \sum_i \left(q_i(x) + \overline{q}_i(x)\right) \quad , \quad i = \mathrm{u, d, s, c}, \dots \quad , \tag{5.61}$$

and by differences between the quark distributions,

$$\Delta_{ij}(x) = q_i(x) - q_j(x) \quad , \tag{5.62}$$

$$\overline{\Delta}_{ij} = \overline{q}_j(x) - \overline{q}_j(x) \quad , \tag{5.63}$$

$$V(x) = \sum_i (q_i(x) - \overline{q}_i(x)) \quad . \tag{5.64}$$

Usually $\Sigma(x)$ is referred to as a singlet distribution function, since it is symmetric in all flavors (it transforms as a singlet in the flavor symmetry group). $\Delta_{ij}, \overline{\Delta}_{ij}$, and V are correspondingly called non singlet distribution functions. The GLAP equations now becomes

$$\frac{\mathrm{d}\Delta_{ij}(x,t)}{\mathrm{d}t} = \frac{\alpha_s(t)}{2\pi} \int\limits_x^1 \frac{\mathrm{d}y}{y} \Delta_{ij}(y,t) P_{qq}\left(\frac{x}{y}\right) \quad , \tag{5.65}$$

$$\frac{\mathrm{d}\Sigma(x,t)}{\mathrm{d}t} = \frac{\alpha_s(t)}{2\pi} \int\limits_x^1 \frac{\mathrm{d}y}{y} \left[\sum(y,t) P_{qq}\left(\frac{x}{y}\right) + 2N_f G(y,t) P_{qG}\left(\frac{x}{y}\right)\right] \quad , \tag{5.66}$$

$$\frac{\mathrm{d}G(x,t)}{\mathrm{d}t} = \frac{\alpha_s(t)}{2\pi} \int\limits_x^1 \frac{\mathrm{d}y}{y} \left[\sum(y,t) P_{Gq}\left(\frac{x}{y}\right) + G(y,t) P_{GG}\left(\frac{x}{y}\right)\right] \quad . \tag{5.67}$$

Here $G(x,t)$ and N_f denote the gluon distribution function and the number of flavors; P_{qG}, P_{Gq}, and P_{GG} correspond to the above graphs respectively. A calculation analogous to the one discussed above yields

$$P_{qG}(z) = \frac{1}{2}\left[z^2 + (1-z)^2\right] \quad , \tag{5.68}$$

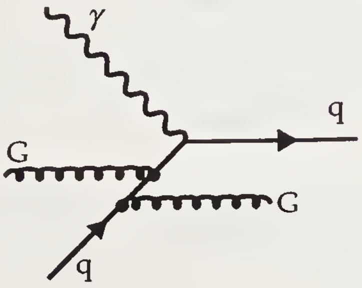
$$P_{Gq}(z) = \frac{4}{3}\left[1 + (1-z)^2\right] \frac{1}{z} \quad , \tag{5.69}$$

$$P_{GG}(z) = 6\left[z\left(\frac{1}{1-z}\right)_+ \frac{1-z}{z} + z(1-z) + \left(\frac{11}{12} - \frac{N_f}{18}\right)\delta(1-z)\right] \quad . \tag{5.70}$$

The derivation of the explicit expressions (5.68–70) is quite cumbersome, the general structure of (5.65–67) on the other hand is very simple. The bigger Q^2, the more partons with decreasing x are resolved. Therefore only such processes occur on the right-hand side, which increases the particle number.

Of course, there are also loss terms. The probability of a quark with a large momentum fraction is decreased for example by the emission of a gluon. This effect is contained in the definition of the splitting function and corresponds to the term that is subtracted owing to $P_{qq} = 4/3\left[(1+z^2)/(1-z)\right]_+$.

However, it should be clear by now that the GLAP equations are an approximation, valid only for large Q^2 and sufficiently large x. The first condition is obvious, since the GLAP equations are a perturbative expansion that becomes completely meaningless if α_s gets too big. Note, for example, the following process:



This graph represents a higher-order correction to the processes in Fig. 5.1. Therefore it contributes to P_{qq} and P_{Gq} in proportion to

$$\alpha_s^2 \int G(x_1) q(x_2) \tilde{P}_{qq}(x_1, x_2, x) \frac{\mathrm{d}x_1}{x_1} \frac{\mathrm{d}x_2}{x_2} \quad , \tag{5.71}$$

with a corresponding function $\tilde{P}_{qq}$. The essential point now is the following. In the case of very small x_1 $G(x_1)$ can increase so much that even α_s $G(x_1)$ remains larger than one (we shall find later that $G(x_1)$ is proportional to $1/x_1$ for small x_1). Hence there are kinematic regions where contributions like those of the last figure are no longer negligible. With the new HERA accelerator at DESY in Hamburg, Germany, these regions have for the first time become accessible to experimental investigations. At HERA an electron beam and a proton beam are collided with an invariant mass of

$$s = (p_\mathrm{e} + p_\mathrm{p})^2 = 4E_\mathrm{e}E_\mathrm{p} \approx 10^5\ \mathrm{GeV}^2 \quad . \tag{5.72}$$

Since the maximum momentum transfer ν_{max} is just $s/2$, events with $Q^2 \geq 5\ \mathrm{GeV}^2$ and $x \geq 10^{-4}$ can be investigated. Currently much effort is being invested into the necessary generalizations of the GLAP equations for small values of x. To this end their structure must be modified, as can already be seen from (5.71). Terms have to be introduced, for example, that depend on products of two or more distribution functions. Up to now there has been no generally accepted solution to this problem (Sect. 6.2). Presumably such a solution will only be found in connection with experimental investigations. The HERA experiments have produced data since 1992.

In this section we have learned how QCD corrections can be evaluated from the corresponding Feynman graphs. This procedure is of great clarity, but it requires extensive calculations. Therefore one frequently chooses another way to derive the GLAP equations that is much more elegant but unfortunately less obvious. The latter disadvantage occurs because a part of the result, namely the general structure of the GLAP equations, has to be more or less assumed. We should discuss this alternative derivation in the following section, since it shows another side to the meaning of the GLAP equations.

5.2 An Alternative Approach to the GLAP Equations

In this section we shall be concerned with the derivation of the GLAP equations using only our knowledge about the properties of QCD. We know already that scattering experiments using leptons as projectiles and hadrons as targets reveal information about the hadronic structure. The structure functions or momentum distributions of the partons inside the hadron depend on the four-momentum q_μ carried by the exchanged photon (in general the vector meson). If the momentum transfer Q^2 is increased, more and more details of the hadronic structure become visible (see Fig. 5.10).

Q^2 is related to the maximum transverse momentum of a parton in the final state. This fact is easy to understand looking at Fig. 5.10. In the Breit-frame, where the distribution functions are defined, the nucleon is strongly contracted in the direction

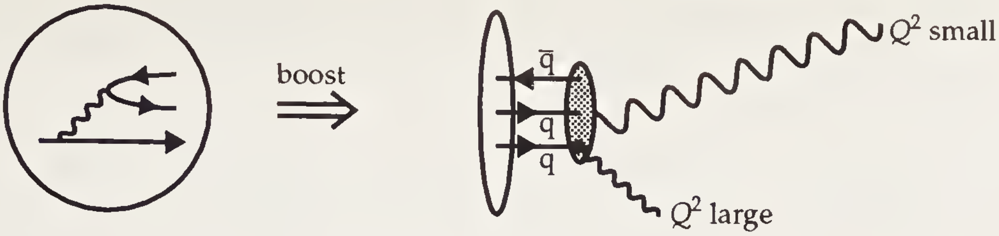


Fig. 5.10. The role of Q^2 as transverse resolving power

of movement. Therefore the resolution with which quarks are seen depends only on the transverse momentum. We denote by $1/\sqrt{Q^2}$ the resolving power, since, owing to the Heisenberg inequality hadronic structures down to $1/\sqrt{Q^2}$ can be resolved. Let us assume that we have chosen Q^2 high enough to enter the perturbative regime. We can resolve partons of size $1/\sqrt{Q^2}$ inside the hadron. If the momentum transfer is increased to $Q'^2 > Q^2$, we can resolve smaller constituents of size $1/\sqrt{Q'^2}$ inside the hadron. For a given resolving power $1/\sqrt{Q^2}$ the momentum distribution of the partons of kind a is denoted by $N_a(x_a, Q^2)$. The smaller constituents b at $1/\sqrt{Q'^2}$ carry some of the parton momenta x_a:

$$\frac{x_b}{x_a} < 1 \quad . \tag{5.73}$$

This results in a decrease of the momentum distribution for large x and an increase for small momentum fractions.

The variation of the distribution functions $\Delta N(x, Q^2)$ when Q^2 is increased by ΔQ^2 can now be treated by using perturbation theory. To lowest order $\Delta N(x, Q^2)$ is proportional to the coupling parameter

$$\alpha_s(Q^2) = \frac{g^2(Q^2)}{4\pi} \quad . \tag{5.74}$$

If the quark masses are neglected, QCD lacks any energy scale. Therefore only momentum ratios can appear in the equations which govern the development of $N_a(x_a, Q^2)$ to higher Q^2. This distribution function for partons of kind a depends for $Q'^2 > Q^2$ on the distribution of partons of all kinds b. We assume this dependence to be linear. The probability of finding a parton a with momentum fraction x_a inside a parton b of momentum fraction x_b is therefore given by

$$\Delta N_b(x_b, Q^2) = \frac{\alpha_s(Q^2)}{2} P_{ba}\left(\frac{x_b}{x_a}\right) N_a(x_a, Q^2) \frac{\Delta Q^2}{Q^2} \quad . \tag{5.75}$$

The quotient $\Delta Q^2/Q^2 = \Delta \ln Q^2$ is introduced to keep the change in momentum transfer dimensionless. In the same way it follows that P_{ba} can only depend on the ratio of the momentum fractions.

To obtain the total change of N_b we have to sum over all partons of kind a that may "contain" the parton b, and we have to integrate over all parton momenta $x_a > x_b$. This leads us, finally, to the GLAP equations:

$$\Delta N_b(x_b, Q^2) = \frac{\alpha_s(Q^2)}{2} \sum_b \int_{x_b}^{1} P_{ba}\left(\frac{x_b}{x_a}\right) N_a(x_a, Q^2) \frac{dx_a}{x_a} \Delta \ln Q^2 \quad . \tag{5.76}$$

(Here we have introduced the ratio $\mathrm{d}x_a/x_a$ instead of $\mathrm{d}x_a$ alone; otherwise a scale would enter the GLAP equations.) This is a set of equations for all types of partons, which in differential form becomes

$$\frac{\partial N_b(x_b,Q^2)}{\partial \ln Q^2} = \frac{\alpha_s(Q^2)}{2} \sum_b \int\limits_{x_b}^{1} P_{ba}\left(\frac{x_b}{x_a}\right) N_a(x_a,Q^2)\frac{\mathrm{d}x_a}{x_a} \quad . \tag{5.77}$$

The functions P_{ba} are called splitting functions since they describe the breaking up of a parton of type a into partons of type b, when the momentum transfer Q^2 is increased.

In the following we will derive the splitting functions P_{ba} for several cases. We make life easy by assuming that there are only two types of partons: quarks and gluons. This is justified because of charge-conjugation symmetry (change of quark distribution equals change of antiquark distribution) and because of our treating quarks as massless objects (change of u, d, s quarks proceeds in the same way). This reduces the number of splitting functions to four:

$$P_{qq} = -\frac{4}{3}(1+x) + 2\delta(1-x) + \frac{8}{3}\left(\frac{1}{1-x} - \delta(1-x)\int\limits_0^1 \mathrm{d}x'\frac{1}{1-x'}\right) \quad , \tag{5.78}$$

$$P_{qG} = \frac{1}{2}\left[(1-x)^2 + x^2\right] \quad , \tag{5.79}$$

$$P_{Gq} = \frac{4}{3}\frac{1+(1-x)^2}{x} \quad , \tag{5.80}$$

$$\begin{aligned} P_{GG} = {} & 6\left[\frac{1}{x} - 2 + x(1-x)\right] + \left(\frac{11}{2} - \frac{N_f}{3}\right)\delta(1-x) \\ & + 6\left(\frac{1}{1-x} - \delta(1-x)\int\limits_0^1 \mathrm{d}x'\frac{1}{1-x'}\right) \quad . \end{aligned} \tag{5.81}$$

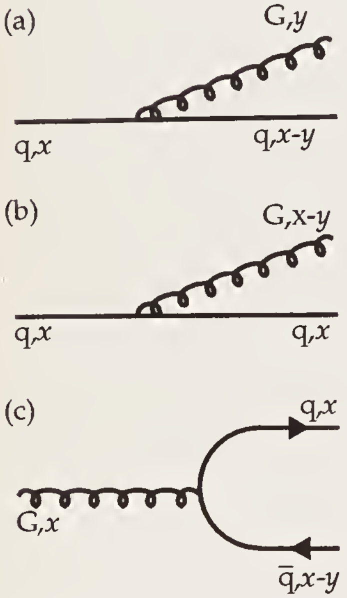


Fig. 5.11a–c. The processes described by the splitting functions (**a**) P_{Gq}, (**b**) P_{qq}, and (**c**) P_{qG}

We shall evaluate the functions P_{Gq}, P_{qq}, and P_{qG} in Examples 5.5–7. The effect of these functions is depicted graphically in Figs. 5.11a,b, and c.

EXERCISE

5.4 The Maximum Transverse Momentum

Problem. Determine the maximum transverse momentum of a parton occurring in the final state if the photon momentum before the collision is given in the Breit system by

$$q_\mu = (0;0,0,-Q) \tag{1}$$

and the initial parton momentum in the Breit system is

$$p_\mu = (p;0,0,p) \quad . \tag{2}$$

Exercise 5.4.

Solution. We first assume that two partons escape after the collision. These partons carry momenta

$$p_1' = \left(\frac{p}{2}; \boldsymbol{p}_\perp, \frac{p-Q}{2}\right) \quad ,$$
$$p_2' = \left(\frac{p}{2}; -\boldsymbol{p}_\perp, \frac{p-Q}{2}\right) \quad . \tag{3}$$

Obviously the four-momentum is conserved:

$$q + p = p_1' + p_2' \quad . \tag{4}$$

Furthermore squaring the parton momentum p_1' yields

$$\begin{aligned}(p_1')^2 = 0 &= \left(\frac{p}{2}\right)^2 - (\boldsymbol{p}_\perp)^2 - \left(\frac{p-Q}{2}\right)^2 \\ &= \left(\frac{p}{2}\right)^2 - (\boldsymbol{p}_\perp)^2 - \left(\frac{p}{2}\right)^2 + \frac{2p \cdot Q}{4} - \left(\frac{Q}{2}\right)^2 \\ &= -(\boldsymbol{p}_\perp)^2 + \frac{p \cdot Q}{2} - \left(\frac{Q}{2}\right)^2 \quad .\end{aligned} \tag{5}$$

Introducing the momentum fraction $x = Q^2/(2p \cdot Q)$, we get

$$-(\boldsymbol{p}_\perp)^2 - \left(\frac{Q}{2}\right)^2 + \frac{Q^2}{4x} = 0 \tag{6}$$

or

$$|\boldsymbol{p}_\perp| = \frac{Q}{2}\sqrt{\frac{1-x}{x}} \quad . \tag{7}$$

For more than two partons in the final state this is the maximum transverse momentum that can be achieved.

EXAMPLE

5.5 Derivation of the Splitting Function P_{Gq}

For given momentum transfer Q^2 – in the perturbative region – we define the Hamiltonian

$$\hat{H}_{Q^2} = \hat{H}_{Q^2}^{\text{kin}} + \hat{V} \quad . \tag{1}$$

The potential $\hat{V}$ describes the binding of the partons. It is the nonperturbative part, including interactions with small Q^2. The kinetic part $\hat{H}_{Q^2}^{\text{kin}}$ yields when applied to the parton wave functions the kinetic parton energies

$$\hat{H}_{Q^2}^{\text{kin}}|q_{Q^2}(p)\rangle = E|q_{Q^2}(p)\rangle = |\boldsymbol{p}||q_{Q^2}(p)\rangle \quad ,$$
$$\hat{H}_{Q^2}^{\text{kin}}|G_{Q^2}(k)\rangle = \omega|G_{Q^2}(k)\rangle = |\boldsymbol{k}||G_{Q^2}(k)\rangle \quad . \tag{2}$$

Example 5.5.

$|q_{Q^2}(p)\rangle$ and $|G_{Q^2}(k)\rangle$ denote respectively the quark and gluon wave functions at momentum transfer Q^2. Again Q is related to the maximum transverse momentum by

$$|\boldsymbol{k}_\perp|_{\max} = \frac{Q}{2}\sqrt{\frac{1-x}{x}} = \varepsilon Q \quad . \tag{3}$$

At $Q'^2 > Q^2$ the kinetic part of the Hamiltonian can be written as

$$\hat{H}^{\text{kin}}_{Q'^2} = \hat{H}^{\text{kin}}_{Q^2} + \hat{H}_{\Delta Q^2} \quad . \tag{4}$$

Application of $H^{\text{kin}}_{Q'^2}$ to the quark and gluon wave functions at Q'^2 again yields

$$\begin{aligned}\hat{H}^{\text{kin}}_{Q'^2}|q_{Q'^2}(p)\rangle &= |\boldsymbol{p}||q_{Q'^2}(p)\rangle \quad ,\\ \hat{H}^{\text{kin}}_{Q'^2}|G_{Q'^2}(k)\rangle &= |\boldsymbol{k}||G_{Q'^2}(k)\rangle \quad .\end{aligned} \tag{5}$$

The additional operator $\hat{H}_{\Delta Q^2}$ just describes the interactions leading to transverse momenta $|\boldsymbol{k}_\perp|$ between εQ and $\varepsilon Q'$.

The interaction part of the QCD Hamiltonian can be derived in the temporal Weyl gauge, $A_0^\alpha = 0$ for all gluons. The Hamiltonian $\hat{H}_{\Delta Q^2}$ up to first order in the coupling parameter g is

$$\begin{aligned}\hat{H}_{\Delta Q^2} = &-g(Q^2)\int \mathrm{d}^3r \sum_f \hat{\bar{q}}_f(\boldsymbol{r})\left[\gamma\cdot\hat{\boldsymbol{A}}^\alpha(\boldsymbol{r})\frac{\lambda^\alpha}{2}\right]\hat{q}_f(\boldsymbol{r})\\ &+ g(Q^2)\int \mathrm{d}^3r f_{\alpha\beta\gamma}\hat{A}_j^\alpha(\boldsymbol{r})\hat{A}_k^\beta(\boldsymbol{r})\frac{\partial}{\partial r_j}\hat{A}_k^\gamma(\boldsymbol{r}) \quad .\end{aligned} \tag{6}$$

The first part describes the interaction between quarks and gluons, the second part handles the 3-gluon interaction. In the first part the sum is over all quark flavors $u, d, s, \ldots$. In both parts the integration is performed over the region in space that is in accordance with the transverse momentum $\boldsymbol{k}_\perp$.

We turn now to the quark and gluon wave functions. Since for higher resolution Q'^2 smaller constituents of the hadron can be seen, the wave function describing big partons at Q^2 can be written as a superposition of wave functions for the constituents at Q'^2. For the quark and gluon wave functions at Q^2 we get the expansion

$$\begin{aligned}|q_{Q^2}(p)\rangle &= C_q(p)|q_{Q'^2}(p)\rangle + \sum_{k',p'} C_{Gq}(k',p')|G_{Q'^2}(k')q_{Q'^2}(p')\rangle \quad ,\\ |G_{Q^2}(k)\rangle &= C_G(k)|G_{Q'^2}(k)\rangle + \sum_{k',p'} C_{qG}(k',p')|G_{Q'^2}(k')q_{Q'^2}(p')\rangle\\ &\quad + \sum_{k',k''} C_{GG}(k',k'')|G_{Q'^2}(k')G_{Q'^2}(k'')\rangle \quad .\end{aligned} \tag{7}$$

The coefficients are given by

Example 5.5.

$$C_q(p) = \langle q_{Q'^2}(p) | q_{Q^2}(p) \rangle \quad ,$$

$$C_G(k) = \langle G_{Q'^2}(k) | G_{Q^2}(k) \rangle \quad ,$$

$$C_{Gq}(k', p') = \langle G_{Q'^2}(k') q_{Q'^2}(p') | q_{Q^2}(p) \rangle \quad , \tag{8}$$

$$C_{qG}(k', p') = \langle G_{Q'^2}(k') q_{Q'^2}(p') | G_{Q^2}(p) \rangle \quad ,$$

$$C_{GG}(k', k'') = \langle G_{Q'^2}(k') G_{Q'^2}(k'') | G_{Q^2}(k) \rangle \quad .$$

The squared coefficients can be related to the momentum distribution functions. Now we have all the ingredients to determine the splitting functions. We start with the calculation of the probability of resolving a gluon inside a quark (Fig. 5.11a). This will lead us to an expression for the splitting function P_{Gq}.

We apply the kinetic part $\hat{H}^{\mathrm{kin}}_{Q'^2}$ of the Hamiltonian to the quark wave function at resolution Q^2. This gives on the one hand

$$\begin{aligned} \hat{H}^{\mathrm{kin}}_{Q'^2} | q_{Q^2}(p) \rangle &= \hat{H}^{\mathrm{kin}}_{Q^2} | q_{Q^2}(p) \rangle + \hat{H}_{\Delta Q^2} | q_{Q^2}(p) \rangle \\ &= |\boldsymbol{p}| \, | q_{Q^2}(p) \rangle + \hat{H}_{\Delta Q^2} | q_{Q^2}(p) \rangle \quad . \end{aligned} \tag{9}$$

Expansion of the quark wave function yields on the other hand

$$\begin{aligned} &\hat{H}^{\mathrm{kin}}_{Q'^2} | q_{Q^2}(p) \rangle \\ &\quad = C_q(p) \hat{H}^{\mathrm{kin}}_{Q'^2} | q_{Q'^2}(p) \rangle + \sum_{k', p'} C_{Gq}(k', p') \hat{H}^{\mathrm{kin}}_{Q'^2} | G_{Q'^2}(k') q_{Q'^2}(p') \rangle \\ &\quad = C_q(p) |\boldsymbol{p}| \, | q_{Q'^2}(p) \rangle + \sum_{k', p'} (|\boldsymbol{k}'| + |\boldsymbol{p}'|) C_{Gq}(k', p') | G_{Q'^2}(k') q_{Q'^2}(p') \rangle \; . \end{aligned} \tag{10}$$

We project (9) onto the final state wave function $|G_{Q'^2}(k') q_{Q'^2}(p')\rangle$:

$$\begin{aligned} &|\boldsymbol{p}| \langle G_{Q'^2}(k') q_{Q'^2}(p') | q_{Q^2}(p) \rangle + \langle G_{Q'^2}(k') q_{Q'^2}(p') | \hat{H}_{\Delta Q^2} | q_{Q^2}(p) \rangle \\ &\quad = (|\boldsymbol{k}'| + |\boldsymbol{p}'|) C_{Gq}(k', p') \quad , \end{aligned} \tag{11}$$

employing the orthogonality between the wave functions. Thus we obtain an expression for C_{Gq}:

$$C_{Gq}(k', p') = \frac{\langle G_{Q'^2}(k') q_{Q'^2}(p') | \hat{H}_{\Delta Q^2} | q_{Q^2}(p) \rangle}{|\boldsymbol{k}'| + |\boldsymbol{p}'| - |\boldsymbol{p}|} \quad . \tag{12}$$

The absolute square of this coefficient is proportional to the probability of finding a gluon of longitudinal momentum k' inside a quark of momentum p when the resolution Q^2 is increased. The momentum fraction of the gluon relative to the quark is therefore

$$x = \frac{k'_{\parallel}}{|\boldsymbol{p}|} = \frac{\boldsymbol{k}' \cdot \boldsymbol{p}}{|\boldsymbol{p}|^2} \quad . \tag{13}$$

Example 5.5.

The gluon distribution can also be written as

$$N_G(x,Q^2)=\sum_a \frac{1}{2}\sum_{ss'\varepsilon^a}\int\frac{\mathrm{d}^3p'}{(2\pi)^3 2E'}\int\frac{\mathrm{d}^3k'}{(2\pi)^3 2\omega'}$$
$$\times\,\delta\left(x-\frac{\boldsymbol{k}'\cdot\boldsymbol{p}}{|\boldsymbol{p}|^2}\right)|C_{Gq}(k',p')|^2\frac{1}{2E}\quad, \tag{14}$$

where the $\boldsymbol{k}$ integration is performed for $|\boldsymbol{k}_\perp|$ in the range from εQ^2 to $\varepsilon Q'^2$. The eight color degrees of freedom of the gluons are enumerated by a. We now evaluate the squared coefficient $|C_{Gq}(k',p')|^2$. To do this, only the first part of Hamiltonian (6) is required:

$$C_{Gq}(k',p')=-g(Q^2)\int\mathrm{d}^3r\sum_f\langle G_{Q'^2}(k')q_{Q'^2}(p')|\hat{\bar{q}}_f(\boldsymbol{r})$$
$$\times\left(\boldsymbol{\gamma}\cdot\hat{\boldsymbol{A}}^a(\boldsymbol{r})\frac{\lambda^a}{2}\right)\hat{q}_f(\boldsymbol{r})|q_{Q^2}(p)\rangle\frac{1}{|\boldsymbol{k}'|+|\boldsymbol{p}'|-|\boldsymbol{p}|}$$
$$=-g(Q^2)\delta^3(\boldsymbol{k}'+\boldsymbol{p}'-\boldsymbol{p})(2\pi)^3\overline{u(p',s')}\boldsymbol{\varepsilon}^{a*}\cdot\boldsymbol{\gamma}\frac{\lambda^a}{2}u(p,s)$$
$$\times\frac{1}{|\boldsymbol{k}'|+|\boldsymbol{p}'|-|\boldsymbol{p}|}\quad. \tag{15}$$

From this it follows that

$$N_G(x,Q^2)=\int\frac{\mathrm{d}^3p'}{(2\pi)^3}\int\frac{\mathrm{d}^3k'}{(2\pi)^3}\frac{1}{8EE'\omega'}\delta\left(x-\frac{\boldsymbol{k}'\cdot\boldsymbol{p}}{|\boldsymbol{p}|^2}\right)$$
$$\times\,\delta^3(\boldsymbol{k}'+\boldsymbol{p}'-\boldsymbol{p})(2\pi)^3 g(Q^2)^2\frac{1}{(|\boldsymbol{k}'|+|\boldsymbol{p}'|-|\boldsymbol{p}|)^2}$$
$$\times\frac{1}{2}\sum_a\sum_{s,s',\varepsilon^a}\left|\overline{u(p',s')}(\boldsymbol{\varepsilon}^{a*}\cdot\boldsymbol{\gamma})\frac{\lambda^a}{2}u(p,s)\right|^2\quad. \tag{16}$$

We consider the expression for the squared matrix element. Performing the summation over color and spin, where we have used the relation $\not\varepsilon^a=-\boldsymbol{\varepsilon}^a\cdot\boldsymbol{\gamma}$ for $\varepsilon^a=(0,\boldsymbol{\varepsilon}^a)$, we get

$$\frac{1}{2}\sum_s\frac{1}{3}\sum_c\sum_a\sum_{\varepsilon^a}\sum_{s',c'}\left|\overline{u(p',s')}\not\varepsilon^a\frac{\lambda^a}{2}u(p,s)\right|^2$$
$$=\frac{1}{6}\sum_a\sum_{\varepsilon^a}\mathrm{tr}\left\{\frac{(\lambda^a)^2}{4}\right\}\mathrm{tr}\{\not p'\not\varepsilon^{a*}\not p\not\varepsilon^a\}$$
$$=\frac{1}{6}\sum_a\sum_{\varepsilon^a}\frac{1}{2}\cdot 4\left[(p'\cdot\varepsilon^{a*})(p\cdot\varepsilon^a)+(p'\cdot\varepsilon^a)(p\cdot\varepsilon^{a*})-(p\cdot p')(\varepsilon^a\cdot\varepsilon^{a*})\right]$$
$$=\frac{1}{3}\sum_a\sum_{\varepsilon^a}\left[(\boldsymbol{p}'\cdot\boldsymbol{\varepsilon}^{a*})(\boldsymbol{p}\cdot\boldsymbol{\varepsilon}^a)\right.$$
$$\left.+(\boldsymbol{p}'\cdot\boldsymbol{\varepsilon}^a)(\boldsymbol{p}\cdot\boldsymbol{\varepsilon}^{a*})-(p_0'p_0-(\boldsymbol{p}'\cdot\boldsymbol{p}))(\boldsymbol{\varepsilon}^a\cdot\boldsymbol{\varepsilon}^{a*})\right]$$
$$=\frac{1}{3}\sum_a\sum_{\varepsilon^a}(\varepsilon^a)_i(\varepsilon^{a*})_j\left\{p_ip_j'+p_i'p_j-\left[(\boldsymbol{p}'\cdot\boldsymbol{p})-|\boldsymbol{p}||\boldsymbol{p}'|\right]\delta_{ij}\right\}$$

Example 5.5.

$$= \frac{1}{3} 8 \cdot \left(\delta_{ij} - \frac{k'_i k'_j}{|\boldsymbol{k}'|^2}\right) \left\{p_i p'_j + p'_i p_j - \left[(\boldsymbol{p}' \cdot \boldsymbol{p}) - |\boldsymbol{p}||\boldsymbol{p}'|\right] \delta_{ij}\right\}$$

$$= \frac{16}{3} \frac{1}{|\boldsymbol{k}'|^2} [|\boldsymbol{p}||\boldsymbol{p}'||\boldsymbol{k}'|^2 - (\boldsymbol{k}' \cdot \boldsymbol{p})(\boldsymbol{k}' \cdot \boldsymbol{p}')] \quad , \tag{17}$$

which results in

$$\begin{aligned} N_G(x, Q^2) = \frac{2}{3} \frac{1}{(2\pi)^3} g^2(Q^2) \int \mathrm{d}^3 p' \int \mathrm{d}^3 k' \frac{1}{EE'\omega'} \delta\left(x - \frac{\boldsymbol{k}' \cdot \boldsymbol{p}}{|\boldsymbol{p}|^2}\right) \\ \times \delta^3(\boldsymbol{k}' + \boldsymbol{p}' - \boldsymbol{p}) \cdot \frac{1}{(|\boldsymbol{k}'| + |\boldsymbol{p}'| - |\boldsymbol{p}|)^2} \\ \times \frac{1}{|\boldsymbol{k}'|^2} [|\boldsymbol{p}||\boldsymbol{p}'||\boldsymbol{k}'|^2 - (\boldsymbol{k}' \cdot \boldsymbol{p})(\boldsymbol{k}' \cdot \boldsymbol{p}')] \quad . \end{aligned} \tag{18}$$

The momenta can be written as

$$\begin{aligned} \boldsymbol{p} &= (0, 0, p) \quad , \\ \boldsymbol{k}' &= (\boldsymbol{k}'_\perp, xp) \quad , \\ \boldsymbol{p}' &= (-\boldsymbol{k}'_\perp, (1-x)p) \quad . \end{aligned} \tag{19}$$

Obviously the three-momentum is conserved: $\boldsymbol{p} = \boldsymbol{k}' + \boldsymbol{p}'$. The transverse momentum is expected to be small compared to the longitudinal momenta:

$$|\boldsymbol{k}'_\perp| \ll p \quad , \quad |\boldsymbol{k}'_\perp| \ll xp \quad , \quad |\boldsymbol{k}'_\perp| \ll (1-x)p \quad . \tag{20}$$

The contribution to $N_G(x, Q^2)$ can be approximated by

$$\begin{aligned} |\boldsymbol{k}'| + |\boldsymbol{p}'| - |\boldsymbol{p}| &\approx xp + \frac{|\boldsymbol{k}'_\perp|^2}{2x|\boldsymbol{p}|} + (1-x)p + \frac{|\boldsymbol{k}'_\perp|^2}{2(1-x)|\boldsymbol{p}|} - |\boldsymbol{p}| \\ &= \frac{|\boldsymbol{k}'_\perp|^2}{2p} \frac{1}{x(1-x)} \quad , \end{aligned} \tag{21}$$

and therefore

$$\frac{1}{|\boldsymbol{k}'| + |\boldsymbol{p}'| - |\boldsymbol{p}|} \approx \frac{2p}{|\boldsymbol{k}'_\perp|^2} x(1-x) \quad .$$

Furthermore

$$\begin{aligned} &\frac{1}{|\boldsymbol{k}'|} [|\boldsymbol{p}||\boldsymbol{p}'||\boldsymbol{k}'|^2 - (\boldsymbol{k}' \cdot \boldsymbol{p})(\boldsymbol{k}' \cdot \boldsymbol{p}')] \\ &\quad \approx \frac{1}{x^2 p^2} [p(1-x)p(|\boldsymbol{k}'_\perp|^2 + x^2 p^2) \\ &\quad + \frac{1}{2(1-x)p^2} |\boldsymbol{k}'_\perp| p(|\boldsymbol{k}'_\perp|^2 + x^2 p^2) + xp^2 |\boldsymbol{k}'_\perp|^2 - x^2 p^2 (1-x) p^2] \\ &\quad = \frac{|\boldsymbol{k}'_\perp|^2}{x^2} \left[1 + \frac{x^2}{2(1-x)}\right] \end{aligned} \tag{22}$$

and

$$\frac{1}{EE'\omega'} = \frac{1}{|\boldsymbol{p}||\boldsymbol{p}'||\boldsymbol{k}'|} \approx \frac{1}{p^3 x(1-x)} \quad . \tag{23}$$

Inserting Eq. (21–23) into the gluon distribution (18) results in

Example 5.5.

$$\begin{aligned} N_G(x,Q^2) &= \frac{2}{3}\frac{1}{(2\pi)^3}g^2(Q^2)\int \mathrm{d}^3k' \int \mathrm{d}^3p' \\ &\quad \times \delta\left(x - \frac{\boldsymbol{k}'\cdot\boldsymbol{p}}{|\boldsymbol{p}|^2}\right)\delta^3(\boldsymbol{k}'+\boldsymbol{p}'-\boldsymbol{p})\frac{1}{p^3x(1-x)} \\ &\quad \times \frac{4p^2}{|\boldsymbol{k}'_\perp|^4}x^2(1-x)^2\cdot\frac{|\boldsymbol{k}'_\perp|^2}{x^2}\left(1+\frac{x^2}{2(1-x)}\right) \\ &= \frac{4}{3}\frac{1}{(2\pi)^3}g^2(Q^2)\frac{1+(1-x)^2}{x}\int\frac{\mathrm{d}^2k'_\perp}{|\boldsymbol{k}'_\perp|^2} \quad . \end{aligned} \tag{24}$$

In the last step we wrote $\mathrm{d}^3k' = \mathrm{d}^2k'_\perp \mathrm{d}k_\parallel$ and performed the integration over $\boldsymbol{p}'$ and $k_\parallel$, thus cancelling the δ functions.

The last integration has to be performed for transverse momenta $|\boldsymbol{k}'_\perp|$ between εQ and $\varepsilon Q'$:

$$\begin{aligned} \int_{\varepsilon Q}^{\varepsilon Q'}\frac{\mathrm{d}^2k'_\perp}{|\boldsymbol{k}'_\perp|^2} &= 2\pi\int_{\varepsilon Q}^{\varepsilon Q'}\frac{dk'_\perp}{|\boldsymbol{k}'_\perp|} = 2\pi\ln|\boldsymbol{k}'_\perp|\Big|_{\varepsilon Q}^{\varepsilon Q'} \\ &= \pi(\ln\varepsilon^2Q'^2 - \ln\varepsilon^2Q^2) \\ &= \pi\Delta\ln\varepsilon^2Q^2 = \pi\Delta\ln Q^2 \quad . \end{aligned} \tag{25}$$

Since ε was introduced as an arbitrary constant $\Delta\ln\varepsilon = 0$. We are left with

$$N_G(x,Q^2) = \frac{g^2(Q^2)}{8\pi^2}\frac{4}{3}\frac{1+(1-x)^2}{x}\Delta\ln Q^2 \quad . \tag{26}$$

This result now has to be compared to the GLAP equation, where we have to impose the initial condition

$$\begin{aligned} N_q(x,Q^2) &= \delta(1-x) \quad , \\ N_G(x,Q^2) &= 0 \quad . \end{aligned} \tag{27}$$

From the GLAP equations it follows that

$$\begin{aligned} N_G(x,Q'^2) &= N_G(x,Q^2) + \Delta N_G(x,Q^2) \\ &= \frac{g^2}{8\pi^2}P_{Gq}(x)\Delta\ln Q^2 \quad . \end{aligned} \tag{28}$$

For the P_{Gq} splitting function we get the desired result:

$$P_{Gq}(x) = \frac{4}{3}\frac{1+(1-x)^2}{x} \quad . \tag{29}$$

EXAMPLE

5.6 Derivation of the Splitting Function P_{qq}

To determine the quark momentum distribution we have to keep in mind that the production of a quark of momentum fraction x corresponds to the production of a gluon of momentum fraction $1-x$. For $x=1$ the original quark is obtained with a probability $|C_q(p)|^2$. We can therefore write

$$\begin{aligned} N_q(x,Q'^2) &= N_q(x,Q^2)+\Delta N_q(x,Q^2) \\ &= \Delta N_G(1-x,Q^2)+\delta(1-x)|C_q(p)|^2 \quad . \end{aligned} \tag{1}$$

The change of the quark distribution function is then given by

$$\begin{aligned} \Delta N_q(x,Q^2) &= N_q(x,Q'^2)-N_q(x,Q^2) \\ &= N_G(1-x,Q^2)-\delta(1-x)\left[1-|C_q(p)|^2\right] \quad . \end{aligned} \tag{2}$$

$\Delta N_G(1-x,Q^2)$ is already known. Furthermore the longitudinal momentum is conserved. The expectation value of the longitudinal momentum should equal zero:

$$\int\limits_0^1 \mathrm{d}x\, x(\Delta N_q(x,Q^2)+\Delta N_G(x,Q^2))=0 \quad . \tag{3}$$

If we use the result of the last section, that the change of the gluon distribution is given by the splitting function P_{Gq},

$$\begin{aligned} \Delta N_G(1-x,Q^2) &= \frac{g^2(Q^2)}{8\pi^2}\frac{4}{3}\frac{1+x^2}{1-x}\Delta\ln Q^2 \quad , \\ \Delta N_G(x,Q^2) &= \frac{g^2(Q^2)}{8\pi^2}\frac{4}{3}\frac{1+(1-x)^2}{x}\Delta\ln Q^2 \quad , \end{aligned} \tag{4}$$

momentum conservation allows as to determine the coefficient $|C_q(p)|^2$:

$$0=\int\limits_0^1 \mathrm{d}x\, x\left[\Delta N_G(1-x,Q^2)-\delta(1-x)(1-|C_q(p)|^2)+\Delta N_G(x,Q^2)\right] \tag{5}$$

or

$$\begin{aligned} |C_q(p)|^2 &= 1-\int\limits_0^1 \mathrm{d}x\, x\left[\Delta N_G(1-x,Q^2)+\Delta N_G(x,Q^2)\right] \\ &= 1-\frac{g^2(Q^2)}{8\pi^2}\Delta\ln Q^2\frac{4}{3}\int\limits_0^1 \mathrm{d}x\, x\left(\frac{1+(1-x)^2}{x}+\frac{1+x^2}{1-x}\right) \\ &= 1-\frac{g^2(Q^2)}{8\pi^2}\Delta\ln Q^2\left(\frac{8}{3}\int\limits_0^1 \mathrm{d}x\frac{1}{1-x}-2\right) \quad . \end{aligned} \tag{6}$$

Example 5.6.

Finally we get for the change in the quark distribution (2)

$$\Delta N_q(x,Q^2) = \frac{g^2(Q^2)}{8\pi^2}\Delta \ln Q^2 \left[\frac{4}{3}\frac{1+x^2}{1-x} + 2\delta(1-x) - \frac{8}{3}\,\delta(1-x)\int\limits_0^1 \mathrm{d}x\,\frac{1}{1-x}\right] \quad . \tag{7}$$

Comparing this to the GLAP equation

$$\Delta N_q(x,Q^2) = \frac{g^2(Q^2)}{8\pi^2}P_{qq}(x)\Delta \ln Q^2 \tag{8}$$

allows us to extract the splitting function

$$\begin{aligned} P_{qq} &= \frac{4}{3}\frac{1+x^2}{1-x} + 2\delta(1-x) - \frac{8}{3}\delta(1-x)\int\limits_0^1 \mathrm{d}x\,\frac{1}{1-x} \\ &= -\frac{4}{3}(x+1) + 2\delta(1-x) + \frac{8}{3}\left(\frac{1}{1-x} - \delta(1-x)\int\limits_0^1 \mathrm{d}x\,\frac{1}{1-x}\right) \quad . \end{aligned} \tag{9}$$

EXAMPLE

5.7 Derivation of the Splitting Function P_{qG}

Next we calculate the splitting function that describes the quark fraction inside a gluon when the resolving power is increased from Q^2 to Q'^2 (Fig. 5.11c). The final state wave function is given by

$$|q^2_{Q'}(p)\overline{q}_{Q'^2}(p')\rangle \quad . \tag{1}$$

The initial gluon wave function at Q^2 can be expanded into these quarks states:

$$|G_{Q^2}(k)\rangle = \sum_{p,p'} C_{q\overline{q}}(p,p')|q_{Q'^2}(p)\overline{q}_{Q'^2}(p')\rangle \quad , \tag{2}$$

where the coefficients are given by

$$C_{q\overline{q}}(p,p') = \langle q_{Q'^2}(p)\overline{q}_{Q'^2}(p')|G_{Q^2}(k)\rangle \quad . \tag{3}$$

Applying the kinetic part of the Hamiltonian yields

$$\hat{H}_{Q^2}|G_{Q^2}(k)\rangle + \Delta\hat{H}_{Q^2}|G_{Q^2}(k)\rangle = \sum_{p,p'} C_{q\overline{q}}(p,p')\hat{H}_{Q'^2}|q_{Q'^2}(p)\overline{q}_{Q'^2}(p')\rangle \quad , \tag{4}$$

or

$$k|G_{Q^2}(k)\rangle + \Delta\hat{H}_{Q^2}|G_{Q^2}(k)\rangle = \sum_{p,p'}(|p| + |p'|)C_{q\overline{q}}(p,p')|q_{Q'^2}(p)\overline{q}_{Q'^2}(p')\rangle \quad . \tag{5}$$

The coefficients can be obtained by projecting with $\langle q_{Q'^2}(p)\overline{q}_{Q'^2}(p')|$:

Example 5.7.

$$C_{q\overline{q}}(p,p') = \frac{\langle q_{Q'^2}(p)\overline{q}_{Q'^2}(p')|\Delta\hat{H}_{Q^2}|G_{Q^2}(k)\rangle}{|\boldsymbol{p}|+|\boldsymbol{p}'|-|\boldsymbol{k}|} \quad . \tag{6}$$

To determine $C_{q\overline{q}}(pp')$ we need from the Hamiltonian $\Delta\hat{H}_{Q^2}$ the part that describes the quark–gluon interaction:

$$\Delta\hat{H}_{Q^2} = -g^2(Q^2)\sum_a \int \mathrm{d}^3x \sum_f \overline{q}_f(\boldsymbol{r})A^a(\boldsymbol{r})\gamma\frac{\lambda^a}{2}q_f(\boldsymbol{r}) \, . \tag{7}$$

It follows for the matrix element that

$$\begin{aligned}&\langle q_{Q'^2}(p)\overline{q}_{Q'^2}(p')|\Delta\hat{H}_{Q^2}|G_{Q^2}(k)\rangle \\ &\quad = -\sum_a g(Q^2)(2\pi)^3\delta^3(\boldsymbol{p}+\boldsymbol{p}'-\boldsymbol{k})\overline{u(\boldsymbol{p},s)}\varepsilon^{\,a}\cdot\gamma\frac{\lambda^a}{2}v(\boldsymbol{p}',s') \quad .\end{aligned} \tag{8}$$

Again we can relate the squared coefficient $C_{q\overline{q}}(p,p')$ to the quark distribution function by calculating the “decay probability” of a gluon if the resolving power Q^2 is increased:

$$\begin{aligned}\Delta N_q(c,Q^2) = {}& \frac{1}{8}\sum_a \frac{1}{2}\sum_{\varepsilon^a}\sum_{cs}\sum_{c's'}\frac{1}{2\omega}\int\frac{\mathrm{d}^3p}{(2\pi)^3 2E}\int\frac{\mathrm{d}^3p'}{(2\pi)^3 2E'} \\ & \times\delta\left(x-\frac{\boldsymbol{p}\cdot\boldsymbol{k}}{|\boldsymbol{k}|^2}\right)|C_{q\overline{q}}(p,p')|^2 \quad .\end{aligned} \tag{9}$$

This is summed over spins and colors of the outgoing quarks and it is averaged over the spin and color of the initial gluon. The δ function guarantees that the quark longitudinal momentum is x times the gluon momentum:

$$x = \frac{p^{\prime\prime}}{|\boldsymbol{k}|} = \frac{\boldsymbol{p}\cdot\boldsymbol{k}}{|\boldsymbol{k}|^2} \quad . \tag{10}$$

Inserting $C_{q\overline{q}}(p,p')$ from (6) into (9), we find that (9) becomes

$$\begin{aligned}\Delta N_q(c,Q^2) = {}& g(Q^2)^2\frac{1}{16}(2\pi)^3\int\frac{\mathrm{d}^3p}{(2\pi)^3}\int\frac{\mathrm{d}^3p'}{(2\pi)^3} \\ & \times\delta\left(x-\frac{\boldsymbol{p}\cdot\boldsymbol{k}}{|\boldsymbol{k}|^2}\right)\delta^3(\boldsymbol{p}+\boldsymbol{p}'-\boldsymbol{k})\frac{1}{8\omega EE'}\left(\frac{1}{|\boldsymbol{p}|+|\boldsymbol{p}'|-|\boldsymbol{k}|}\right)^2 \\ & \times\sum_a\sum_{\varepsilon^a}\sum_{cs}\sum_{c's'}|\overline{u(\boldsymbol{p},s)}\varepsilon^{\,a}\cdot\gamma\frac{\lambda^a}{2}v(\boldsymbol{p}',s')|^2 \quad .\end{aligned} \tag{11}$$

We first performt the spin and color summations again using $\not{\varepsilon}^a = -\gamma\cdot\varepsilon$ for $\varepsilon^a = (0,\varepsilon^{\,a})$:

Example 5.7.

$$
\begin{aligned}
&\sum_{a}\sum_{\varepsilon^a}\sum_{cs}\sum_{c's'}|\overline{u(\boldsymbol{p},s)}\varepsilon^{\,a}\cdot\gamma\frac{\lambda^a}{2}v(\boldsymbol{p}',s')|^2\\
&=\sum_{a,b}\mathrm{tr}\left(\frac{\lambda^a\lambda^b}{4}\right)\sum_{\varepsilon^a}\mathrm{tr}[\not{p}(\varepsilon^{\,a}\cdot\gamma)\not{p}'(\varepsilon^{\,b*}\cdot\gamma)]\\
&=\sum_a\frac{1}{2}\sum_{\varepsilon^a}\mathrm{tr}\left(\not{p}\not{\varepsilon}^a\not{p}'\not{\varepsilon}^{a*}\right)\\
&=\sum_a\frac{1}{2}\sum_{\varepsilon^a}4\cdot[(p\cdot\varepsilon^a)(p'\cdot\varepsilon^{a*})+(p'\cdot\varepsilon^a)(p\cdot\varepsilon^{a*})-(p\cdot p')(\varepsilon^a\cdot\varepsilon^{a*})]\\
&=2\sum_a\sum_{\varepsilon^a}[(\boldsymbol{p}\cdot\varepsilon^{\,a})(\boldsymbol{p}'\cdot\varepsilon^{\,a*})\\
&\quad+(\boldsymbol{p}'\cdot\varepsilon^{\,a})(\boldsymbol{p}\cdot\varepsilon^{\,a*})-(p_0p_0'-\boldsymbol{p}\cdot\boldsymbol{p}')(\varepsilon^{\,a}\cdot\varepsilon^{\,a*})]\\
&=16\left(\delta_{ij}-\frac{k_ik_j}{|\boldsymbol{k}|^2}\right)(p_ip_j'+p_i'p_j+(|\boldsymbol{p}||\boldsymbol{p}'|-\boldsymbol{p}\cdot\boldsymbol{p}')\delta_{ij})\\
&=\frac{32}{|\boldsymbol{k}|^2}\left[|\boldsymbol{k}|^2|\boldsymbol{p}||\boldsymbol{p}'|-(\boldsymbol{k}\cdot\boldsymbol{p})(\boldsymbol{k}\cdot\boldsymbol{p}')\right]\quad.
\end{aligned}
\tag{12}
$$

The quark distribution function can now be written as

$$
\begin{aligned}
\Delta N_q(c,Q^2)&=g(Q^2)^2\frac{1}{4}(2\pi)^3\int\mathrm{d}^3p\,\delta\left(x-\frac{\boldsymbol{p}\cdot\boldsymbol{k}}{|\boldsymbol{k}|^2}\right)\frac{1}{8\omega EE'}\left(\frac{1}{|\boldsymbol{p}|+|\boldsymbol{p}'|-|\boldsymbol{k}|}\right)^2\\
&\quad\times\frac{1}{|\boldsymbol{k}|^2}(|\boldsymbol{k}|^2|\boldsymbol{p}||\boldsymbol{p}'|-(\boldsymbol{k}\cdot\boldsymbol{p})(\boldsymbol{k}\cdot\boldsymbol{p}'))\quad.
\end{aligned}
\tag{13}
$$

We introduce explicit forms for the quark and gluon momenta ,

$$
\begin{aligned}
\boldsymbol{k}&=(0,0,k)\quad,\\
\boldsymbol{p}&=(\boldsymbol{k}_\perp,xk)\quad,\\
\boldsymbol{p}'&=(-\boldsymbol{k}_\perp,(1-x)k)\quad,
\end{aligned}
\tag{14}
$$

so that the three-momentum is conserved: $\boldsymbol{k}=\boldsymbol{p}+\boldsymbol{p}'$. As in the last sections we take $|\boldsymbol{k}_\perp|$ to lie between εQ and $\varepsilon Q'$ and assume that all momenta are much greater than $|\boldsymbol{k}_\perp|^2$:

$$
|\boldsymbol{k}|\gg|\boldsymbol{k}_\perp|\quad,\quad|\boldsymbol{p}|\gg|\boldsymbol{k}_\perp|\quad,\quad|\boldsymbol{p}'|\gg|\boldsymbol{k}_\perp|\quad.
\tag{15}
$$

The terms that enter (13) can now be approximated as in previous sections:

$$
\frac{1}{|\boldsymbol{p}|+|\boldsymbol{p}'|-|\boldsymbol{k}|}=\frac{2k}{|\boldsymbol{k}_\perp|}x(1-x)\quad,
\tag{16}
$$

$$
\frac{1}{\omega EE'}=\frac{1}{x(1-x)k^3}\quad,
\tag{17}
$$

$$
\frac{1}{|\boldsymbol{k}|^2}(|\boldsymbol{k}|^2|\boldsymbol{p}||\boldsymbol{p}'|-(\boldsymbol{k}\cdot\boldsymbol{p})(\boldsymbol{k}\cdot\boldsymbol{p}'))=\frac{1}{2}|\boldsymbol{k}_\perp|^2\frac{(1-x)^2+x^2}{x(1-x)}\quad.
\tag{18}
$$

Inserting these expressions into (13) and rewriting

$$
\mathrm{d}^3p=\mathrm{d}^2k_\perp\mathrm{d}p_{\|}
\tag{19}
$$

Example 5.7.

we can perform the integration over the longitudinal momentum, obtaining

$$\Delta N_q(x,Q^2) = \frac{g(Q^2)^2}{8\pi^3}\int \frac{\mathrm{d}^2k_\perp}{|k_\perp|^2}\frac{1}{2}\left((1-x)^2+x^2\right)$$
$$= \frac{g(Q^2)^2}{4\pi^2}\int \frac{\mathrm{d}k_\perp}{|k_\perp|}\frac{1}{2}\left((1-x)^2+x^2\right) \quad . \tag{20}$$

Keeping in mind that $|k_\perp|$ is in the range εQ to $\varepsilon Q'$, we can perform the last integration. The result is

$$\Delta N_q(x,Q^2) = \frac{g(Q^2)^2}{4\pi^2}\,\frac{1}{2}\left((1-x)^2+x^2\right)\ln|k_\perp|\Bigg|_{\varepsilon Q}^{\varepsilon Q'}$$
$$= \frac{g(Q^2)^2}{8\pi^2}\frac{1}{2}\left((1-x)^2+x^2\right)\Delta\ln Q^2 \quad , \tag{21}$$

since ε is an arbitrary constant $\Delta\ln\epsilon = 0$. Comparing this result with the GLAP equation yields, with the initial condition

$$N_G(x,Q^2) = \delta(1-x) \quad , \quad N_q(x,Q^2) = 0 \quad , \tag{22}$$

the expression for the splitting function:

$$P_{qG}(x) = \frac{1}{2}\left((1-x)^2+x^2\right) \quad . \tag{23}$$

5.3 Common Parametrizations of the Distribution Functions and Anomalous Dimensions

Instead of dealing with the GLAP equations we can transform the set of integral equations into a set of linear equations by introducing the moments of the structure functions. The nth moment is defined as [1]

$$N_a(n,Q^2) = \int_0^1 \mathrm{d}x\; x^{n-1}N_a(x,Q^2) \quad , \tag{5.82}$$

where N_a is the structure function of the parton of kind a. Differentiating this with respect to $\ln(Q^2)$ and inserting the GLAP equations results in

---

[1] Usually one defines the nth moment as the expectation value of x^n rather than x^{n-1}, which is commonly chosen in QCD.

$$\begin{aligned}
\frac{\mathrm{d}N_a(n,Q^2)}{\mathrm{d}\ln Q^2} &= \int\limits_0^1 \mathrm{d}x\ x^{n-1}\frac{\mathrm{d}N_a(x,Q^2)}{\mathrm{d}\ln Q^2} \\
&= \int\limits_0^1 \mathrm{d}x\ x^{n-1}\frac{g^2(Q^2)}{8\pi^2}\sum_b\int\limits_x^1 \frac{\mathrm{d}x'}{x'}P_{ab}\left(\frac{x}{x'}\right)N_b(x',Q^2) \\
&= \frac{g_s^2(Q^2)}{8\pi^2}\sum_b\int\limits_0^1 \mathrm{d}x\ x^{n-1}\int\limits_x^1 \frac{\mathrm{d}x'}{x'}P_{ab}\left(\frac{x}{x'}\right)N_b(x',Q^2) \quad . \qquad (5.83)
\end{aligned}$$

The second integration can be performed from $x' = 0$ to $x' = 1$ if we introduce the Heavyside function in the integrand. Thus the integrand vanishes for $x' < x$:

$$\frac{\mathrm{d}N_a(n,Q^2)}{\mathrm{d}\ln Q^2} = \frac{g^2(Q^2)}{8\pi^2}\sum_b\int\limits_0^1 \mathrm{d}x\ x^{n-1}\int\limits_0^1 \frac{\mathrm{d}x'}{x'}P_{ab}\left(\frac{x}{x'}\right)\Theta(x'-x)N_b(x',Q^2) \quad . \qquad (5.84)$$

Now the integrations can be exchanged. Substituting $z = x/x'$ we get:

$$\begin{aligned}
\frac{\mathrm{d}N_a(n,Q^2)}{\mathrm{d}\ln Q^2} &= \frac{g^2(Q^2)}{8\pi^2}\sum_b\int\limits_0^1 \frac{\mathrm{d}x'}{x'}N_b(x',Q^2)\int\limits_0^1 \mathrm{d}x\ x^{n-1}P_{ab}\left(\frac{x}{x'}\right)\Theta(x'-x) \\
&= \frac{g^2(Q^2)}{8\pi^2}\sum_b\int\limits_0^1 \frac{\mathrm{d}x'}{x'}N_b(x',Q^2)\int\limits_0^{1/x'} \mathrm{d}z\ x'\,(zx')^{n-1}P_{ab}(z)\Theta(x'-x'z) \\
&= \frac{g^2(Q^2)}{8\pi^2}\sum_b\int\limits_0^1 \mathrm{d}x'x'^{n-1}\ N_b(x',Q^2)\int\limits_0^{1/x'} \mathrm{d}z\ z^{n-1}P_{ab}(z)\Theta(1-z) \\
&= \frac{g^2(Q^2)}{8\pi^2}\sum_b\int\limits_0^1 \mathrm{d}x'x'^{n-1}\ N_b(x',Q^2)\int\limits_0^1 \mathrm{d}z\ z^{n-1}\ P_{ab}(z) \quad . \qquad (5.85)
\end{aligned}$$

The last step can be performed since $1/x' \geq 1$ (because $x' \leq 1$) and the Θ function contributes only for $z = 1/x' \leq 1$. Therefore the upper bound of the integration can be taken as 1.

In the high Q^2 regime $\alpha_s(Q^2) = g_s(Q^2)^2/4\pi$ is given in leading-log order by

$$\alpha_s(Q^2) = \frac{12}{33-2f}\frac{1}{\ln(Q^2/\Lambda^2)} \quad , \qquad (5.86)$$

where f denotes the number of quark flavors and Λ is the QCD cut off parameter. We obtain

$$\begin{aligned}
\frac{\mathrm{d}N_a(n,Q^2)}{\mathrm{d}\ln Q^2} &= \frac{6}{33-2f}\frac{1}{\ln(Q^2/\Lambda^2)}\sum_b N_b(n,Q^2)\int\limits_0^1 \mathrm{d}z\ z^{n-1}\ P_{ab}(z) \\
&= -\frac{1}{\ln(Q^2/\Lambda^2)}\sum_b N_b(n,Q^2)d_{ab}(n) \quad , \qquad (5.87)
\end{aligned}$$

where we have abbreviated the moment of the splitting function $P_{ab}(z)$ by

$$d_{ab}(n) = -\frac{6}{33-2f}\int\limits_0^1 \mathrm{d}z\, z^{n-1}\, P_{ab}(z) \quad . \tag{5.88}$$

Thus we have derived a system of linear equations between the moments of the structure functions and their derivatives. The moments $d_{ab}(n)$ of the splitting functions are the anomalous dimensions. Once the splitting functions are given, the moments $d_{ab}(n)$ can easily be obtained (see Exercise 5.8):

$$\begin{aligned} d_{Gq}(n) &= -\frac{8}{33-2f}\frac{n^2+n+2}{n(n^2-1)} \quad , \\ d_{qq}(n) &= \frac{4}{33-2f}\left(1-\frac{2}{n(n+1)}+4\sum_{j=2}^{n}\frac{1}{j}\right) \quad , \\ d_{qG}(n) &= -\frac{3}{33-2f}\frac{n^2+n+2}{n(n+1)(n+2)} \quad , \\ d_{GG}(n) &= \frac{9}{33-2f}\left(\frac{1}{3}+\frac{2f}{9}-\frac{4}{n(n-1)}-\frac{4}{(n+1)(n+2)}+4\sum_{j=2}^{n}\frac{1}{j}\right) \quad . \end{aligned} \tag{5.89}$$

The moments $N(n,Q^2)$ of the parton distributions $N(x,Q^2)$ are given by experiment. However, it is more difficult to calculate the parton distributions $N(x)$ for given moments $N(n)$. Since the polynomials x^n do not form an orthonormal system of functions in the range $0 \le x \le 1$, one has to transform the moments $N(n)$ into coefficients

$$c_i = f\big(N(n)\big) \quad , \tag{5.90}$$

so that

$$N(x,Q^2) = \sum_i c_i x^i \quad . \tag{5.91}$$

The coefficients c_i are functions of the moments $N(n)$ and can be obtained from the relation

$$N(n) = \sum_i \frac{1}{i+n} c_i \quad , \tag{5.92}$$

which is proven as follows.

Let $f(x)$ be the function that is to be approximated by $g(x) = \sum_i c_i x^i$. Again the moments of $f(x)$ are denoted by

$$f_n = \int\limits_0^1 \mathrm{d}x\, x^{n-1} f(x) \quad . \tag{5.93}$$

We require that

$$\mathcal{M} = \int_0^1 \mathrm{d}x \left(f(x) - g(x)\right)^2 \tag{5.94}$$

becomes minimal, in which case the variation of $\mathcal{M}$ with respect to the coefficients c_k vanishes:

$$0 = \delta\mathcal{M} = \frac{\partial\mathcal{M}}{\partial c_k}\delta c_k \quad , \tag{5.95}$$

from which it follows that

$$\frac{\partial\mathcal{M}}{\partial c_k} = 0 \quad \text{for all } k \quad . \tag{5.96}$$

Inserting the series expansion for $g(x)$ into (5.94), we can express $\mathcal{M}$ by the coefficients c_i and the moments f:

$$\begin{aligned}
\mathcal{M} &= \int_0^1 \mathrm{d}x \left(f(x)^2 + g(x)^2 - 2f(x)g(x)\right) \\
&= \int_0^1 \mathrm{d}x\, f(x)^2 + \int_0^1 \mathrm{d}x\; g(x)^2 - 2\int_0^1 \mathrm{d}x\, f(x)g(x) \\
&= C + \sum_{i,j} c_i c_j \int_0^1 \mathrm{d}x\; x^{i+j} - 2\sum_i c_i \int_0^1 \mathrm{d}x\; x^i f(x) \\
&= C + \sum_{i,j} \frac{c_i c_j}{i+j+1} - 2\sum_i c_i f_{i+1} \quad .
\end{aligned} \tag{5.97}$$

$C = \int_0^1 \mathrm{d}x\, f(x)^2$ is a constant.

Performing the derivation with respect to c_k we get the desired relation between the coefficients c_i and the moments f_i of the function f:

$$\begin{aligned}
0 &= \frac{\partial\mathcal{M}}{\partial c_k} \\
&= \frac{\partial}{\partial c_k}\left(C + \sum_{i,j}\frac{c_i c_j}{i+j+1} - 2\sum_i c_i f_{i+1}\right) \\
&= \sum_{i,j}\frac{\partial}{\partial c_k}\frac{c_i c_j}{i+j+1} - 2\sum_i \frac{\partial c_i}{\partial c_k} f_{i+1} \\
&= \sum_{i,j}\frac{\delta_{ik}c_j + c_i\delta_{jk}}{i+j+1} - 2\sum_i \delta_{ik} f_{i+1} \\
&= 2\sum_i \frac{c_i}{i+k+1} - 2f_{k+1} \quad ,
\end{aligned} \tag{5.98}$$

or

$$f_k = \sum_i \frac{c_i}{i+k} = \sum_i A_{ki} c_i \quad . \tag{5.99}$$

In order to compute the coefficients c_i from the known moments f_k, one has to determine the inverse of the matrix A_{ki}:

$$c_i = \left(A^{-1}\right)_{ik} f_k \quad . \tag{5.100}$$

Obviously the coefficients c_i change their values if one more moment f_k is added in the calculation.

It is possible to give an upper and lower bound to $f(x)$ by employing the Padé approximation method for the moments f_k. Recent parametrizations for the parton distributions inside the nucleon are given by CTEQ (The CTEQ group provides continuously updated sets of distribution functions, obtainable via electronic mail from Botts@msupa.pa.msu.edu), which is regularly updated. Here we give another parametrization. Writing for s in a leading-log approximation

$$s = \ln \frac{\ln\left[Q^2/(0.232\ \mathrm{GeV})^2\right]}{\ln\left[\mu^2/(0.232\ \mathrm{GeV})^2\right]} \quad , \tag{5.101}$$

where $\mu^2 = 0.25\,\mathrm{GeV}^2$, the parton distributions are parametrized as follows. For the u and d valence quarks in a proton we use

$$x v_p(x, Q^2) = N x^n \left(1 + A\sqrt{x} + bx\right)(1 - x)^D \quad , \tag{5.102}$$

with

$$\begin{aligned} N &= 0.663 + 0.191s + 0.041s^2 + 0.031s^3 \quad , \\ a &= 0.326 \quad , \\ A &= -1.97 + 6.74s - 1.96s^2 \quad , \\ B &= 24.4 - 20.7s + 4.08s^2 \quad , \\ D &= 2.86 + 0.70s - 0.02s^2 \quad , \end{aligned} \tag{5.103}$$

for $v = u_v + d_v$ and

$$\begin{aligned} N &= 0.579 + 0.283s + 0.047s^2 \quad , \\ a &= 0.523 - 0.015s \quad , \\ A &= 2.22 - 0.59s - 0.27s^2 \quad , \\ B &= 5.95 - 6.19s + 1.55s^2 \quad , \\ D &= 3.57 + 0.94s - 0.16s^2 \quad , \end{aligned} \tag{5.104}$$

for $v = d_v$. The distributions for a neutron are obtained by interchanging u and d quarks.

The gluon and light sea-quark distributions are given by

$$\begin{aligned} x w_{\mathrm{p}}'(x, Q^2) = \Bigg[& x^\alpha \left(A + Bx + Cx^2\right) \ln \left(\frac{1}{x}\right)^b \\ & + s^\alpha \exp\left(-E + \sqrt{E' s^\beta \ln\left(\frac{1}{x}\right)}\right)\Bigg](1 - x)^D \quad , \end{aligned} \tag{5.105}$$

with

$$
\begin{aligned}
\alpha &= 0.558 \quad , \\
\beta &= 1.218 \quad , \\
a &= 1.00 - 0.17s \quad , \\
b &= 0 \quad , \\
A &= 4.879s - 1.383s^2 \quad , \\
B &= 25.92 - 28.97s + 5.596s^2 \quad , \\
C &= -25.69 + 23.68s - 1.975s^2 \quad , \\
D &= 2.537 + 1.718s + 0.353s^2 \quad , \\
E &= 0.595 + 2.138s \quad , \\
E' &= 4.066
\end{aligned}
\tag{5.106}
$$

for $w = G$ and

$$
\begin{aligned}
\alpha &= 1.396 \quad , \\
\beta &= 1.331 \quad , \\
a &= 0.412 - 0.171s \quad , \\
b &= 0.566 - 0.496s \quad , \\
A &= 0.363 \quad , \\
B &= -1.196 \quad , \\
C &= 1.029 + 1.785s - 0.459s^2 \quad , \\
D &= 4.696 + 2.109s \quad , \\
E &= 3.838 + 1.944s \quad , \\
E' &= 2.845
\end{aligned}
\tag{5.107}
$$

for $w = \mathrm{u}$ and $w = \mathrm{d}$. Finally for the s, c, b quark distribution we employ the relation

$$
\begin{aligned}
x w'_{\mathrm{p}}(x, Q^2) = {} & \frac{(s - s_{w'})^\alpha}{[\ln(1/x)]^\alpha} \left(1 + A\sqrt{x} + Bx\right)(1-x)^D \\
& \times \exp\left(-E + \sqrt{E' s^\beta \ln\left(\frac{1}{x}\right)}\right) \quad ,
\end{aligned}
\tag{5.108}
$$

where the s-quark distribution is given by

$$
\begin{aligned}
\alpha &= 0.803 \quad , \\
\beta &= 0.563 \quad , \\
a &= 2.082 - 0.577s \quad , \\
A &= -3.055 + 1.024s^{0.67} \quad , \\
B &= 27.4 - 20.0s^{0.154} \quad , \\
D &= 6.22 \quad , \\
E &= 4.33 + 1.408s \quad , \\
E' &= 8.27 - 0.437s \quad , \\
s_{w'} &= s_{\mathrm{s}} = 0 \quad .
\end{aligned}
\tag{5.109}
$$

The c, c̄ distribution is parametrized by

$$\begin{aligned}
\alpha &= 1.01 \quad , \\
\beta &= 0.37 \quad , \\
a &= 0 \quad , \\
A &= 0 \quad , \\
B &= 4.24 - 0.804s \quad , \\
D &= 3.46 + 1.076s \quad , \\
E &= 4.61 + 1.49s \quad , \\
E' &= 2.555 + 1.961s \quad , \\
s_{w'} &= s_c = 0.888 \quad ,
\end{aligned} \tag{5.110}$$

and the b,b̄ distribution by

$$\begin{aligned}
\alpha &= 1.00 \quad , \\
\beta &= 0.51 \quad , \\
a &= 0 \quad , \\
A &= 0 \quad , \\
B &= 1.848 \quad , \\
D &= 2.929 + 1.396s \quad , \\
E &= 4.71 + 1.514s \quad , \\
E' &= 4.02 + 1.239s \quad , \\
s_{w'} &= s_b = 1.351 \quad .
\end{aligned} \tag{5.111}$$

The parametrizations are valid in the x range

$$10^{-5} \leq x < 1 \tag{5.112}$$

and the scale

$$\mu^2 \leq Q^2 \leq 10^8\,\mathrm{GeV}^2 \tag{5.113}$$

with an averaged inaccuracy of 2–3 %.

EXERCISE

5.8 Anomalous Dimensions

Problem. Calculate the moments of the splitting functions P_{Gq}, P_{qq}, P_{qG}, and P_{GG}. The splitting functions are explicitly

(a) $$P_{Gq} = \frac{4}{3}\,\frac{1+(1-x)^2}{x} \quad ,$$

Exercise 5.8.

$$\text{(b)}\quad P_{qq} = -\frac{4}{3}(1+x) + 2\,\delta(1-x) + \lim_{\varepsilon\to 0}\frac{8}{3}\left[\frac{1}{1-x+\varepsilon} - \delta(1-x)\int\limits_0^1 \mathrm{d}y\,\frac{1}{1-y+\varepsilon}\right] \quad ,$$

$$\text{(c)}\quad P_{qG} = \frac{1}{2}\left[x^2 + (1-x)^2\right] \quad ,$$

$$\text{(d)}\quad P_{GG} = 6\left[\frac{1}{x} - 2 + x(1-x)\right] + \left(\frac{11}{2} - \frac{f}{3}\right)\delta(1-x) + \lim_{\varepsilon\to 0} 6\left[\frac{1}{1-x+\varepsilon} - \delta(1-x)\int\limits_0^1 \mathrm{d}y\,\frac{1}{1-y+\varepsilon}\right] \quad .$$

Solution. (a) Inserting the explicit expression for the splitting function P_{Gq} we get for the nth moment

$$\begin{aligned} d_{Gq}(n) &= -\frac{6}{33-2f}\int\limits_0^1 \mathrm{d}x\; x^{n-1} P_{Gq}(x) \\ &= -\frac{6}{33-2f}\int\limits_0^1 \mathrm{d}x\; x^{n-1}\,\frac{4}{3}\,\frac{1+(1-x)^2}{x} \\ &= -\frac{8}{33-2f}\int\limits_0^1 \mathrm{d}x\;\left(2x^{n-2} - 2x^{n-1} + x^n\right) \\ &= -\frac{8}{33-2f}\left(\frac{2}{n-1}x^{n-1} - \frac{2}{n}x^n + \frac{1}{n+1}x^{n+1}\right)\Bigg|_0^1 \quad . \end{aligned} \tag{1}$$

Since $n \geq 1$ we get

$$\begin{aligned} d_{Gq}(n) &= -\frac{8}{33-2f}\left(\frac{2}{n-1} - \frac{2}{n} + \frac{1}{n+1}\right) \\ &= -\frac{8}{33-2f}\,\frac{n^2+n+2}{n(n^2-1)} \quad . \end{aligned} \tag{2}$$

(b) The nth moment of the splitting function P_{qq} is given by

$$\begin{aligned} d_{qq}(n) &= -\frac{6}{33-2f}\int\limits_0^1 \mathrm{d}x\; x^{n-1} P_{\mathrm{qq}}(x) \\ &= -\frac{6}{33-2f}\int\limits_0^1 \mathrm{d}x\; x^{n-1}\Bigg[-\frac{4}{3}(1+x) + 2\delta(1-x) \\ &\qquad + \lim_{\varepsilon\to 0}\frac{8}{3}\left(\frac{1}{1-x+\varepsilon} - \delta(1-x)\int\limits_0^1 \mathrm{d}y\,\frac{1}{1-y+\varepsilon}\right)\Bigg] \end{aligned}$$

$$= -\frac{6}{33-2f}\left[-\frac{4}{3}\int\limits_0^1 \mathrm{d}x\,\left(x^{n-1}+x^n\right)+2\right.$$

$$\left.+\lim_{\varepsilon\to 0}\frac{8}{3}\left(\int\limits_0^1 \mathrm{d}x\,\frac{x^{n-1}}{1-x+\varepsilon}-\int\limits_0^1 \mathrm{d}y\,\frac{1}{1-y+\varepsilon}\right)\right] . \tag{3}$$

With the expansion

$$\frac{x^{n-1}}{1-x+\varepsilon}=-\sum_{i=0}^{n-2}x^i+\frac{1}{1-x+\varepsilon} \tag{4}$$

this equation can be put into a simple form:

$$\begin{aligned}
d_{qq}(n) &= -\frac{6}{33-2f}\left[-\frac{8}{3}\sum_{i=0}^{n}\int\limits_0^1 \mathrm{d}x\,x^i+\frac{4}{3}\int\limits_0^1 \mathrm{d}x\,\left(x^{n-1}+x^n\right)+2\right]\\
&= -\frac{6}{33-2f}\left[-\frac{8}{3}\sum_{i=0}^{n}\frac{1}{i+1}+\frac{4}{3}\left(\frac{1}{n}+\frac{1}{n+1}\right)+2\right]\\
&= -\frac{6}{33-2f}\left[-\frac{8}{3}\sum_{i=2}^{n}\frac{1}{i}-\frac{8}{3}\frac{1}{n+1}-\frac{8}{3}+\frac{4}{3}\frac{1}{n}+\frac{4}{3}\frac{1}{n+1}+2\right]\\
&= \frac{4}{33-2f}\left[4\sum_{i=2}^{n}\frac{1}{i}+1-\frac{2}{(n+1)n}\right] .
\end{aligned} \tag{5}$$

(c) For the moment $d_{qG}(n)$ we get

$$\begin{aligned}
d_{qG}(n) &= -\frac{6}{33-2f}\int\limits_0^1 \mathrm{d}x\,x^{n-1}P_{qG}(x)\\
&= -\frac{6}{33-2f}\int\limits_0^1 \mathrm{d}x\,x^{n-1}\,\frac{1}{2}\left[x^2+(1-x)^2\right]\\
&= -\frac{3}{33-2f}\int\limits_0^1 \mathrm{d}x\,\left(2x^{n+1}-2x^n+x^{n-1}\right)\\
&= -\frac{3}{33-2f}\left(\frac{2}{n+2}x^{n+2}-\frac{2}{n+1}x^{n+1}+\frac{1}{n}x^n\right)\Bigg|_0^1\\
&= -\frac{3}{33-2f}\,\frac{n^2+n+2}{(n+2)(n+1)n} .
\end{aligned} \tag{6}$$

(d) The remaining moments $d_{GG}(n)$ of the splitting function P_{GG} are obtained in the same way as in (b):

Exercise 5.8.

$$\begin{aligned} d_{GG}(n) &= -\frac{6}{33-2f}\int_0^1 \mathrm{d}x\, x^{n-1}P_{GG}(x) \\ &= -\frac{6}{33-2f}\left[6\int_0^1 \mathrm{d}x\, x^{n-1}\left(\frac{1}{x}-2+x(1-x)\right)+\left(\frac{11}{2}-\frac{f}{3}\right)\right. \\ &\quad \left.+\,6\lim_{\varepsilon\to 0}\left(\int_0^1 \mathrm{d}x\,\frac{x^{n-1}}{1-x+\varepsilon}-\int_0^1 \mathrm{d}y\,\frac{1}{1-y+\varepsilon}\right)\right] . \end{aligned} \tag{7}$$

Again we employ the relation

$$\frac{x^{n-1}}{1-x+\varepsilon} = -\sum_{i=0}^{n-2} x^i + \frac{1}{1-x+\varepsilon} \tag{8}$$

and get, after integrating and rearranging,

$$\begin{aligned} d_{GG}(n) &= -\frac{6}{33-2f}\left[6\left(\frac{1}{n-1}-\frac{2}{n}+\frac{1}{n+1}-\frac{1}{n+2}\right)\right. \\ &\quad \left.+\left(\frac{11}{2}-\frac{f}{3}\right)-6\sum_{i=0}^{n-2}\int_0^1 \mathrm{d}x\, x^i\right] \\ &= -\frac{6}{33-2f}\left[6\left(\frac{1}{n-1}-\frac{2}{n}+\frac{1}{n+1}-\frac{1}{n+2}-\sum_{i=0}^{n-2}\frac{1}{i+1}\right)+\frac{11}{2}-\frac{f}{3}\right] \\ &= -\frac{6}{33-2f}\left[-6\sum_{i=2}^{n}\frac{1}{i}-6\right. \\ &\quad \left.+\,6\left(\frac{1}{n-1}-\frac{1}{n}+\frac{1}{n+1}-\frac{1}{n+2}\right)+\frac{11}{2}-\frac{f}{3}\right] \\ &= \frac{9}{33-2f}\left[4\sum_{i=2}^{n}\frac{1}{i}+\frac{1}{3}+\frac{2f}{9}-4\left(\frac{1}{n-1}-\frac{1}{n}+\frac{1}{n+1}-\frac{1}{n+2}\right)\right] \\ &= \frac{9}{33-2f}\left[4\sum_{i=2}^{n}\frac{1}{i}+\frac{1}{3}+\frac{2f}{9}-\frac{4}{n(n-1)}-\frac{4}{(n+2)(n+1)}\right] . \end{aligned} \tag{9}$$

5.4 Renormalization and the Expansion Into Local Operators

The Q^2 dependence of the distribution functions, and deep inelastic scattering in general, is frequently formulated by means of the so-called operator-product expansion. This formalism will be discussed in this section. However, the operator-product method is of general validity and not restricted to the context of perturbative

QCD. In fact we shall encounter it again during the discussion of QCD sum rules in the Chap. 7, on nonperturbative QCD.

The following analysis again starts with (5.1):

$$W_{\mu\mu'} = \frac{1}{2\pi}\int \mathrm{d}^4x \exp(\mathrm{i}qx)\frac{1}{2}\sum_{\text{pol.}}\langle \mathrm{N}|J_\mu(x)J_{\mu'}(0)|\mathrm{N}\rangle \quad . \tag{5.114}$$

In the rest system of the nucleon

$$P_\mu = (M,0,0,0) \quad , \quad q_\mu = \left(\frac{\nu}{M},0,0,\sqrt{\frac{\nu^2}{M^2}+Q^2}\right) \quad . \tag{5.115}$$

Here we are interested in the asymptotic region $\nu, Q^2 \gg M^2, \quad Q^2/2\nu = x$, where the exponential factor in (5.114) is

$$\exp\left\{\mathrm{i}\frac{\nu}{M}\left[x^0 - \left(1+\frac{Q^2M^2}{\nu^2}\right)x^3\right]\right\} \approx \exp\left[\mathrm{i}\frac{\nu}{M}\left(x^0-x^3\right)\right] \quad . \tag{5.116}$$

Provided that $|x^0 - x^3|$ is much larger than M/ν, its contribution to (5.114) averages to zero:

$$|x^0 - x^3| \sim O\left(\frac{M}{\nu}\right) \quad . \tag{5.117}$$

Therefore only x_μ values with

$$x^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2 \le x_0^2 - x_3^2 \sim O\left(\frac{M}{\nu}\right) \tag{5.118}$$

contribute to the integral (5.114). On the other hand one can prove that the matrix element $\langle N|J_\mu(x)J_{\mu'}(0)|N\rangle$ only contributes for $x^2 \ge 0$. To this end recall that this matrix element can be written as

$$\sum_X \int d^4x \langle \mathrm{N}|J_\mu(x)|X\rangle\langle X|J_{\mu'}(0)|\mathrm{N}\rangle \exp(\mathrm{i}qx) \tag{5.119}$$

with *physical* final states X. The crucial point is that the expression with exchanged coordinates

$$\sum_X \int d^4x \langle \mathrm{N}|J_\mu(0)|X\rangle\langle X|J_{\mu'}(x)|\mathrm{N}\rangle \exp(\mathrm{i}qx) \sim \sum_X \ldots \delta^4(P_X - P_N + q) \tag{5.120}$$

is equal to zero, since there is no physical state with baryon number one and energy $E_X = P_{\mathrm{N}}^0 - q^0 = M_N - \nu/M_{\mathrm{N}} < M_{\mathrm{N}}$. Hence the matrix element in (5.114) can be replaced by

$$\langle \mathrm{N}|J_\mu(x)J_{\mu'}(0)|\mathrm{N}\rangle - \langle \mathrm{N}|J_\mu(0)J_{\mu'}(x)|\mathrm{N}\rangle = \langle N|\left[J_\mu(x)J_{\mu'}(0)\right]_-|N\rangle \quad . \tag{5.121}$$

Since the commutator vanishes for spacelike distances $x^2 < 0$, only time and lightlike x_μ contribute to (5.114):

$$W_{\mu\mu'} = \frac{1}{2\pi} \int\limits_{x^2 \geq 0} \mathrm{d}^4x \ \exp(iqx) \ \frac{1}{2} \sum_{\text{pol.}} \langle \mathrm{N} | [J_\mu(x) J_{\mu'}(0)]_- | \mathrm{N} \rangle \quad . \tag{5.122}$$

It follows with the help of (5.117) that $W_{\mu\mu'}$ is dominated by the matrix element of the current commutator for lightlike x values. Therefore the following expansion is also known as the *light-cone expansion.*

It is well known that a commutator of currents with a lightlike distance diverges. This divergence also occurs in the free case

$$\hat{J}_\mu(x) = \hat{\bar{\Psi}}(x)\gamma_\mu \hat{\Psi}(x) \quad , \tag{5.123}$$

where it can be reduced to the divergence of the free propagator $S(x)$ for $x^2 \to 0$:

$$\begin{aligned} [\hat{J}_\mu(x) \hat{J}_\nu(0)]_- &= [\hat{\bar{\Psi}}(x)\gamma_\mu \hat{\Psi}(x), \hat{\bar{\Psi}}(0)\gamma_\nu \hat{\Psi}(0)]_- \\ &= \hat{\bar{\Psi}}(x)\gamma_\mu \hat{\Psi}(x) \hat{\bar{\Psi}}(0)\gamma_\nu \hat{\Psi}(0) + [\hat{\bar{\Psi}}(x)\gamma_\mu]_\alpha \hat{\bar{\Psi}}_\beta(0) \hat{\Psi}_\alpha(x) [\gamma_\nu \hat{\Psi}(0)]_\beta \\ &\quad - \hat{\bar{\Psi}}(0)\gamma_\nu \hat{\Psi}(0) \hat{\bar{\Psi}}(x)\gamma_\mu \hat{\Psi}(x) \\ &\quad - \hat{\bar{\Psi}}_\beta(0) [\hat{\bar{\Psi}}(x)\gamma_\mu]_\alpha [\gamma_\nu \hat{\Psi}(0)]_\beta \hat{\Psi}_\alpha(x) \quad . \end{aligned} \tag{5.124}$$

Owing to the anticommutation properties of $\hat{\Psi}$ and $\hat{\bar{\Psi}}$ the second and the fourth term cancel. The first two and the last two terms are combined and the anticommutators that occur are replaced by the propagator:

$$\begin{aligned} &[\hat{\bar{\Psi}}(x)\gamma_\mu]_\alpha S_{\alpha\beta}(x) [\gamma_\nu \hat{\Psi}(0)]_\beta - [\hat{\bar{\Psi}}(0)\gamma_\nu]_\beta S_{\beta\alpha}(-x) [\gamma_\mu \hat{\Psi}(x)]_\alpha \quad , \\ &\qquad \{\hat{\Psi}_\alpha(x), \hat{\bar{\Psi}}_\beta(0)\} = S_{\alpha\beta}(x) = \left(\mathrm{i}\gamma_\mu \partial^\mu + m\right)_{\alpha\beta} \cdot \mathrm{i}\Delta(x) \quad . \end{aligned} \tag{5.125}$$

In the case of massless fields and $x^2 \to 0 \quad S(x)$ is given by

$$S_{\alpha\beta}(x) \to \gamma_\mu \partial_x{}^\mu \frac{1}{2\pi} \varepsilon(x_0) \delta\left(x^2\right) \tag{5.126}$$

with

$$\varepsilon(x_0) = \mathrm{sgn}(x_0) \quad , \tag{5.127}$$

$$\begin{aligned} [\hat{J}_\mu(x), \hat{J}_\nu(0)]_- \to &\left(\overline{\Psi}(x)\gamma_\mu \gamma_\lambda \gamma_\nu \Psi(0) - \overline{\Psi}(0)\gamma_\nu \gamma_\lambda \gamma_\mu \Psi(x)\right) \\ &\times \frac{1}{2\pi} \partial^\lambda \varepsilon(x_0) \delta(x^2) \quad . \end{aligned} \tag{5.128}$$

In the following only the leading divergence of $S(x)$, i.e., the free part, is considered. The remaining terms are discussed later. For the electromagnetic interaction treated here $W_{\mu\nu}$ is symmetric in μ and ν (see (3.18)). Therefore we can write the current commutator as

$$\begin{aligned} &\frac{1}{2}\left(\left[\hat{J}_\mu\left(\frac{x}{2}\right), \hat{J}_\nu\left(-\frac{x}{2}\right)\right]_- + \left[\hat{J}_\nu\left(\frac{x}{2}\right), \hat{J}_\mu\left(-\frac{x}{2}\right)\right]_-\right) \\ &\quad = \left[\hat{\bar{\Psi}}\left(\frac{x}{2}\right) \frac{1}{2}(\gamma_\mu \gamma_\lambda \gamma_\nu + \gamma_\nu \gamma_\lambda \gamma_\mu) \hat{\Psi}\left(-\frac{x}{2}\right)\right. \\ &\qquad \left. - \hat{\bar{\Psi}}\left(-\frac{x}{2}\right) \frac{1}{2}(\gamma_\nu \gamma_\lambda \gamma_\mu + \gamma_\mu \gamma_\lambda \gamma_\nu) \hat{\Psi}\left(\frac{x}{2}\right)\right] \frac{1}{2\pi} \partial^\lambda \varepsilon(x_0) \delta(x^2) \\ &\quad = s_{\mu\lambda\nu\beta} \left[\hat{\bar{\Psi}}\left(\frac{x}{2}\right) \gamma^\beta \hat{\Psi}\left(-\frac{x}{2}\right) - \hat{\bar{\Psi}}\left(-\frac{x}{2}\right) \gamma^\beta \hat{\Psi}\left(\frac{x}{2}\right)\right] \\ &\qquad \times \frac{1}{2\pi} \partial^\lambda \varepsilon(x_0) \delta(x^2) \quad , \end{aligned} \tag{5.129}$$

with

$$s_{\mu\lambda\nu\beta} = g_{\mu\lambda}g_{\nu\beta} + g_{\mu\beta}g_{\nu\lambda} - g_{\mu\nu}g_{\lambda\beta} \quad . \tag{5.130}$$

Here we have shifted the position arguments of the current operators from "x" and "0" to "$\pm x/2$". This operation does not change anything, because (5.122) is invariant under such translations, but it emphasizes the symmetry properties of the current operator. In the case of neutrino–nucleon scattering, i.e., for a parity-violating process, $W_{\mu\mu'}$ is no longer symmetric in μ and μ' (see Exercise 3.3). Because

$$\gamma_\mu\gamma_\lambda\gamma_\nu = (s_{\mu\lambda\nu\beta} + \mathrm{i}\varepsilon_{\mu\lambda\nu\beta}\gamma_5)\gamma^\beta \quad , \tag{5.131}$$

(as we shall show in Exercise 5.9) we obtain the following additional terms:

$$\begin{aligned}
&\left[\hat{J}_\mu\left(\frac{x}{2}\right), \hat{J}_\nu\left(-\frac{x}{2}\right)\right]_- \\
&\quad = \left[\hat{\bar{\Psi}}\left(\frac{x}{2}\right)(s_{\mu\lambda\nu\beta} + \mathrm{i}\varepsilon_{\mu\lambda\nu\beta}\gamma_5)\gamma^\beta\hat{\Psi}\left(-\frac{x}{2}\right)\right. \\
&\qquad \left. - \hat{\bar{\Psi}}\left(-\frac{x}{2}\right)(s_{\nu\lambda\mu\beta} + \mathrm{i}\varepsilon_{\nu\lambda\mu\beta}\gamma_5)\gamma^\beta\hat{\Psi}\left(\frac{x}{2}\right)\right]\frac{1}{2\pi}\partial^\lambda\varepsilon(x_0)\delta(x^2) \\
&\quad = \left\{s_{\mu\lambda\nu\beta}\left[\hat{\bar{\Psi}}\left(\frac{x}{2}\right)\gamma^\beta\hat{\Psi}\left(-\frac{x}{2}\right) - \hat{\bar{\Psi}}\left(-\frac{x}{2}\right)\gamma^\beta\hat{\Psi}\left(\frac{x}{2}\right)\right]\right. \\
&\qquad \left. + \mathrm{i}\varepsilon_{\mu\lambda\nu\beta}\left[\hat{\bar{\Psi}}\left(\frac{x}{2}\right)\gamma_5\gamma^\beta\hat{\Psi}\left(-\frac{x}{2}\right) + \hat{\bar{\Psi}}\left(-\frac{x}{2}\right)\gamma_5\gamma^\beta\hat{\Psi}\left(\frac{x}{2}\right)\right]\right\} \\
&\qquad \times \frac{1}{2\pi}\partial^\lambda\varepsilon(x_0)\delta(x^2) \quad .
\end{aligned} \tag{5.132}$$

Inserting (5.132) into the matrix element (5.122) yields

$$\begin{aligned}
&\langle\mathrm{N}|\left[\hat{J}_\mu\left(\frac{x}{2}\right), \hat{J}_\nu\left(-\frac{x}{2}\right)\right]_-|\mathrm{N}\rangle \\
&\quad = \frac{1}{2\pi}\left[\partial^\lambda\varepsilon(x_0)\delta(x^2)\right] \\
&\qquad \times \left\{s_{\mu\lambda\nu\beta}\langle\mathrm{N}|\hat{\bar{\Psi}}\left(\frac{x}{2}\right)\gamma^\beta\hat{\Psi}\left(-\frac{x}{2}\right) - \hat{\bar{\Psi}}\left(-\frac{x}{2}\right)\gamma^\beta\hat{\Psi}\left(\frac{x}{2}\right)|\mathrm{N}\rangle\right. \\
&\qquad \left. + \mathrm{i}\varepsilon_{\mu\lambda\nu\beta}\langle\mathrm{N}|\hat{\bar{\Psi}}\left(\frac{x}{2}\right)\gamma_5\gamma^\beta\hat{\Psi}\left(-\frac{x}{2}\right) + \hat{\bar{\Psi}}\left(-\frac{x}{2}\right)\gamma_5\gamma^\beta\hat{\Psi}\left(\frac{x}{2}\right)|\mathrm{N}\rangle\right\} ,
\end{aligned} \tag{5.133}$$

which is a typical form for the *operator-product expansion* (OPE).

The first factor on the right hand side of (5.133) is a divergent expression, which can be derived from perturbative QCD and which does not depend on the hadron considered. The terms in the curly brackets are finite matrix elements, which contain all the nonperturbative information. The perturbative factors are called *Wilson coefficients*. Thus the Q^2 expansion, which can be evaluated in deep inelastic scattering, corresponds to a series in these coefficients. On the other hand the nonperturbative information contained in distribution functions and more generally in all possible field correlators is now contained in the matrix elements.

EXERCISE

5.9 Decomposition Into Vector and Axialvector Couplings

Problem. Prove relation (5.139).

$$\gamma_\mu \gamma_\alpha \gamma_\nu = (s_{\mu\alpha\nu\beta} + \mathrm{i}\varepsilon_{\mu\alpha\nu\beta})\gamma^\beta \quad . \tag{1}$$

Solution. $\gamma_\mu \gamma_\alpha \gamma_\nu$ anticommutes with γ_5. On the other hand it must be a linear combination of the 16 matrices which span the Clifford space, namely 1, γ_5, γ_μ, $\gamma_\mu \gamma_5$, and $\sigma_{\mu\nu}$. Obviously this implies

$$\gamma_\mu \gamma_\alpha \gamma_\nu = a_{\mu\alpha\nu\beta}\gamma^\beta + b_{\mu\alpha\nu\beta}\gamma_5\gamma^\beta \quad . \tag{2}$$

Multiplying by γ_δ and taking the trace we find

$$\begin{aligned} \mathrm{tr}\{\gamma_\mu \gamma_\alpha \gamma_\nu \gamma_\delta\} &= 4(g_{\mu\alpha}g_{\nu\delta} + g_{\mu\delta}g_{\nu\alpha} + g_{\mu\nu}g_{\alpha\delta}) \\ &= 4a_{\mu\alpha\nu\delta} \quad . \end{aligned} \tag{3}$$

The second coefficient is determined by multiplying (2) by $\gamma_\delta \gamma_5$ and taking the trace

$$4b_{\mu\alpha\nu\delta} = 4\mathrm{i}\varepsilon_{\mu\alpha\nu\delta} \quad . \tag{4}$$

Thus we have proven (1)

$$\gamma_\mu \gamma_\alpha \gamma_\nu = g_{\mu\alpha}\gamma_\nu + g_{\nu\alpha}\gamma_\mu - g_{\mu\nu}\gamma_\alpha + \mathrm{i}\varepsilon_{\mu\alpha\nu\delta}\gamma_5\gamma^\delta \quad . \tag{5}$$

We have shown above that only short (or more precisely lightlike) distances contribute to the scattering tensor. It is therefore natural to consider a Taylor expansion in "x_μ" and in fact the operator product expansion is this precisely, as we shall now make clear. The only difference from a usual Taylor expansion is that the x-dependent terms are not just $x, x^2, x^3 \ldots$ but divergent expressions (or more precisely distributions)(see 5.129 and 5.132). A Taylor expansion of (5.129) generates a sum over products of operators at the same position; this means that we expand a nonlocal operator into local operators:

$$\begin{aligned} \hat{\bar{\Psi}}\left(\frac{x}{2}\right)\hat{\Psi}\left(-\frac{x}{2}\right) = \sum_n \frac{(-1)^n}{n!}\left(\frac{x}{2}\right)^{\mu_1}\left(\frac{x}{2}\right)^{\mu_2} \cdots \left(\frac{x}{2}\right)^{\mu_n} \\ \times \hat{\bar{\Psi}}(0)\overleftrightarrow{\partial}_{\mu_1}\overleftrightarrow{\partial}_{\mu_2} \ldots \overleftrightarrow{\partial}_{\mu_n}\hat{\Psi}(0) \quad , \end{aligned} \tag{5.134}$$

$$\overleftrightarrow{\partial}_\mu = \overrightarrow{\partial_\mu} - \overleftarrow{\partial_\mu} \quad . \tag{5.135}$$

With the free current operator, (5.122) therefore becomes

$$\begin{aligned}
W_{\mu\mu'} &= -\frac{1}{2\pi}\int\limits_{x^2\geq 0} \mathrm{d}x^4 \exp(\mathrm{i}q\cdot x)\sum_n \frac{(-1)^n}{n!}\left(\frac{x}{2}\right)^{\mu_1}\left(\frac{x}{2}\right)^{\mu_2}\cdots\left(\frac{x}{2}\right)^{\mu_n} \\
&\qquad \times\left[1-(-1)^n\right]\frac{1}{2}\sum_{\text{pol.}}\langle \mathrm{N}|\hat{O}^n_{\mu_1\ldots\mu_n\beta}|\mathrm{N}\rangle s_{\mu\alpha\mu'}{}^{\beta}\frac{1}{2\pi}\partial^\alpha\left[\varepsilon(x^0)\delta(x^2)\right] \\
&= \frac{1}{2\pi}\int\limits_{x^2\geq 0} \mathrm{d}x^4 \exp(\mathrm{i}q\cdot x)\sum_{\text{n odd}} \frac{1}{n!}\left(\frac{x}{2}\right)^{\mu_1}\left(\frac{x}{2}\right)^{\mu_2}\cdots\left(\frac{x}{2}\right)^{\mu_n} s_{\mu\alpha\mu'}{}^{\beta} \\
&\qquad \times\frac{1}{2}\sum_{\text{pol.}}\langle \mathrm{N}|\hat{O}^n_{\mu_1\ldots\mu_n\beta}|\mathrm{N}\rangle \frac{1}{\pi}\partial^\alpha\left[\varepsilon(x^0)\delta(x^2)\right] \quad , \qquad (5.136)
\end{aligned}$$

with

$$\hat{O}^n_{\mu_1\ldots\mu_n\beta} = \hat{\bar{\Psi}}(0)\,\partial_{\mu_1}\partial_{\mu_2}\ldots\partial_{\mu_n}\gamma_\beta\,\hat{\Psi}(0) \quad . \qquad (5.137)$$

The essential point is that the operators $\hat{O}^n_{\mu_1\mu_2\ldots\mu_n\beta}$ are no longer singular. Instead we have

$$\begin{aligned}
\frac{1}{2}\sum_{\text{pol.}}\langle \mathrm{N}(P)|\hat{O}^n_{\mu_1\ldots\mu_n\beta}|\mathrm{N}(P)\rangle &= A^{(n)}\left[P_{\mu_1}P_{\mu_2}\ldots P_{\mu_n}P_\beta\right. \\
&\qquad \left. + \,(\text{terms with } M^2 g_{\mu_i\mu_j} \text{ etc.})\right] \qquad (5.138)
\end{aligned}$$

with some constant $A^{(n)}$.

The x integration in (5.136) leads to a replacement of $x^{\mu_1}x^{\mu_2}\ldots x^{\mu_n}$ by the corresponding q components. A Fourier transformation gives

$$x^{\mu_1}x^{\mu_2}\ldots x^{\mu_n}\longrightarrow(\ldots)\cdot q^{\mu_1}q^{\mu_2}\ldots q^{\mu_n} \quad . \qquad (5.139)$$

Hence (5.138) is contracted with $q^{\mu_1}q^{\mu_2}\ldots q^{\mu_n}$. It then follows that

$$\frac{q^{\mu_i}q^{\mu_j}P_{\mu_i}P_{\mu_j}}{q^{\mu_i}q^{\mu_j}M^2 g_{\mu_i\mu_j}} = \frac{\nu^2}{-Q^2M^2} = -\frac{\nu}{2M^2}\frac{1}{x}\longrightarrow\infty \qquad (5.140)$$

for $\nu, Q^2\longrightarrow\infty$ and x fixed. Thus all the remaining terms on the right hand side of (5.138) are suppressed by powers of $1/Q^2$ (higher twist):

$$\frac{1}{2}\sum_{\text{pol.}}\langle \mathrm{N}(P)|\hat{O}^n_{\mu_1\cdots\mu_n\beta}|\mathrm{N}(P)\rangle\longrightarrow A^{(n)}P_{\mu_1}P_{\mu_2}\ldots P_{\mu_n}P_\beta+\ldots \quad , \qquad (5.141)$$

$$\begin{aligned}
W_{\mu\mu'} &= \frac{1}{2\pi}\int\limits_{x^2\geq 0}\mathrm{d}^4x \exp(\mathrm{i}q\cdot x)\sum_{n\text{odd}}\frac{1}{n!}\left[\frac{x\cdot P}{2}\right]^n s_{\mu\alpha\mu'}{}^{\beta}P_\beta A^{(n)} \\
&\qquad \times\frac{1}{\pi}\partial^\alpha\left[\varepsilon(x^0)\delta(x^2)\right]+\ldots \qquad (5.142) \\
&= \frac{1}{2\pi}\int\limits_{x^2\geq 0}\mathrm{d}^4x \exp(\mathrm{i}q\cdot x)\sum_{n\text{odd}}(\text{finite term})\times(\text{divergent term})+\ldots
\end{aligned}$$

We have succeeded in splitting up the contributions to $W_{\mu\mu'}$ into a finite and a divergent factor. The whole dependence on the specific process under consideration is related to the finite factor while the divergent factor always remains the same. This seperation of divergent current commutators into an operator with finite matrix elements specific for the hadron under consideration and a constant divergent function is the basic idea of the expansion into local operators.

One can ask what the practical use of the formal expansion (5.142) is. As a first simple answer we wish to show that the main statements of the parton model can be immediately derived from this equation. The simple parton model assumes noninteracting constituents, which correspond exactly to the use of the free current commutator. We substitute

$$\sum_{n\mathrm{odd}} \left[\frac{x\cdot P}{2}\right]^n \frac{A^{(n)}}{n!} = -\int \mathrm{d}\xi \exp(\mathrm{i}x\cdot\xi P) f(\xi) \quad , \tag{5.143}$$

$$f(\eta) = -\frac{1}{2\pi}\int \mathrm{d}(x\cdot P) \sum_{n\mathrm{odd}} \left[\frac{x\cdot P}{2}\right]^n \frac{A^{(n)}}{n!} \exp(-\mathrm{i}x\cdot P\eta) \quad . \tag{5.144}$$

In the hadron scattering tensor which then assumes the form

$$W_{\mu\mu'} = \frac{-1}{2\pi}\int \mathrm{d}^4x \int \mathrm{d}\xi \exp(\mathrm{i}x\cdot(q+\xi P))\, s_{\mu\alpha\mu'}{}^{\beta} P_\beta f(\xi) \frac{1}{\pi}\partial^\alpha \left[\varepsilon(x^0)\delta(x^2)\right] \, . \tag{5.145}$$

Using (see Exercise 5.8)

$$\int \mathrm{d}^4x \exp(\mathrm{i}x\cdot(q+\xi P))\, \delta(x^2)\varepsilon(x^0) = \mathrm{i}4\pi^2\delta\left[(q+\xi P)^2\right]\varepsilon(q_0+\xi P_0) \tag{5.146}$$

and partial integration yields

$$W_{\mu\mu'} = -2\int \mathrm{d}\xi\, s_{\mu\alpha\mu'}{}^{\beta} P_\beta f(\xi)(q+\xi P)^\alpha \delta\left[(q+\xi P)^2\right]\varepsilon(q_0+\xi P_0) \, . \tag{5.147}$$

The argument of the δ function simplifies to

$$(q+\xi P)^2 = q^2 + \xi^2 M^2 + 2\xi q\cdot P \approx -Q^2 + \xi 2\nu = \frac{Q^2}{x}(\xi - x) \quad , \tag{5.148}$$

$$\begin{aligned} W_{\mu\mu'} &= -2 s_{\mu\alpha\mu'}{}^{\beta} P_\beta f(x) \frac{x}{Q^2}(q+xP)^\alpha \\ &= -2(g_{\mu\alpha}g_{\mu'}{}^{\beta} + g_\mu{}^\beta g_{\mu'\alpha} - g_{\mu\mu'}g_\alpha{}^\beta)P_\beta(q+xP)^\alpha \frac{x}{Q^2}f(x) \\ &= -2\frac{x}{Q^2}f(x)\left[(q+xP)_\mu P_{\mu'} + P_\mu(q+xP)_{\mu'} - g_{\mu\mu'}(xM^2 + P\cdot q)\right] \, . \end{aligned} \tag{5.149}$$

Since $W_{\mu\mu'}$ is multiplied by the lepton scattering tensor $L^{\mu\mu'}$, we can omit, because of

$$q^\mu L_{\mu\mu'} = q^{\mu'} L_{\mu\mu'} = 0 \quad , \tag{5.150}$$

all terms proportional to q_μ and $q_{\mu'}$:

$$
\begin{aligned}
W_{\mu\mu'} &\to -2\frac{x}{Q^2}f(x)\left[-g_{\mu\mu'}(xM^2+\nu)+2xP_\mu P_{\mu'}\right] \\
&\approx -2\frac{x}{Q^2}f(x)\left[-g_{\mu\mu'}\nu+2xP_\mu P_{\mu'}\right] \quad . \qquad (5.151)
\end{aligned}
$$

In an analogous way, i.e., by neglecting all terms proportional to q_μ or $q_{\mu'}$, we obtain from (3.18)

$$
W_{\mu\mu'} \to -g_{\mu\mu'}W_1(Q^2,\nu)+P_\mu P_{\mu'}\frac{W_2(Q^2,\nu)}{M_{\mathrm{N}}^2} \quad . \qquad (5.152)
$$

Comparison of (5.151) with (5.152) shows that $W_1(Q^2,x)$ and $W_2(Q^2,x)$ in fact depend only on x. Furthermore we get the Callan–Gross relation:

$$
\begin{aligned}
&W_1(Q^2,\nu) = -2\frac{x}{Q^2}f(x)\nu =: F_1(x) \quad , \qquad (5.153) \\
&\frac{W_2(Q^2,\nu)}{M_N^2} = -2\frac{x}{Q^2}f(x)\cdot 2x =: \frac{F_2(x)}{\nu} \\
\Rightarrow \quad & F_2(x) = 2xF_1(x) \quad . \qquad (5.154)
\end{aligned}
$$

This was just a simple example. In general the operator product expansion (OPE) provides two major results:

1. It gives a formal expression for all the terms that can occur. This allows us to analyze in a systematic way the terms contributing to a given power of $1/Q^2$. Furthermore it generates relations between various phenomenological expressions such as the structure functions.
2. For the individual terms apppearing in the $1/Q^2$ expansion the OPE gives the corresponding correlators. This shows to what property of the (unknown) exact wave function a given observable is sensitive and allows us to calculate it in an appropriate model (i.e., a model which should describe the specific correlation in question realistically).

EXERCISE

5.10 The Proof of (5.146)

Problem. Proof (5.146) by replacing $(q+\xi P)_\mu$ by k_μ and evaluating the Fourier transformation of (5.146), i.e., by proving that

$$
\frac{\mathrm{i}}{4\pi^2}\int \exp(-\mathrm{i}k\cdot x)\,\delta(k^2)\varepsilon(k_0)\mathrm{d}^4k = \delta(x^2)\varepsilon(x_0) \quad . \qquad (1)
$$

Solution. Performing the k_0 integration turns the lefthand side into

$$
\begin{aligned}
I = \frac{\mathrm{i}}{4\pi^2}\int \frac{1}{2k}&\left[\exp\left(-\mathrm{i}k(x^0-r\cos\theta)\right)\right. \\
&\left.-\exp\left(+\mathrm{i}k(x^0+r\cos\theta)\right)\right]\mathrm{d}^3k \qquad (2)
\end{aligned}
$$

with

Exercise 5.10.

$$
\begin{aligned}
k &:= \sqrt{\boldsymbol{k}^2} \quad , \\
r &= \sqrt{\boldsymbol{x}^2} \quad , \qquad\qquad (3)\\
\boldsymbol{k}\cdot\boldsymbol{x} &= kr\cos\theta \quad .
\end{aligned}
$$

We evaluate the angular integration

$$
\begin{aligned}
I &= \frac{\mathrm{i}}{2\pi}\int\limits_0^\infty \mathrm{d}k\,\frac{k^2}{2k}\left[\frac{1}{\mathrm{i}kr}\left(\exp\left(-\mathrm{i}k(x^0-r)\right)-\exp\left(-\mathrm{i}k(x^0+r)\right)\right)\right.\\
&\quad\left.-\frac{1}{\mathrm{i}kr}\left(\exp\left(\mathrm{i}k(x^0+r)\right)-\exp\left(\mathrm{i}k(x^0-r)\right)\right)\right]\\
&= \frac{1}{4\pi r}\int\limits_{-\infty}^{\infty}\mathrm{d}k\left[\exp\left(-\mathrm{i}k(x^0-r)\right)-\exp\left(\mathrm{i}k(x^0+r)\right)\right] \qquad (4)
\end{aligned}
$$

and obtain a difference of δ functions, which yields the postulated result:

$$
\begin{aligned}
I &= \frac{2\pi}{4\pi r}\left[\delta(x^0-r)-\delta(x^0+r)\right]\\
&= \delta(x^2)\big|_{x^0=r}-\delta(x^2)\big|_{x^0=-r}\\
&= \delta(x^2)\varepsilon(x_0) \quad . \qquad (5)
\end{aligned}
$$

Let us discuss next how the expansion of $W_{\mu\mu'}$ changes if in (5.142) we replace the free by the exact current commutator, i.e., if we take the quark interactions into account. This corresponds in lowest order to the transition from the simple parton model to the GLAP equations.

First the derivatives in $\hat{O}^n_{\mu_1\mu_2\ldots\mu_n\beta}$ have to be replaced by covariant derivatives. For the sake of simplicity all derivatives that operate to the left, are partially integrated such that they act to the right:

$$
\hat{O}^n_{\mu_1\mu_2\ldots\mu_n\beta}\to 2^n\hat{D}_{\mu_1}\hat{D}_{\mu_2}\ldots\hat{D}_{\mu_n}\gamma_\beta \quad . \qquad (5.155)
$$

In the general interacting case some modified divergent function takes the place of $\partial^\alpha[\varepsilon(x_0)\delta(x^2)]$. The leading divergence remains the same, but additional terms occur. Also the appearance of $s_{\mu\alpha\mu'\beta}$ was due to the specific situation of the free case (see (5.129)). Now the general form of (5.142) is

$$
\begin{aligned}
W_{\mu\mu'} &= \frac{1}{2\pi}\int \mathrm{d}^4x\ \exp(\mathrm{i}q\cdot x)\sum_j\sum_n\left[-g_{\mu\mu'}\,n!\,x^{\mu_1}x^{\mu_2}\ldots x^{\mu_n}\mathrm{i}^n C^{(n)}_{1,j}(x)\right.\\
&\quad\left.+\left(g_{\mu\mu_1}g_{\mu'\mu_2}x^{\mu_3}x^{\mu_4}\ldots x^{\mu_n}+\text{permutations}\right)\mathrm{i}^{n-2}C^{(n)}_{2,j}(x)\right]\\
&\quad\times\frac{1}{2}\sum_{\text{pol.}}\langle \mathrm{N}|\hat{O}^{(n)j}_{\mu_1\mu_2\ldots\mu_n}|\mathrm{N}\rangle \quad . \qquad (5.156)
\end{aligned}
$$

Instead of the tensor $s_{\mu\alpha\mu'\beta}$ all possible tensors occur, where μ and μ' are contracted with one of the x indices and a term proportional to $g_{\mu\mu'}$. Again in leading order all terms containing g_{ij} can be neglected, since the matrix element of the operator $\hat{O}^{(n)j}$ is proportional to $P_{\mu_1}P_{\mu_2}\dots P_{\mu_n}$. Therefore all terms with $g_{\mu_i\mu_j}$ yield much smaller contributions than those with $x_{\mu_i}x_{\mu_j}$:

$$\frac{x^2 g_{\mu_i\mu_j}P^{\mu_i}P^{\mu_j}}{x_{\mu_i}x_{\mu_j}P^{\mu_i}P^{\mu_j}} \xrightarrow{\text{Fourier transformation}} \frac{q^2P^2}{(q\cdot P)^2} = -\frac{M^2Q^2}{\nu^2} = -\frac{2M^2x}{\nu} \rightarrow 0 \quad . \tag{5.157}$$

Since we do not know the expansion analogous to (5.138), we have to regard the divergent functions of the two terms $C_1^{(n)}$ and $C_2^{(n)}$ that occur as different. Furthermore we must take into account that the operator $\hat{O}^{(n)}$ can now change the color of a quark and can in addition act on the gluons. Consequently we have to distinguish between three operators.

$$j = q \quad : \quad \hat{O}^{(n),q}_{\mu_1\mu_2\dots\mu_n} = \frac{\mathrm{i}^{n-1}}{2n!}\gamma_{\mu_1}\hat{D}_{\mu_2}\hat{D}_{\mu_3}\dots\hat{D}_{\mu_n} \tag{5.158}$$

acts on quark states (SU(3)-singlet);

$$j = \mathrm{NS} \quad : \quad \hat{O}^{(n),\mathrm{NS}}_{\mu_1\mu_2\dots\mu_n,a} = \frac{\mathrm{i}^{n-1}}{2n!}\frac{\hat{\lambda}^a}{2}\gamma_{\mu_1}\hat{D}_{\mu_2}\hat{D}_{\mu_3}\dots\hat{D}_{\mu_n} \tag{5.159}$$

acts on quark states and can change the flavor (SU(3)-octet); and

$$j = \mathrm{G} \quad : \quad \hat{O}^{(n),\mathrm{G}}_{\mu_1\mu_2\dots\mu_n} \tag{5.160}$$

acts on the gluon states

$$\langle \bar{F}_{\mu\nu}|\hat{O}^{(n),\mathrm{G}}_{\mu_1\mu_2\dots\mu_n}|\bar{F}_{\mu\nu}\rangle = \frac{\mathrm{i}^{n-2}}{n!}\mathrm{tr}\left\{\bar{F}_{\mu\mu_1}\hat{D}_{\mu_2}\hat{D}_{\mu_3}\dots\hat{D}_{\mu_{n-1}}\bar{F}^{\mu}_{\mu_n}\right\} \quad . \tag{5.161}$$

Here we have chosen the standard normalizations for C_1 and C_2 , which are called coefficient functions. Since P_μ is the only four-vector that occurs in

$$\frac{1}{2}\sum_{\text{pol.}}\langle N(P)|\hat{O}^{(n),j}_{\mu_1\mu_2\dots\mu_n}|N(P)\rangle \quad , \tag{5.162}$$

we can expand (5.162) into powers of P_μ. Again $g_{\mu_i\mu_j}P^2$ is negligible compared to $P_{\mu_i\mu_j}$ and therefore the leading term is

$$\frac{1}{2}\sum_{\text{pol.}}\langle N(P)|\hat{O}^{(n),j}_{\mu_1\mu_2\dots\mu_n}|N(P)\rangle = A^{(n),j}\frac{1}{n!}P_{\mu_1}P_{\mu_2}\dots P_{\mu_n} \quad , \tag{5.163}$$

and

$$\begin{aligned} W_{\mu\mu'} = \frac{1}{2\pi}\int \mathrm{d}^4x \exp(\mathrm{i}q\cdot x)\sum_j\sum_n & \Big[-g_{\mu\mu'}(x\cdot P)^n\mathrm{i}^nC^{(n)}_{1,j}(x) \\ & + n!P_\mu P_{\mu'}\frac{(x\cdot P)^{n-2}}{n!}\mathrm{i}^{n-2}C^{(n)}_{2,j}(x)\Big]A^{(n),j} \quad . \end{aligned} \tag{5.164}$$

Now we substitute for

$$x_\mu \exp(\mathrm{i}q \cdot x) = -\mathrm{i}\frac{\partial}{\partial q^\mu} \exp(\mathrm{i}q \cdot x) = -2\mathrm{i}q^\mu \frac{\partial}{\partial q^2} \exp(q \cdot x) \quad , \tag{5.165}$$

giving

$$\begin{aligned} W_{\mu\mu'} = \frac{1}{2\pi} \sum_{j,n} \Bigg[& -g_{\mu\mu'}(2q \cdot P)^n \left(\frac{\partial}{\partial q^2}\right)^n \int \mathrm{d}^4x \exp(\mathrm{i}q \cdot x)\, C_{1,j}^{(n)}(x) \\ & + P_\mu P_{\mu'}(2q \cdot P)^{n-2} \left(\frac{\partial}{\partial q^2}\right)^{n-2} \int \mathrm{d}^4x \exp(\mathrm{i}q \cdot x)\, C_{2,j}^{(n)}(x) \Bigg] A^{(n),j} \quad . \end{aligned} \tag{5.166}$$

Just as in the free case we can read off the structure functions $F_1(x)$ and $F_2(x)$, which in their most general form are now

$$F_1(x) = \frac{1}{2\pi} \sum_{j,n} (2q \cdot P)^n \left(\frac{\partial}{\partial q^2}\right)^n \int \mathrm{d}^4x \exp(\mathrm{i}q \cdot x)\, C_{1,j}^{(n)}(x) A^{(n),j} \quad , \tag{5.167}$$

$$F_2(x) = \frac{2\nu}{4\pi} \sum_{j,n} (2q \cdot P)^{n-2} \left(\frac{\partial}{\partial q^2}\right)^n \int \mathrm{d}^4x \exp(\mathrm{i}q \cdot x)\, C_{2,j}^{(n)}(x) A^{(n),j} \quad . \tag{5.168}$$

To simplify these expressions we define the nth moment of the coefficient functions

$$\tilde{C}_{1,j}^{(n)} = \frac{1}{4\pi} (Q^2)^n \left(\frac{\partial}{\partial q^2}\right)^n \int \mathrm{d}^4x \exp(\mathrm{i}q \cdot x)\, C_{1,j}^{(n)}(x) \tag{5.169}$$

and

$$\tilde{C}_{1,j}^{(n)} = \frac{1}{16\pi} (Q^2)^{n-1} \left(\frac{\partial}{\partial q^2}\right)^{n-2} \int \mathrm{d}^4x \exp(\mathrm{i}q \cdot x)\, C_{2,j}^{(n)}(x) \quad . \tag{5.170}$$

Now the structure functions are

$$F_1(x, Q^2) = 2 \sum_{j,n} x^{-n} \tilde{C}_{1,j}^{(n)} A^{(n),j} \quad , \tag{5.171}$$

$$F_2(x, Q^2) = 4 \sum_{j,n} x^{-n+1} \tilde{C}_{2,j}^{(n)} A^{(n),j} \quad . \tag{5.172}$$

Note that (5.171) and (5.172) now have the form of a Taylor expansion in Bjorken x. What have we gained by this general formulation? We have expressed the structure functions by sums over products of divergent functions $\tilde{C}_{1,j}$, $\tilde{C}_{2,j}$ and unknown constants $A^{(n),j}$. The only thing we know is that the $C_{1,j}$, $C_{2,j}$ are independent of the hadron considered. Therefore the $\tilde{C}_{1,j}$, $\tilde{C}_{2,j}$ can only depend on Q^2 and the constants of the theory, which are the coupling constant g and the renormalization point μ:

$$\begin{aligned} \tilde{C}_{1,j}^{(n)} &= \tilde{C}_{1,j}^{(n)}(Q^2, g, \mu) \quad , \\ \tilde{C}_{2,j}^{(n)} &= \tilde{C}_{2,j}^{(n)}(Q^2, g, \mu) \quad . \end{aligned} \tag{5.173}$$

In addition we know that the constants $A^{(n),j}$ are characteristic for the process under consideration. Of course, all quantities can be approximately evaluated, the $\tilde{C}_{1,j}$, $\tilde{C}_{2,j}$ reliably by means of perturbation theory from the free current commutator $\left[\hat{J}_\mu(x), \hat{J}_\nu(0)\right]$ (sec (5.124)) and the QCD Feynman rules. The $A^{(n),j}$, however, are trucly nonperturbative and can only be obtained from a phenomenological model of the nucleon or from lattice QCD or from a sumrule calculation, using (5.163). The finite matrix elements $A^{(n),j}$ thus contain information about the inner structure of the nucleons, for example.

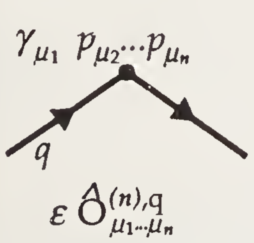
To obtain the perturbative expansion of the $\tilde{C}^{(n)}_{1,j}$, $\tilde{C}^{(n)}_{2,j}$ for the structure functions, we have to repeat the calculation in Sect. 5.1, which again leads to the GLAP equations. An advantage of this general OPE formulation is that formal features of the $\tilde{C}_{1,j}$, $\tilde{C}_{2,j}$ can be deduced and relationships between different quantities can be established for the full nonperturbative expressions. The most important way to deduce information about the coefficient functions is to consider their renormalization behavior.

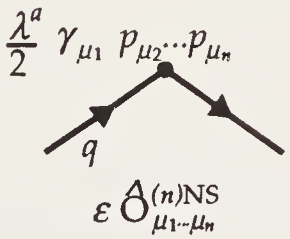
Up to now we have not discussed the renormalization of gauge theories, which we treat in a separate volume on "Gauge Theories". Here we only need to know that QCD is renormalizable and that for a given renormalization scheme carried out for a specific kinematic the bare coupling constants, Green functions, and so on are replaced by the renormalized ones. The renormalization schemes differ by the chosen kinematics, which can in general be characterized by a momentum parameter μ^2. Correspondingly the renormalized functions $\tilde{C}_1$, $\tilde{C}_2$ in (5.173) can also be written as

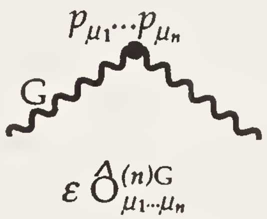
$$\tilde{C}^{(n)}_{i,j}(Q^2,g,\mu) = \sum_k Z^{(n)}_{jk}(g_0,\mu)\tilde{C}^{(n)}_{i,k}(Q^2,g_0)_{\text{unren.}} \quad , \tag{5.174}$$

$$g = g(g_0,\mu,Q^2) \quad . \tag{5.175}$$

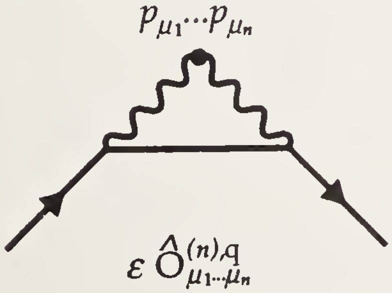
Equation (5.174) takes care of the fact that renormalization in general mixes different $\tilde{C}^{(n)}_j$, $j = q$, NS, G. This can be understood qualitatively by taking into account that not only do the graphs


$\gamma_{\mu_1} p_{\mu_2}\cdots p_{\mu_n}$
q
$\varepsilon\, \hat{O}^{(n),q}_{\mu_1\cdots\mu_n}$


$\frac{\lambda^a}{2}\gamma_{\mu_1} p_{\mu_2}\cdots p_{\mu_n}$
q
$\varepsilon\, \hat{O}^{(n)\mathrm{NS}}_{\mu_1\cdots\mu_n}$


$p_{\mu_1}\cdots p_{\mu_n}$
G
$\varepsilon\, \hat{O}^{(n)\mathrm{G}}_{\mu_1\cdots\mu_n}$

occur, but also, for example, the graph


$p_{\mu_1}\cdots p_{\mu_n}$
$\varepsilon\, \hat{O}^{(n),q}_{\mu_1\cdots\mu_n}$

Hence the sum over $k = q$, NS, G occurs on the right-hand side of (5.174). Since the physical theory must not depend on the renormalization point μ, we can demand that

$$\frac{\mathrm{d}\tilde{C}_{ij}^{(n)}(Q^2, g, \mu)}{\mathrm{d}\mu} = 0 \quad . \tag{5.176}$$

Now the total derivative is split up into partial derivatives.

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\mu}\tilde{C}_{i,j}^{(n)}(Q^2, g, \mu) &= \left(\frac{\partial g}{\partial \mu}\frac{\partial}{\partial g} + \frac{\partial}{\partial \mu}\right)\tilde{C}_{i,j}^{(n)}(Q^2, g, \mu) \\ &= \sum_k \frac{\partial Z_{jk}^{(n)}}{\partial \mu}\tilde{C}_{i,k}^{(n)}(Q^2, g_0)_{\text{unren.}} \\ &= \sum_{klm} \frac{\partial Z_{jk}^{(n)}}{\partial \mu}\left(Z^{(n)}\right)_{kl}^{-1}\left(Z^{(n)}\right)_{lm}\tilde{C}_{i,m}^{(n)}(Q^2, g_0)_{\text{unren.}} \\ &= \sum_l \left[\sum_k \frac{\partial Z_{jk}^{(n)}}{\partial \mu}\left(Z^{(n)}\right)_{kl}^{-1}\right]\tilde{C}_{i,l}^{(n)}(Q^2, g, \mu) \quad . \end{aligned} \tag{5.177}$$

Defining

$$\gamma_{jl}^{(n)} := -\mu\sum_k \frac{\partial Z_{jk}^{(n)}}{\partial \mu}\left(Z^{(n)}\right)_{kl}^{-1} = \gamma_{jl}^{(n)}(g, \mu) \tag{5.178}$$

yields

$$\sum_l \left[\left(\mu\frac{\partial g}{\partial \mu}\frac{\partial}{\partial g} + \mu\frac{\partial}{\partial \mu}\right)\delta_{jl} + \gamma_{jl}^{(n)}\right]\tilde{C}_{i,l}^{(n)}(Q^2, g, \mu) = 0 \quad . \tag{5.179}$$

The term $\partial g/\partial\mu$, i.e., the dependence of the coupling constant on the renormalization point, is a fixed characteristic function for every field theory. It is referred to as the β function:

$$\beta(g) = \mu\frac{\partial g}{\partial \mu} \quad . \tag{5.180}$$

In the case of QCD, perturbation theory yields (see Exercise 5.11)

$$\beta(g) = -\left(11 - \frac{2}{3}N_f\right)\frac{g^3}{(4\pi)^2} \quad - \quad \left(102 - \frac{38}{3}N_f\right)\frac{g^5}{(4\pi)^4} + O(g^7) \ , \tag{5.181}$$

where N_f denotes the number of quark flavors.

EXERCISE

5.11 The Lowest Order Terms of the β Function

Problem. Show that (5) in Example 4.3 indeed yields the lowest order contribution to the β function.

$$\alpha_s(-q^2) = \frac{\alpha_s(\mu^2)}{1 + \frac{11-\frac{2}{3}N_f(-q^2)}{4\pi}\alpha_s(\mu^2)\ln\left(-\frac{q^2}{\mu^2}\right)} \quad . \tag{1}$$

Here μ^2 denotes the renormalization point where $\alpha_s(-q^2)$ assumes the renormalized value $\alpha_s(\mu^2)$. $N_f(-q^2)$ is the number of light quarks with masses smaller than $-q^2$.

Solution. Equation (1) yields for $g(-q^2)$

$$g(-q^2) = \frac{g(\mu^2)}{\sqrt{1 + \frac{11-\frac{2}{3}N_f(-q^2)}{(4\pi)^2}g^2(\mu^2)\ln\left(-\frac{q^2}{\mu^2}\right)}} \tag{2}$$

with $\alpha_s = g^2/4\pi$.

We multiply (2) by the demominator on the right hand-side and differentiate the resulting equation with respect to μ^2:

$$\begin{aligned} g'(\mu^2) &= g(-q^2)\frac{\partial}{\partial\mu^2}\sqrt{1 + \frac{11-\frac{2}{3}N_f(-q^2)}{(4\pi)^2}g^2(\mu^2)\ln\left(-\frac{q^2}{\mu^2}\right)} \\ &= g(-q^2)\frac{\frac{1}{2}\frac{11-\frac{2}{3}N_f(-q^2)}{(4\pi)^2}}{\sqrt{1 + \frac{11-\frac{2}{3}N_f(-q^2)}{(4\pi)^2}g^2(\mu^2)\ln\left(-\frac{q^2}{\mu^2}\right)}} \\ &\quad\times\left[2g(\mu^2)g'(\mu^2)\ln\left(-\frac{g^2}{\mu^2}\right) - \frac{1}{\mu^2}g^2(\mu^2)\right] \quad . \end{aligned} \tag{3}$$

Once more we insert (2), into this expression and collect the terms containing g':

$$\begin{aligned} &g'(\mu^2)\left[1 + \frac{11-\frac{2}{3}N_f(-q^2)}{(4\pi)^2}g^2(\mu^2)\ln\left(-\frac{q^2}{\mu^2}\right)\right] \\ &= \frac{11-\frac{2}{3}N_f(-q^2)}{(4\pi)^2}\left[g^2(\mu^2)g'(\mu^2)\ln\left(-\frac{g^2}{\mu^2}\right) - \frac{g^3(\mu^2)}{2\mu^2}\right] \quad , \end{aligned} \tag{4}$$

$$g'(\mu^2) = -\frac{1}{2\mu^2}\frac{11-\frac{2}{3}N_f(-q^2)}{(4\pi)^2}g^3(\mu^2) \quad . \tag{5}$$

By means of (5) the β function now assumes the form

$$\beta = \frac{\partial g(\mu)}{\partial\mu} = 2\mu^2\frac{\partial g(\mu^2)}{\partial\mu^2} = -\left[11 - \frac{2}{3}N_f(-q^2)\right]\frac{g^3(\mu^2)}{(4\pi)^2} \quad . \tag{6}$$

In fact this is identical with the first term in (5.181). We can see immediately where the higher terms come from. From Example 4.3, (1) is derived from vacuum

Exercise 5.11.

polarization graphs. Consequently we obtain the higher-order contributions to the β function if higher graphs are taken into account in this calculation.

Finally we summarize the basic ideas of this little calculation. The renormalization is carried out for a given value $-q^2$ of the momentum transfer, i.e., all divergencies are subtracted. This procedure yields a renormalized coupling constant $\alpha_s(\mu^2)$. If the formerly divergent graphs are evaluated for different values of $-q^2$, we only obtain finite contributions, that depend on the renormalized quantities. Equation (1) describes this remaining finite renormalization. Since $\alpha_s(-q^2)$ represents the physical coupling constant, it must not depend on the renormalization scale μ^2. Therefore $\alpha_s(\mu^2)$ has to depend on μ^2 in such a way that the right-hand side of (1) is no longer a function of μ^2. This demand determines $\alpha_s(\mu^2)$ and consequently also the β function.

Obviously (5.179) yields a restriction for $\tilde{C}^{(n)}_{i,\ell}$, from which one can derive using (5.171) and (5.172) a constraint for the observable structure functions. In this way our formal analysis leads to measurable predictions. Equations of the type (5.179) are called *renormalization group equations* or *Callan–Symanzik equations*. They play an important role in formal considerations of field theories, because they yield nonperturbative results.

We now wish to explain the meaning of the β function in some more detail. As shown in Exercise 5.9, the coupling constant g in fact depends on the dimensionless quantity $t = +\frac{1}{2}\ln Q^2/\mu^2$. This enables us to write the defining equation of the dimensionless β function as

$$\beta = \mu\frac{\partial g(\mu)}{\partial\mu} = -\mu\frac{\partial t}{\partial\mu}\frac{\partial g'(t)}{\partial t} = -\frac{\partial g'(t)}{\partial t} \quad . \tag{5.182}$$

The β function describes how the coupling depends on the momentum transfer, given a fixed renormalization point μ^2 (the notations g and g' indicate that the functional dependence is altered by the replacement of μ by t). In fact there are two definitions known in the literature. The first one is expression (5.182) and the second one is

$$\tilde{\beta}(g) = t\frac{\partial g'(t)}{\partial t} = Q^2\ln\frac{Q^2}{\mu^2}\frac{\partial g''(Q^2)}{\partial Q^2} \quad . \tag{5.183}$$

A great advantage of the latter definition is that its features are not dependent on the sign of t and they can be analyzed in a more general manner. Because of these two different definitions, the renormalization group equations found in the literature sometimes differ by a factor t in front of the β function.

The zeros of the β function are crucial for the general behavior of the coupling constant in the case of very large momentum transfers (the so-called ultraviolet limit) and very small momentum transfers (the so-called infrared limit) and thus for the most basic properties of the theory. This can be easily understood by analyzing the two examples in Fig. 5.12.

Case (a). If $\beta(g)$ is positive, then g becomes larger with increasing Q^2. In the region $g < g_0$ g therefore approaches the value g_0 with increasing Q^2. On the other hand

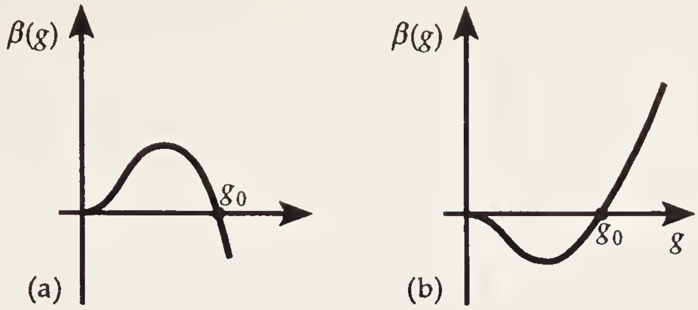


Fig. 5.12a,b. Two possible forms of the β function

$\beta(g)$ is negative for $g > g_0$ and here g becomes smaller with increasing Q^2. g_0 is therefore called a stable ultra-violet fixpoint. This point, however, is not stable in the infrared limit. As soon as g is different from g_0, and consequently $\tilde{\beta}(g)$ no longer vanishes, g moves away from the value g_0 for decreasing Q^2. Correspondingly the behavior at the point $g = 0$ is just the opposite. This is a stable infrared fixpoint and it is unstable in the ultraviolet limit. Hence Fig. 5.12a characterizes a theory whose coupling constant vanishes for small momentum transfers and assumes a constant value for large momentum transfers.

Case (b). In this case the points $g = 0$ and $g = g_0$ have exchanged their meanings compared to *Case (a)*. The coupling constant vanishes for large momentum transfers. This behavior is called *asymptotic freedom*. For small momentum transfers g assumes the constant value g_0. If g_0 is very large, which is known as *infrared slavery*, such a theory exhibits similar features to QCD.

For our problem we want to evaluate (5.179) using perturbation theory. To this end we use

$$\beta = -\left(11 - \frac{2}{3}N_f\right)\frac{g^3}{(4\pi)^2} = -bg^3 \tag{5.184}$$

as an approximation for $\beta(g)$. With the definition

$$\bar{g}^2 = \frac{g^2}{1 + 2bg^2t} \quad , \tag{5.185}$$

(5.179) becomes

$$\sum_\ell \left[\left(-bg^3\frac{\partial}{\partial g} + \mu\frac{\partial}{\partial \mu}\right)\delta_{j\ell} + \gamma_{j\ell}^{(n)}\right]\tilde{C}_{i,\ell}^{(n)}(Q^2, g, \mu) = 0 \quad . \tag{5.186}$$

Furthermore it holds that

$$\frac{\partial \bar{g}}{\partial t} = \frac{1}{2\bar{g}}\frac{\partial \bar{g}^2}{\partial t} = \frac{1}{2\bar{g}}\frac{-g^2 2bg^2}{(1 + 2bg^2t)^2} = -b\bar{g}^3 \quad , \tag{5.187}$$

$$\frac{\partial \bar{g}}{\partial g} = \frac{g}{\bar{g}}\frac{\partial \bar{g}^2}{\partial g^2} = \frac{g}{\bar{g}}\frac{1}{(1 + 2bg^2t)^2} = \frac{\bar{g}^3}{g^3} \quad , \tag{5.188}$$

and hence the derivatives in (5.186) acting on $\bar{g}$ yield zero:

$$\left(-bg^3\frac{\partial}{\partial g}-\frac{\partial}{\partial t}\right)\bar{g}(g,t) = -bg^3\frac{\bar{g}^3}{g^3}+b\bar{g}^3 = 0 \quad . \tag{5.189}$$

The solution of (5.186) for $\gamma^{(n)}_{i,j\ell} = 0$ is therefore simply

$$\tilde{C}^{(n)}_{i,\ell}(Q^2,g,\mu) = f_{i,\ell}(\bar{g}(g,t)) \quad , \tag{5.190}$$

with an unknown function $f_\ell(\bar{g}(g,t))$. For $\gamma^{(n)}_{i,j\ell} \neq 0$ this must only be multiplied by a corresponding exponential factor:

$$\tilde{C}^{(n)}_{i,\ell}(Q^2,g,\mu) = \sum_k f_{i,k}\left(\bar{g}(g,t)\right)\exp\left[\int_0^t \mathrm{d}t'\gamma^{(n)}_{k,\ell}(\bar{g}(g,t'))\right] \quad . \tag{5.191}$$

Now we also expand the $\gamma^{(n)}_{k,\ell}$ into a power series:

$$\gamma^{(n)}_{k,\ell} = b^{(n)}_{k,\ell} + c^{(n)}_{k,\ell}\bar{g} - d^{(n)}_{k,\ell}\bar{g}^2 + \quad \cdots \tag{5.192}$$

A perturbative calculation shows that both the constant and linear terms are equal to zero. The lowest-order contribution is proportional to $\bar{g}^2$. Hence the t' integral can be approximately evaluated:

$$\int_0^t \mathrm{d}t'\gamma^{(n)}_{k,\ell} \approx -d^{(n)}_{k,\ell}\int_0^t \frac{g^2}{1+2bg^2t'}\mathrm{d}t' = -\frac{d^{(n)}_{k,\ell}}{2b}\ln(1+2bg^2t) \quad , \tag{5.193}$$

$$\begin{aligned}\tilde{C}^{(n)}_{i,\ell}(t,g) &= \sum_k f_{i,k}\left(\bar{g}(g,t)\right)(1+2bg^2t)^{-d^{(n)}_{k,\ell}/2b} \\ &= \sum_k f_{i,k}\left(\bar{g}(g,t)\right)\left(\bar{g}^2(g,t)/g^2\right)^{d^{(n)}_{k,\ell}/2b} \quad .\end{aligned} \tag{5.194}$$

In the limiting case $t \gg 1, \bar{g} \to 0$ the function $f_{i,k}\left(\bar{g}(g,t)\right)$ assumes a constant value $a^{(n)}_{i,k}$:

$$t \gg 1\ , \quad x \text{ constant} \quad \Rightarrow \quad \tilde{C}^{(n)}_{i,\ell}(t,g) = \sum_k a^{(n)}_{i,k}\left(\bar{g}^2(g,t)/g^2\right)^{d^{(n)}_{k,\ell}/2b} \ . \tag{5.195}$$

We insert this into (5.171) and (5.172), which we slightly modify for this purpose. Let $F_i(z,Q^2)$ be the complex continuation of $F_i(x,Q^2)$. Then we obviously have

$$\frac{1}{2\pi \mathrm{i}}\int_\ell F_1(z,Q^2)z^{n-1}\mathrm{d}z = 2\sum_j \tilde{C}^{(n)}_{1,j}A^{(n),j} \quad , \tag{5.196}$$

$$\frac{1}{2\pi \mathrm{i}}\int_\ell F_2(z,Q^2)z^{n-2}\mathrm{d}z = 4\sum_j \tilde{C}^{(n)}_{2,j}A^{(n),j} \quad , \tag{5.197}$$

with the integration contour depicted in Fig. 5.13.

Since $F_1(z)$ and $F_2(z)$ can become singular only in the case $z \to 0$, C can be deformed and replaced by an ε prescription:

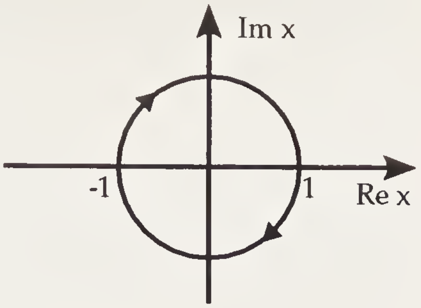


Fig. 5.13. Integration contour ℓ in (5.196) and (5.197)

$$\int_\ell F_1(z,Q^2)z^{n-1}\mathrm{d}z = \int_{-1}^{1} F_1(x+\mathrm{i}\varepsilon,Q^2)x^{n-1}\mathrm{d}x - \int_{-1}^{1} F_1(x-\mathrm{i}\varepsilon,Q^2)x^{n-1}\mathrm{d}x$$
$$= \int_{-1}^{1} \left(F_1(x+\mathrm{i}\varepsilon,Q^2) - F_1^*(x+\mathrm{i}\varepsilon,Q^2)\right) x^{n-1}\mathrm{d}x \ . \qquad (5.198)$$

Now we have to determine how the analytically continued structure function behaves under complex conjugation. Thus we have to take into account that $W_{\mu\nu}$ is the imaginary part of the scattering amplitude in the forward direction $T_{\mu\nu}$ (this is a special case of the so-called optical theorem):

$$W_{\mu\nu} = \frac{1}{\pi}\mathrm{Im}\left(T_{\mu\nu}\right) \quad . \qquad (5.199)$$

Correspondingly we have

$$F_1\left(x+\mathrm{i}\varepsilon,Q^2\right) - F_1^*\left(x+\mathrm{i}\varepsilon,Q^2\right) = 2F_1\left(x,Q^2\right) \quad , \qquad (5.200)$$

$$\int_\ell F_1\left(z,Q^2\right) z^{n-1}\mathrm{d}z = 4\int_0^1 F_1\left(x,Q^2\right) x^{n-1}\mathrm{d}x =: 4M_1^{(n+1)}\left(Q^2\right) \quad , \qquad (5.201)$$

$$\int_\ell F_2\left(z,Q^2\right) z^{n-2}\mathrm{d}z = 4\int_0^1 F_2\left(x,Q^2\right) x^{n-2}\mathrm{d}x =: 4M_2^{(n)}\left(Q^2\right) \quad . \qquad (5.202)$$

The quantities $M_1^{(n)}$ and $M_2^{(n)}$ are called *moments of the structure functions*. Owing to (5.195) they obey the evolution equations

$$M_1^{(n+1)}(t) = \frac{1}{2}\sum_{jk} a_{1,k}^{(n)} \left(\frac{\bar{g}^2(g,t)}{g^2}\right)^{d_{kj}^{(n)}/2b} A^{(n),j} \quad , \qquad (5.203)$$

$$M_2^{(n)}(t) = \sum_{jk} a_{2,k}^{(n)} \left(\frac{\bar{g}^2(g,t)}{g^2}\right)^{d_{kj}^{(n)}/2b} A^{(n),j} \quad . \qquad (5.204)$$

In fact these equations correspond to the GLAP equations. We shall prove this for the nonsinglet contributions to the moments of the structure functions. Since $d_{kj}^{(n)}$

is diagonal for $j = \mathrm{NS}$, the manipulations required are quite simple. The moments of the NS-quark distribution functions are defined as

$$M_{1,ij}^{(n+1)}(t) = \int_0^1 x^{n-1}\Delta_{ij}(x,t)\,\mathrm{d}x \quad . \tag{5.205}$$

(5.65) then yields (we substitute $x = yz$ in the second step)

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}M_{1,ij}^{(n+1)}(t) &= \int_0^1 \mathrm{d}x\, x^{n-1}\frac{\alpha_s(t)}{2\pi}\int_x^1 \frac{\mathrm{d}y}{y}\Delta_{ij}(y,t)P_{qq}\left(\frac{x}{y}\right) \\
&= \frac{\alpha_s(t)}{2\pi}\int_0^1 \mathrm{d}y \int_0^y \frac{\mathrm{d}x}{y}x^{n-1}\Delta_{ij}(y,t)P_{qq}\left(\frac{x}{y}\right) \\
&= \frac{\alpha_s(t)}{2\pi}\int_0^1 \mathrm{d}y\,\Delta_{ij}(y,t)y^{n-1}\int_0^1 \mathrm{d}z P_{qq}z^{n-1} \\
&= \frac{\alpha_s(t)}{2\pi}D^{(n)}M_{1,ij}^{(n+1)} \quad ,
\end{aligned} \tag{5.206}$$

with

$$D^{(n)} = \int_0^1 \mathrm{d}z P_{qq} z^{n-1} \quad . \tag{5.207}$$

From (5.209) it follows that

$$M_{1,\mathrm{NS}}^{(n+1)}(t) = M_{1,\mathrm{NS}}^{(n+1)}\left(\frac{\bar{g}^2(g,t)}{g^2}\right)^{d_{\mathrm{NS}}^{(n)}/2b} \quad , \tag{5.208}$$

$$\begin{aligned}
\frac{\partial M_{1,\mathrm{NS}}^{(n+1)}(t)}{\partial t} &= \frac{d_{\mathrm{NS}}^{(n)}}{2b}\frac{1}{\bar{g}^2}2\bar{g}\left(-b\bar{g}^3\right)M_{1,\mathrm{NS}}^{(n+1)}(t) \\
&= -\bar{g}^2 d_{\mathrm{NS}}^{(n)}M_{1,\mathrm{NS}}^{(n+1)}(t) = \frac{\alpha_s(t)}{2\pi}\left(-8\pi^2 d_{\mathrm{NS}}^{(n)}\right)M_{1,\mathrm{NS}}^{(n+1)}(t) \quad .
\end{aligned} \tag{5.209}$$

Clearly (5.209) and (5.206) are the same relations, provided $-8\pi^2 d_{\mathrm{NS}}^{(n)}$ is identified with $D^{(n)}$. If the $\gamma_{\mathrm{NS}}^{(n)}$ are evaluated by means of perturbation theory, we obtain the expression (5.207) for $d_{\mathrm{NS}}^{(n)}$ with $P_{qq}(z)$ given by (5.152). But also without resorting to perturbation theory we can see that a comparison of (5.209) with (5.206) shows that the GLAP equations must have the form

$$\frac{\partial\Delta_{ij}}{\partial t} \approx \alpha_s(t)\int_x^1 \frac{\mathrm{d}y}{y}\Delta_{ij}(y,t)P\left(\frac{x}{y}\right) \quad . \tag{5.210}$$

EXAMPLE

5.12 The Moments of the Structure Functions

In this example we discuss the physical meaning of the moments of structure functions. Looking at equations (5.171), (5.172), (5.196), (5.197), (5.201), and (5.202) one could get the impression that knowing the structure function is equivalent to knowledge of all its moments. But this is only true in the limit $Q^2 \to \infty$. For every finite Q^2, deviations occur owing to the corrections that were neglected on the way from (5.140) to (5.141).

One can prove that the corrections to $\hat{O}^n_{\mu_1 \cdots \mu_n \mu_\beta}$, i.e., to $C^{(n)}$ and therefore to the nth moment of the structure function F_2 and to the $(n+1)$th moment of F_1, are suppressed by a factor

$$\frac{n\mu^2}{Q^2} \quad . \tag{1}$$

Equation (1) shows already that for every finite Q^2 the corrections become large at some value of n, i.e.,each perturbative calculation of the moments is only valid up to a specific maximum number n. The factor (1) can be understood quite easily. Equation (5.155) is inserted into the expression for the scattering tensor and contracted with the leptonic scattering tensor. Then we count how often terms of the kind

$$q^{\mu_1} \cdots q^{\mu_n} g_{\mu_i \mu_j} p_{\mu_1} \cdots p_{\mu_{i-1}} p_{\mu_{i+1}} \cdots p_{\mu_{j-1}} p_{\mu_{j+1}} \cdots p_{\mu_n} \tag{2}$$

occur compared to the term

$$q^{\mu_1} \cdots q^{\mu_n} p_{\mu_1} \cdots p_{\mu_n} \quad . \tag{3}$$

This procedure yields the factor n [2]. Thus a perturbative QCD calculation is well suited for the lower moments, for example, the integral over the structure functions, but fails in the case of the higher moments. From (5.201) and (5.202) these higher moments are determined by the structure functions near $x = 1$. The value $x = 1$, however, represents elastic scattering. Correspondingly here we find the elastic peak and for x slightly below 1 we find peaks due to the different resonances. Since these resonance structures depend on the individual features of the hadronic bound states, there is no way to describe this region by perturbative QCD. On the other hand this analysis clearly shows that the nonperturbative effects must be included in higher-order terms of the operator product expansion.

The points just discussed (usually referred to as *precocious scaling*) are explained again in Fig. 5.14. This figure depicts the experimenally observed structure function $\nu W_2(x, Q^2)$ (full line), which converges to $F_2(x)$ for large Q^2. The Nachtmann variable

$$\xi = \frac{2x}{1 + \left(1 + 4x^2 \frac{M_N^2}{Q^2}\right)^{\frac{1}{2}}} \tag{4}$$

---

[2] Since the explicit calculation is quite cumbersome, we refer to A. de Rujula, H. Georgi, and H.D. Politzer: Ann. Phys. **103**, 315 (1977).

Example 5.12.

has been used instead of x. In the limit $Q^2 \to \infty$, ξ is equivalent to x. At finite Q^2 this variable takes the effects of the nucleon mass into account. The contributions of the various resonances can be clearly identified. The elastic peak at ξ_p is not plotted.

The dashed curve results from a perturbative QCD calculation. Clearly the agreement becomes better for increasing Q^2 values, since here the effect of the individual hadron resonances can be neglected. But also for small values of Q^2 the lowest moments

$$M_2^{(1)}(Q^2) = \int_0^1 F_2(x, Q^2)\, x \,\mathrm{d}x \tag{5}$$

and

$$M_2^{(2)}(Q^2) = \int_0^1 F_2(x, Q^2)\,\mathrm{d}x \tag{6}$$

are quite well described by the QCD calculation.

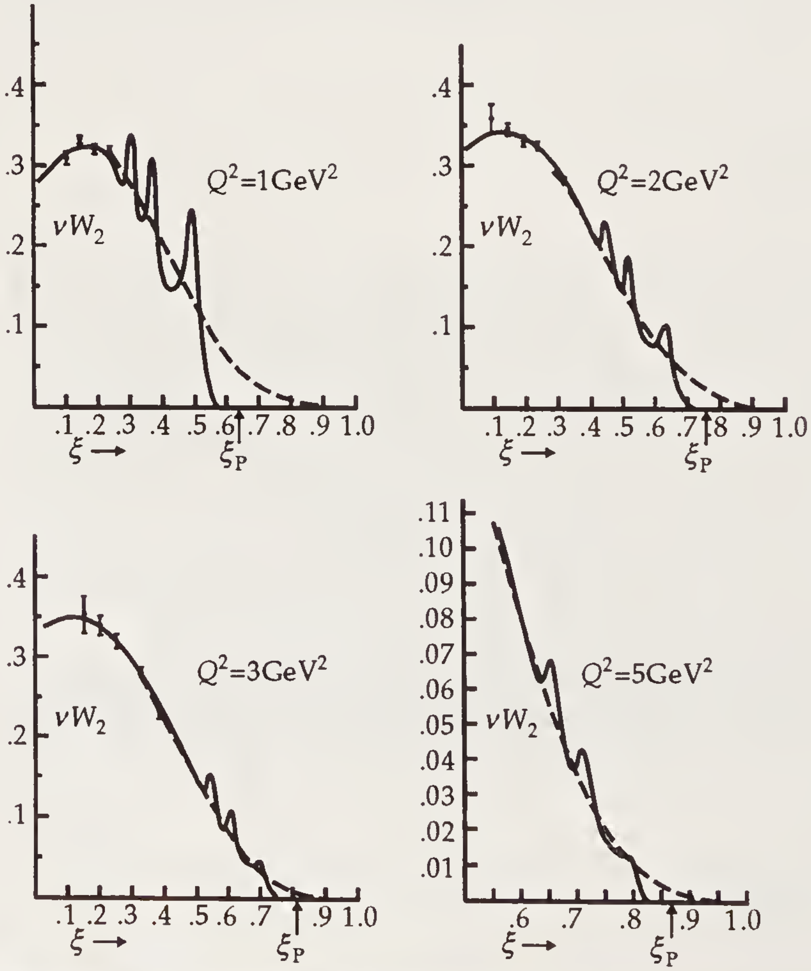


Fig. 5.14. Comparison of the experimental structure function νW_2 for electron–proton scattering with the result of a perturbative QCD calculation. ξ denotes the Nachtmann variable and ξ_p the position of the elastic peak. From A. de Rujula, H. Georgi, and H.D. Politzer: Ann. Phys. **103**, 315 (1977)

Before we continue our discussion of the OPE we briefly summarize the derivations carried out so far. Employing deep inelastic scattering as an example, we explained how scattering tensors can be expanded into products of divergent coefficient functions, which can be evaluated by means of perturbation theory and finite matrix elements not dependent on the momentum transfer, for example. In the case of $F_2(x,Q^2)$ we had

$$\int_0^1 \mathrm{d}x\, x^{n-2} F_2(x,Q^2) = 2\pi\mathrm{i} \sum_j \tilde{C}_{2,j}^{(n)}(Q^2,g,\mu) A^{(n),j} \quad . \tag{5.211}$$

This so-called *factorization* into a Q^2-dependent perturbative part and fixed numbers $A^{(n),j}$ that contain information about the distribution functions at a given value of Q_0^2 is of fundamental importance to most QCD applications. We showed the possibility of factorization for deep inelastic scattering, but in fact one must prove that this method is not destroyed by higher-order terms. Since each *soft* (Q^2 is small) gluon line couples strongly ($\alpha_s(Q^2)$ is large), higher orders in QCD are in general as important as the lowest order. Therefore the validity of the expansion (5.211), or of analogous expansions for other processes, has to be investigated very carefully. Because of its fundamental importance we shall discuss this question in the following section.

The nonperturbative matrix elements $A^{(n),j}$ can be either treated as pure parameters or calculated with help of nonperturbative models, e.g., the bagmodel. To clarify the last statement we now return to the starting point of our discussion and investigate the matrix elements in some more detail. The basic equations were (5.123), (5.129), (5.136), and (5.141).

$$\begin{aligned} \langle \mathrm{N}| \left[J_\mu\left(\frac{x}{2}\right), J_\nu\left(-\frac{x}{2}\right) \right]_- |\mathrm{N}\rangle &= \langle \mathrm{N}| \hat{\bar{\Psi}}\left(\frac{x}{2}\right) \gamma_\mu \left[\hat{\Psi}(x), \hat{\bar{\Psi}}(0) \right]_+ \gamma_\nu \hat{\Psi}\left(-\frac{x}{2}\right) \\ &\quad - \hat{\bar{\Psi}}\left(-\frac{x}{2}\right) \gamma_\nu \left[\hat{\Psi}(0), \hat{\bar{\Psi}}(x) \right]_+ \gamma_\mu \hat{\Psi}\left(\frac{x}{2}\right) |\mathrm{N}\rangle \\ &= s_{\mu\alpha\nu\beta} \frac{1}{2\pi} \partial^\alpha \left[\varepsilon(x_0)\delta(x^2) \right] \\ &\quad \times \sum_n \frac{(-)^n}{n!} \left(\frac{x}{2}\right)^{\mu_1} \cdots \left(\frac{x}{2}\right)^{\mu_n} \left[1-(-)^n \right] \\ &\quad \times \frac{1}{2} \sum_{\mathrm{pol.}} \langle \mathrm{N}| \hat{O}^n_{\mu_1\cdots\mu_n\beta} |\mathrm{N}\rangle \quad , \end{aligned} \tag{5.212}$$

$$\frac{1}{2} \sum_{\mathrm{pol.}} \langle \mathrm{N}| \hat{O}^n_{\mu_1\cdots\mu_n\beta} |\mathrm{N}\rangle \to A^{(n)} P_{\mu_1} \cdots P_{\mu_n} P_\beta \quad . \tag{5.213}$$

Clearly the $A^{(n)}$ are related to matrix elements of products between field operators. Up to now we have taken only the leading contribution of the noninteracting theory into account.

Now we systematically expand the current commutator $\left[J_\mu\left(\frac{x}{2}\right), J_\nu\left(-\frac{x}{2}\right)\right]_-$. Such a general expansion, already known from quantum electrodynamics, is provided by Wick's theorem for time-ordered products of field operators:

$$
\begin{aligned}
T\left[\Psi\left(\frac{x}{2}\right)\bar{\Psi}\left(-\frac{x}{2}\right)\right] =&: \Psi\left(\frac{x}{2}\right)\bar{\Psi}\left(-\frac{x}{2}\right): \\
&+ \langle 0|T\left[\Psi\left(\frac{x}{2}\right)\bar{\Psi}\left(-\frac{x}{2}\right)\right]|0\rangle \quad \text{etc.}
\end{aligned}
\tag{5.214}
$$

For the product of two currents Wick's theorem yields

$$
\begin{aligned}
T\left[J_\mu\left(\frac{x}{2}\right), J_\nu\left(-\frac{x}{2}\right)\right] = -\mathrm{tr}\Big\{&\langle 0|T\left[\hat{\Psi}\left(-\frac{x}{2}\right)\hat{\bar{\Psi}}\left(\frac{x}{2}\right)\right]|0\rangle\,\gamma_\mu \\
&\times \langle 0|T\left[\hat{\Psi}\left(\frac{x}{2}\right)\hat{\bar{\Psi}}\left(-\frac{x}{2}\right)\right]|0\rangle\gamma_\nu\Big\} \\
&+ :\hat{\bar{\Psi}}\left(\frac{x}{2}\right)\gamma_\mu\langle 0|T\left[\hat{\Psi}\left(\frac{x}{2}\right)\hat{\bar{\Psi}}\left(-\frac{x}{2}\right)\right]|0\rangle\gamma_\nu\hat{\Psi}\left(-\frac{x}{2}\right): \\
&+ :\hat{\bar{\Psi}}\left(-\frac{x}{2}\right)\gamma_\nu\langle 0|T\left[\hat{\Psi}\left(-\frac{x}{2}\right)\hat{\bar{\Psi}}\left(\frac{x}{2}\right)\right]|0\rangle\gamma_\mu\hat{\Psi}\left(\frac{x}{2}\right): \\
&+ :\hat{\bar{\Psi}}\left(\frac{x}{2}\right)\gamma_\mu\hat{\Psi}\left(\frac{x}{2}\right)\hat{\bar{\Psi}}\left(-\frac{x}{2}\right)\gamma_\nu\hat{\Psi}\left(-\frac{x}{2}\right): \quad .
\end{aligned}
\tag{5.215}
$$

Where :. . .: denotes normal-ordered products.

Now we again replace the vacuum expectation values of the T products by the perturbative result:

$$
\langle 0|T\left[\hat{\Psi}\left(\frac{x}{2}\right)\hat{\bar{\Psi}}\left(-\frac{x}{2}\right)\right]|0\rangle = -\frac{1}{4\pi^2}\frac{\gamma_\alpha x^\alpha}{(x^2-\mathrm{i}\varepsilon)^2} \quad .
\tag{5.216}
$$

The difference between the anticommutator (5.125) and the T product is basically the missing $\varepsilon(x_0)$ factor. Hence

$$
\begin{aligned}
T\left[J_\mu\left(\frac{x}{2}\right)J_\nu\left(-\frac{x}{2}\right)\right] =& \frac{x^2 g_{\mu\nu} - 2x_\mu x\nu}{\pi^4(x^2-\mathrm{i}\varepsilon)^4} \\
&- \frac{\mathrm{i}x^\alpha}{2\pi^2(x^2-\mathrm{i}\varepsilon)^2}s_{\mu\alpha\nu\beta} :\hat{\bar{\Psi}}\left(\frac{x}{2}\right)\gamma^\beta\hat{\Psi}\left(-\frac{x}{2}\right) - \hat{\bar{\Psi}}\left(-\frac{x}{2}\right)\gamma^\beta\hat{\Psi}\left(\frac{x}{2}\right): \\
&+ \frac{x^\alpha}{2\pi^2(x^2-\mathrm{i}\varepsilon)^2}\varepsilon_{\mu\alpha\nu\beta} :\hat{\bar{\Psi}}\left(\frac{x}{2}\right)\gamma^\beta\gamma_5\hat{\Psi}\left(-\frac{x}{2}\right) - \hat{\bar{\Psi}}\left(-\frac{x}{2}\right)\gamma^\beta\gamma_5\hat{\Psi}\left(\frac{x}{2}\right): \\
&+ :\hat{\bar{\Psi}}\left(\frac{x}{2}\right)\gamma_\mu\hat{\Psi}\left(\frac{x}{2}\right)\hat{\bar{\Psi}}\left(-\frac{x}{2}\right)\gamma_\nu\hat{\Psi}\left(-\frac{x}{2}\right): \quad .
\end{aligned}
\tag{5.217}
$$

The connection between T product and commutator is given by

$$
\begin{aligned}
&T\left[J_\mu\left(\frac{x}{2}\right)J_\nu\left(-\frac{x}{2}\right)\right] - \left(T\left[J_\mu\left(\frac{x}{2}\right)J_\nu\left(-\frac{x}{2}\right)\right]\right)^\dagger \\
&= \Theta(x^0)J_\mu\left(\frac{x}{2}\right)J_\nu\left(-\frac{x}{2}\right) + \Theta(-x^0)J_\nu\left(-\frac{x}{2}\right)J_\mu\left(\frac{x}{2}\right) \\
&\quad - \Theta(x^0)J_\nu\left(-\frac{x}{2}\right)J_\mu\left(\frac{x}{2}\right) - \Theta(-x^0)J_\mu\left(\frac{x}{2}\right)J_\nu\left(-\frac{x}{2}\right) \\
&= \Theta(x^0)\left[J_\mu\left(\frac{x}{2}\right), J_\nu\left(-\frac{x}{2}\right)\right]_- - \Theta(-x^0)\left[J_\mu\left(\frac{x}{2}\right), J_\nu\left(-\frac{x}{2}\right)\right]_- \\
&= \varepsilon(x^0)\left[J_\mu\left(\frac{x}{2}\right), J_\nu\left(-\frac{x}{2}\right)\right]_- \quad ,
\end{aligned}
\tag{5.218}
$$

with the sign function $\varepsilon(x^0) = \mathrm{sgn}(x^0)$. We multiply (5.218) by $\varepsilon(x^0)$ $\left(\varepsilon^2(x^0) = 1\right)$ and obtain for the expectation value of the current commutator

$$
\begin{aligned}
&\langle \mathrm{N}| \left[J_\mu\left(\frac{x}{2}\right), J_\nu\left(-\frac{x}{2}\right)\right] |\mathrm{N}\rangle \\
&\quad = \varepsilon(x^0)\left(\frac{x^2 g_{\mu\nu} - 2x_\mu x_\nu}{\pi^4(x^2-\mathrm{i}\varepsilon)^4} - \frac{x^2 g_{\mu\nu} - 2x_\mu x_\nu}{\pi^4(x^2+\mathrm{i}\varepsilon)^4}\right) \\
&\qquad - \varepsilon(x^0)\left(\frac{\mathrm{i}x^\alpha}{2\pi^2(x^2-\mathrm{i}\varepsilon)^2} - \frac{\mathrm{i}x^\alpha}{2\pi^2(x^2+\mathrm{i}\varepsilon)^2}\right) s_{\mu\alpha\nu\beta} \\
&\qquad \times \langle \mathrm{N}| : \hat{\bar\Psi}\left(\frac{x}{2}\right)\gamma^\beta\hat\Psi\left(-\frac{x}{2}\right) - \hat{\bar\Psi}\left(-\frac{x}{2}\right)\gamma^\beta\hat\Psi\left(\frac{x}{2}\right) : |\mathrm{N}\rangle \\
&\qquad - \varepsilon(x^0)\left(\frac{x^\alpha}{2\pi^2(x^2-\mathrm{i}\varepsilon)^2} - \frac{x^\alpha}{2\pi^2(x^2+\mathrm{i}\varepsilon)^2}\right) \varepsilon_{\mu\alpha\nu\beta} \\
&\qquad \times \langle \mathrm{N}| : \hat{\bar\Psi}\left(\frac{x}{2}\right)\gamma^\beta\gamma_5\hat\Psi\left(-\frac{x}{2}\right) + \hat{\bar\Psi}\left(-\frac{x}{2}\right)\gamma^\beta\gamma_5\hat\Psi\left(\frac{x}{2}\right) : |\mathrm{N}\rangle \\
&\qquad + \varepsilon(x^0)\Big[\langle \mathrm{N}| : \hat{\bar\Psi}\left(\frac{x}{2}\right)\gamma_\mu\hat\Psi\left(\frac{x}{2}\right)\hat{\bar\Psi}\left(-\frac{x}{2}\right)\gamma_\nu\hat\Psi\left(-\frac{x}{2}\right) : |\mathrm{N}\rangle \\
&\qquad - \langle \mathrm{N}| : \hat{\bar\Psi}\left(-\frac{x}{2}\right)\gamma_\nu\hat\Psi\left(-\frac{x}{2}\right)\hat{\bar\Psi}\left(\frac{x}{2}\right)\gamma_\mu\hat\Psi\left(\frac{x}{2}\right) : |\mathrm{N}\rangle\Big] \quad . \qquad (5.219)
\end{aligned}
$$

Here we have used

$$
\begin{aligned}
&\langle \mathrm{N}\, |: \hat{\bar\Psi}\left(\frac{x}{2}\right)\gamma^\beta\hat\Psi\left(-\frac{x}{2}\right) - \hat{\bar\Psi}\left(-\frac{x}{2}\right)\gamma^\beta\hat\Psi\left(\frac{x}{2}\right) :|\, \mathrm{N}\rangle^\dagger \\
&\quad = \langle \mathrm{N}\, |: \hat{\bar\Psi}\left(-\frac{x}{2}\right)\gamma^\beta\hat\Psi\left(\frac{x}{2}\right) - \hat{\bar\Psi}\left(\frac{x}{2}\right)\gamma^\beta\hat\Psi\left(-\frac{x}{2}\right) :|\, \mathrm{N}\rangle \\
&\quad = -\langle \mathrm{N}\, |: \hat{\bar\Psi}\left(\frac{x}{2}\right)\gamma^\beta\hat\Psi\left(-\frac{x}{2}\right) - \hat{\bar\Psi}\left(-\frac{x}{2}\right)\gamma^\beta\hat\Psi\left(\frac{x}{2}\right) :|\, \mathrm{N}\rangle \quad . \qquad (5.220)
\end{aligned}
$$

Employing the δ distributions we can write the divergent functions in a more compact way:

$$\frac{1}{x^2 \pm \mathrm{i}\varepsilon} \equiv P\left(\frac{1}{x^2}\right) \mp \mathrm{i}\pi\delta(x^2) \quad , \qquad (5.221)$$

$$\delta(x^2) = \frac{\mathrm{i}}{2\pi}\left(\frac{1}{x^2+\mathrm{i}\varepsilon} - \frac{1}{x^2-\mathrm{i}\varepsilon}\right) \quad , \qquad (5.222)$$

$$\frac{\partial}{\partial x_\alpha}\delta(x^2) = \frac{-\mathrm{i}}{2\pi}\left(\frac{x^\alpha}{(x^2+\mathrm{i}\varepsilon)^2} - \frac{x^\alpha}{(x^2-\mathrm{i}\varepsilon)^2}\right) \quad , \qquad (5.223)$$

$$\frac{\partial^3}{\partial(x^2)^3}\delta(x^2) = -\frac{3\mathrm{i}}{\pi}\left(\frac{1}{(x^2+\mathrm{i}\varepsilon)^4} - \frac{1}{(x^2-\mathrm{i}\varepsilon)^4}\right) \quad . \qquad (5.224)$$

Finally we obtain

$$
\begin{aligned}
&\langle \mathrm{N}\, | \left[J_\mu\left(\frac{x}{2}\right), J_\nu\left(-\frac{x}{2}\right)\right]_- |\, \mathrm{N}\rangle \\
&\quad = \varepsilon(x_0)\frac{\mathrm{i}}{3\pi^3}\left(2x_\mu x_\nu - x^2 g_{\mu\nu}\right)\frac{\partial^3}{\partial(x^2)^3}\delta(x^2) \\
&\qquad - \varepsilon(x_0)\frac{\mathrm{i}}{\pi}\left[\frac{\partial}{\partial x_\alpha}\delta(x^2)\right] s_{\mu\alpha\nu\beta} \\
&\qquad \times \langle \mathrm{N}\, |: \hat{\bar\Psi}\left(\frac{x}{2}\right)\gamma^\beta\hat\Psi\left(-\frac{x}{2}\right) - \hat{\bar\Psi}\left(-\frac{x}{2}\right)\gamma^\beta\hat\Psi\left(\frac{x}{2}\right) :|\, \mathrm{N}\rangle \\
&\qquad + \varepsilon(x_0)\frac{\mathrm{i}}{\pi}\left[\frac{\partial}{\partial x_\alpha}\delta(x^2)\right] \varepsilon_{\mu\alpha\nu\beta}
\end{aligned}
$$

$$\begin{aligned}
&\times \langle \mathrm{N} |: \hat{\bar{\Psi}}\left(\frac{x}{2}\right) \gamma^{\beta}\gamma_5 \hat{\Psi}\left(-\frac{x}{2}\right) + \hat{\bar{\Psi}}\left(-\frac{x}{2}\right) \gamma^{\beta}\gamma_5 \hat{\Psi}\left(\frac{x}{2}\right) :| \mathrm{N} \rangle \\
&+ \varepsilon(x_0) \Big[\langle \mathrm{N} |: \hat{\bar{\Psi}}\left(\frac{x}{2}\right) \gamma_{\mu} \hat{\Psi}\left(\frac{x}{2}\right) \hat{\bar{\Psi}}\left(-\frac{x}{2}\right) \gamma_{\nu} \hat{\Psi}\left(-\frac{x}{2}\right) :| \mathrm{N} \rangle \\
&- \langle \mathrm{N} |: \hat{\bar{\Psi}}\left(\frac{-x}{2}\right) \gamma_{\nu} \hat{\Psi}\left(-\frac{x}{2}\right) \hat{\bar{\Psi}}\left(\frac{x}{2}\right) \gamma_{-\mu} \hat{\Psi}\left(\frac{x}{2}\right) :| \mathrm{N} \rangle \Big] \quad . \qquad (5.225)
\end{aligned}$$

Equation (5.225) very clearly shows how the expectation value of the current commutator splits up into terms each of which can be written as a product between a singular function and a finite matrix element. In the case of an interacting theory an infinite number of additional terms occurs. They are formally obtained by inserting $\exp\left(-\mathrm{i}\int \hat{H}^{\mathrm{int}}(t)\mathrm{d}t\right)$ into the expectation value and expanding the exponential function. Doing this yields matrix elements of the form

$$\langle \mathrm{N} |: \hat{\bar{\Psi}}\left(\frac{x}{2}\right) \left[A^a_{\mu}\left(\frac{x}{2}\right) - A^a_{\mu}\left(-\frac{x}{2}\right) \right] \frac{\lambda^a}{2} \hat{\Psi}\left(-\frac{x}{2}\right) :| \mathrm{N} \rangle \quad . \qquad (5.226)$$

To repeat: the matrix elements contain the nonperturbative features of the nucleons, for example, and they are parametrized, for example, by the structure functions. To characterize a nucleon state completely, one has to take an infinite number of matrix elements into account. Practical applications are restricted to the investigation of the dominant contributions. The second and third coefficient functions in (5.226) are divergent while the last one is finite. The corresponding matrix element for four field operators is therefore phenomenologically of no significance. We shall encounter a continuation and modification of this analysis in Sect. 7.2 in the context of QCD sum rules.

5.5 The Spin-Dependent Structure Functions

In recent years it has become possible to measure, in addition to the structure functions $F_1(x, Q^2)$ and $F_2(x, Q^2)$, the so-called spin-dependent structure functions $g_1(x, Q^2)$ and $g_2(x, Q^2)$. The spin-dependent structure functions only play a role when the scattered leptons and hadrons are polarized. The origin of these additional structure functions is quite obvious. For a polarized hadron, e.g., a proton, there is an additional Lorentz vector available, namely the spin vector s^{μ}. Accordingly (3.6) has to be extended by some additional terms. Repeating the analysis of Chap. 3 with the additional spin vector we are led to the following form of the scattering tensor:

$$\begin{aligned}
W_{\mu\nu} = &\left(-g_{\mu\nu} + \frac{q_{\mu}q_{\nu}}{q^2}\right) W_1(x, Q^2) \\
&+ \left(P_{\mu} - q_{\mu}\frac{q\cdot P}{q^2}\right)\left(P_{\nu} - q_{\nu}\frac{q\cdot P}{q^2}\right)\frac{W_2(x, Q^2)}{M^2} \\
&+ \mathrm{i}\varepsilon_{\mu\nu\lambda\sigma} q^{\lambda} s^{\sigma} M G_1(x, Q^2) \\
&+ \mathrm{i}\varepsilon_{\mu\nu\lambda\sigma} q^{\lambda} (s^{\sigma} p\cdot P - P^{\sigma} q\cdot s)\frac{G_2(x, Q^2)}{M} \quad . \qquad (5.227)
\end{aligned}$$

This equation is obviously an extension of (3.18). The factor i guarantees that the transition current is real (the relation $\Gamma^*_{\mu}(p \leftrightarrow p') = \Gamma_{\mu}$ must hold, see Exercise 3.3), while the epsilon tensor ensures that $q^{\mu}W_{\mu\nu} = q^{\nu}W_{\mu\nu} = 0$ holds.

Since the additional contributions change their sign when the hadron spin direction is reversed, they cancel upon spin-averaging. Thus only spin-independent structure functions can be measured with an unpolarized target. Analogously to the unpolarized case, dimensionless functions are introduced according to

$$MG_1(x,Q^2) = \frac{g_1(x,Q^2)}{P\cdot q} \quad , \quad \frac{G_2(x,Q^2)}{M} = \frac{g_2(x,Q^2)}{(P\cdot q)^2} \quad . \tag{5.228}$$

This choice is motivated by the fact that g_1 has a simple interpretation in terms of parton distributions and $g_2(x)$ thus occurs on the same footing as $g_1(x)$. The additional spin-dependent contributions to the hadron scattering tensor are antisymmetric; thus they cannot contribute when contracted with the symmetric, unpolarized lepton tensor of (3.20). In order to measure polarized structure functions in deep inelastic scattering, the lepton must also be polarized. The resulting leptonic scattering tensor is

$$\begin{aligned} L_{\mu\nu} &= \frac{1}{4}\mathrm{tr}[(\not{p}+m)(1-\not{s}\gamma_5)\gamma_\mu(\not{p}'+m)\gamma_\nu] \\ &= \frac{1}{4}[p_\mu p'_\nu + p_\nu p'_\mu - g_{\mu\nu}(p\cdot p' - m^2)] \\ &\quad - \frac{m}{4}\mathrm{tr}(\not{p}\not{s}\gamma_5\gamma_\mu\gamma_\nu) - \frac{m}{4}\mathrm{tr}(\not{s}\gamma_5\gamma_\mu\not{p}'\gamma_\nu) \\ &= \frac{1}{4}[p_\mu p'_\nu + p_\nu p'_\mu - g_{\mu\nu}(p\cdot p' - m^2)] + \mathrm{i}m_{\mu\nu\alpha\beta}q^\alpha s_\mathrm{e}^\beta \quad . \end{aligned} \tag{5.229}$$

Thus the polarized lepton tensor also contains an antisymmetric part. Contracting both, we obtain the additional term

$$\begin{aligned} L_{\mu\nu}W^{\mu\nu} &= \ldots - 2mM(g_{\lambda\alpha}g_{\sigma\beta} - g_{\lambda\beta}g_{\sigma\alpha})q^\lambda s_p^\sigma q^\alpha s_\mathrm{e}^\beta G_1 \\ &\quad - \frac{m}{M}(g_{\lambda\alpha}g_{\sigma\beta} - g_{\lambda\beta}g_{\sigma\alpha})q^\lambda(q\cdot P s_p^\sigma - q\cdot s_p P^\sigma)q^\alpha s_\mathrm{e}^\beta G_2 \; . \end{aligned} \tag{5.230}$$

This poses the question of, what choice of the polarization of electron and hadron, i.e., nucleon, is most suitable. The relevant vectors in the center-of-momentum frame are

$$\begin{aligned} p_\mu = (E_\mathrm{e},0,0,p_\mathrm{e}) \quad &, \quad P_\mu = (E_\mathrm{p},0,0,-P_\mathrm{p}) \\ s_\mathrm{e}^\mu(\text{longitudinal}) = \pm(\mathrm{p_e},0,0,\mathrm{E_e}) \quad &, \quad s_\mathrm{p}^\mu(\text{longitudinal}) = \pm(\mathrm{p_p},0,0,-\mathrm{E_p}) \quad , \\ s_\mathrm{e}^\mu(\text{transverse}) = \pm(0,1,0,0) \quad &\text{and} \quad \pm(0,0,1,0) \quad , \\ s_\mathrm{p}^\mu(\text{transverse}) = \pm(0,1,0,0) \quad &\text{and} \quad \pm(0,0,1,0) \quad . \end{aligned} \tag{5.231}$$

We recognize that the following products are large:

$$s_\mathrm{e}(l)\cdot s_\mathrm{p}(l) \approx \pm 2\frac{E_\mathrm{e}E_\mathrm{p}}{mM} \quad , \quad s_\mathrm{e}(l)\cdot P \approx \pm 2\frac{E_\mathrm{e}E_\mathrm{p}}{m} \quad . \tag{5.232}$$

All the other scalar products in (5.230) are considerably smaller. Thus to measure spin-dependent structure functions, longitudinally polarized leptons must be scattered off longitudinally or transversly polarized nucleons. This property can also be seen from projecting on physical degrees of freedom of the photon instead of using the photon propagator in Feynman gauge. In this case, the hadronic scattering tensor is contracted with $\varepsilon_\mu^*\varepsilon_\nu$.

Here ε is the polarization vector of the virtual photon. In order for spin-dependent terms to be able to contribute, this expression must contain an antisymmetric part. As can easily be checked, this is the case only for longitudinal polarization vectors, e.g., it holds for

$$\varepsilon_\mu = (0,1,\mathrm{i},0)/\sqrt{2} \quad \text{that} \quad \varepsilon_1^*\varepsilon_2 = -\varepsilon_2^*\varepsilon_1 \quad . \tag{5.233}$$

A longitudinally polarized photon is most readily emitted by a longitudinally polarized lepton. For other lepton polarizations its coupling is suppressed by exactly the kinematic factors of (5.230).

In the experiments performed up to now, a longitudinally polarized proton target has been used, and the asymmetry

$$A = \frac{\left[\frac{d^2\sigma}{dE'\,d\Omega}(\uparrow\uparrow) - \frac{d^2\sigma}{dE'\,d\Omega}(\uparrow\downarrow)\right]}{\left[\frac{d^2\sigma}{dE'\,d\Omega}(\uparrow\uparrow) + \frac{d^2\sigma}{dE'\,d\Omega}(\uparrow\downarrow)\right]} \tag{5.234}$$

has been measured.

Inserting (5.236) one finds for longitudinal polarization

$$A_1 = D\left\{\frac{g_1(x) - \frac{2Mx}{Ey}g_2(x)}{F_1(x)} + \eta\sqrt{\frac{2Mx}{Ey}}\,\frac{g_1(x)+g_2(x)}{F_1(x)}\right\} \tag{5.235}$$

with

$$\begin{aligned} D &= \frac{y(2-y)}{y^2+2(1-y)(1+R)} \quad , \\ y &= \frac{E-E'}{E} \quad , \quad R = \frac{\sigma_{\gamma^* L}}{\sigma_{\gamma^* T}} \quad , \\ \eta &= \frac{2\gamma(1-y)}{2-y} \quad , \quad \text{and} \quad \gamma = \sqrt{\frac{Q^2M^2}{\nu^2}} = \sqrt{\frac{2Mx}{Ey}} \quad . \end{aligned} \tag{5.236}$$

For transverse proton polarization the resulting asymmetry reads

$$\begin{aligned} A_2 &= \frac{\left[\frac{d^2\sigma}{dE'\,d\Omega}(\uparrow\rightarrow) - \frac{d^2\sigma}{dE'\,d\Omega}(\downarrow\rightarrow)\right]}{\left[\frac{d^2\sigma}{dE'\,d\Omega}(\uparrow\rightarrow) + \frac{d^2\sigma}{dE'\,d\Omega}(\downarrow\rightarrow)\right]} \\ &= D\sqrt{\frac{2\varepsilon}{1+\varepsilon}}\left(\sqrt{\frac{2Mx}{Ey}}\,\frac{g_1(x)+g_2(x)}{F_1(x)} - \eta\frac{1+\varepsilon}{2\varepsilon}\,\frac{g_1(x)-\frac{2Mx}{Ey}g_2(x)}{F_1(x)}\right) \\ &\longrightarrow \sqrt{\frac{2Mx}{E-E'}}\,\frac{g_1(x)+g_2(x)}{F_1(x)} \qquad \text{for } Q^2 \to \infty,\ y \to 1 \end{aligned} \tag{5.237}$$

with $\varepsilon = (1-y)/(1-y+y^2/2)$.

Since the cross sections for transverse and longitudinal polarization should be of approximately the same size, we deduce that $g_2(x)$ is certainly not larger than $g_1(x)$. Therefore the contribution of $g_2(x)$ at longitudinal polarization is suppressed by a factor Mx/E and constitutes only a small correction. The asymmetry A_2 therefore measures mainly the ratio of $g_1(x)$ to the unpolarized structure function $F_1(x)$.

Next, there is the question of whether $g_1(x)$ can be given a practical meaning like $F_1(x)$? This is indeed the case. To see this, it is sufficient to compare the hadronic and the leptonic scattering tensor of (5.227) and (5.229).

Obviously each Dirac particle gives a contribution with the same form as that proportional to G_1 multiplied by the probability to find a quark with the fitting momentum fraction x and the right polarization. Accordingly, one obtains in analogy with the parton interpretation of F_1,

$$F_1(x) = \frac{1}{2} \sum_{\mathrm{q=u,d,\bar{u},\bar{d}}\ldots} q(x) Q_{\mathrm{q}}^2 \quad , \tag{5.238}$$

where $q(x)$ is the probability of finding a quark q with momentum fraction x, a similar expression for $g_1(x)$

$$g_1(x) = \frac{1}{2} \sum_{\mathrm{q=u,d,\bar{u},\bar{d}}\ldots} Q_{\mathrm{q}}^2 \left[q^{\uparrow}(x) - q^{\downarrow}(x) \right] \quad . \tag{5.239}$$

Here $q^{\uparrow}(x)$ indicates the probability of finding a quark with a momentum fraction x polarized in the same direction as the whole proton. The asymmetry measured is a measure of the distribution of the proton spin among its constituents.

One of the most important properties of unpolarized structure functions is that the momentum fraction deduced from them, which is carried by the quarks, accounts for only half of the total momentum. This is a striking proof for the existence of gluons. Analogously, from polarized structure functions we can ask how much of the proton spin is carried by quarks, how much by gluons and how much is present in angular momentum. This question has, up to now, not been uniquely answered. The present experimental data have caused strong theoretical discussions and led to the design of much improved and completely new experiments. In the near future, experiments of the type

$$\begin{aligned}
&\mathrm{e}^{\uparrow}(\text{long.}) + \mathrm{p}^{\uparrow}, {}^3\mathrm{He}^{\uparrow}, \mathrm{d}^{\uparrow}(\text{long. or trans.}) \to \mathrm{e}' + X \\
&\mathrm{e}^{\uparrow}(\text{long.}) + \mathrm{p}^{\uparrow}, {}^3\mathrm{He}^{\uparrow}, \mathrm{d}^{\uparrow}(\text{long. or trans.}) \to \mathrm{e}' + \pi^{\pm} + X \\
&\mathrm{p} + \mathrm{p}^{\uparrow}(\text{trans.}) \to \gamma + X \quad \text{plus many more channels}
\end{aligned} \tag{5.240}$$

should be performed. In addition polarized proton-proton collisions are a possibility for the future.

This field is extremely active right now and a more detailed discussion of the current situation is not suitable for a textbook. Instead, we consider two particular aspects where crucial concepts of QCD can be exemplified. As it turned out, nearly all techniques of QCD are, in a nearly singular manner, important in analyzing spin structure.

In particular, we consider the following aspects of the discussion:

1. The Bjorken- and Ellis-Jaffe-sum rule: From the knowledge of the axial vector coupling in weak interactions, predictions based on isospin and SU(3) flavor symmetry for $\int dx (g_1^{\mathrm{p}}(x) - g_1^{\mathrm{n}}(x))$ and $\int dx\, g_1^{\mathrm{p}}(x)$ can be obtained. This is a nice example that flavor symmetry continues to play an important role even after the introduction of color SU(3).

2. The axial anomaly and the gluonic contribution to $g_1(x)$: The axial anomaly plays a major role for hadronic physics in general. We have met it in section 4.2 when reviewing the foundations of QCD and will show in Exercise 7.2 its importance in understanding lattice QCD. As it turns out, the anomaly can contribute to the spin structure function in a subtle manner. The analysis of this effect gives exquisite insight into the physical meaning of the anomaly.

Let us start by discussing the Bjorken sum rule. The antisymmetric (spin-dependent) part of the hadronic scattering tensor can be written as an axial-vector forward matrix element of the proton (see Fig. 5.15). As the proton couples to quarks we find that the spin-asymmetric part of the cross section $\Delta\sigma$

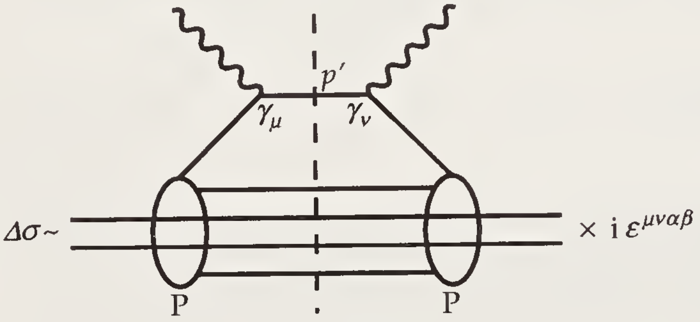


Fig. 5.15. The relation between deep inelastic scattering and the forward matrix element

is proportional to

$$\begin{aligned}
\mathrm{i}\gamma_\mu \not{p}' \gamma_\nu\, \varepsilon^{\mu\nu\alpha\beta} &= p'^\sigma \left(g_{\mu\sigma}\gamma_\nu + g_{\sigma\nu}\gamma_\mu - g_{\mu\nu}\gamma_\sigma + \mathrm{i}\varepsilon_{\mu\sigma\nu\lambda}\gamma^\lambda\gamma_5\right) \mathrm{i}\,\varepsilon^{\mu\nu\alpha\beta} \\
&= p'^\sigma\, \varepsilon_{\mu\nu\sigma\lambda}\, \varepsilon^{\mu\nu\alpha\beta}\, \gamma^\lambda \gamma_5 \\
&= -2p'^\sigma \left(g_\sigma{}^\alpha g_\lambda{}^\beta - g_\sigma{}^\beta g_\lambda{}^\alpha\right) \gamma^\lambda \gamma_5 \\
&= -2p'^\alpha \gamma^\beta \gamma_5 + 2p'^\beta \gamma^\alpha \gamma_5 \quad ,
\end{aligned} \tag{5.241}$$

where we have used the well-known decomposition of the product of three gamma matrices. Thus we find that

$$\Delta\sigma \sim \sum_q \langle \mathrm{p}\,s \,|\, Q_q^2 \bar{q}\, \gamma^\alpha \gamma_5\, q \,|\, \mathrm{p}\,s \rangle \quad . \tag{5.242}$$

It is important to note that, owing to the optical theorem, we have obtained a forward cross section, i.e., the momentum transfer is zero. In contrast, the momentum transfer in the lepton–hadron scattering reaction q_μ is very large, but squaring this diagram to obtain the cross section leads to a graph in which the second photon removes the momentum transferred by the first. (This is just a description of the content of the optical theorem.) Thus this forward matrix element can be related by isospin symmetry to a corresponding matrix element between neutron and proton. More precisely we write

$$Q^2 = \frac{15}{18} + \frac{1}{3}\tau_3 = \begin{cases} \frac{4}{9} & \text{for the up quark} \\ \frac{1}{9} & \text{for the down quark} \end{cases} \quad , \tag{5.243}$$

implying that

$$\Delta\sigma^{\mathrm{p}} - \Delta\sigma^{\mathrm{n}} \sim \frac{1}{3}\langle \mathrm{N}\, s\, | \tau_3\, \bar{q}\, \gamma^\alpha \gamma_5\, q\, | \mathrm{N}\, s \rangle \quad , \tag{5.244}$$

where $\mathrm{N} = \binom{\mathrm{p}}{\mathrm{n}}$ is the usual nucleon doublet. Now we can introduce $\tau_+\tau_- + \tau_-\tau_+ = \mathbb{1}$; we obtain

$$\begin{aligned} \Delta\sigma^{\mathrm{p}} - \Delta\sigma^{\mathrm{n}} &\sim \frac{1}{3}\left\langle \mathrm{N}\, s \left| (\tau_+\tau_- + \tau_-\tau_+)\, \tau_3\, \bar{q}\, \gamma^\alpha \gamma_5\, q \right| \mathrm{N}\, s \right\rangle \\ &= \frac{1}{3}\left\langle \mathrm{N}\, s \left| (\tau_+\tau_- - \tau_-\tau_+) \bar{q}\, \gamma^\alpha \gamma_5\, q \right| \mathrm{N}\, s \right\rangle \\ &= \frac{1}{3}\langle \mathrm{p}\, s\, | \tau_+\, \bar{q}\, \gamma^\alpha \gamma_5\, q\, | \mathrm{n}\, s \rangle + \mathrm{h.c.}\,. \end{aligned} \tag{5.245}$$

This, however, is just the weak-interaction matrix element of neutron beta decay. The corresponding coupling is just g_A/g_V. Putting all the constants together, we end up with

$$\int_0^1 \mathrm{d}x\, \left[g_1^{\mathrm{p}}(x) - g_1^{\mathrm{n}}(x)\right] = \frac{1}{6}\frac{g_A}{g_V} \quad . \tag{5.246}$$

This is the famous Bjorken sum rule, which allows us to connect the proton and the neutron results. It is strictly valid for $Q^2 \to \infty$. Various perturbative and higher-twist corrections have been calculated:

$$\begin{aligned} &\int_0^1 \mathrm{d}x\, \left[g_1^{\mathrm{p}}(x,Q^2) - g_1^{\mathrm{n}}(x,Q^2)\right] \\ &\quad = \frac{1}{6}\frac{g_A}{g_V}\left[1 - \frac{\alpha_s}{\pi} - \frac{43}{12}\left(\frac{\alpha_s}{\pi}\right)^2\right] \\ &\quad + \frac{M^2}{Q^2}\int_0^1 \mathrm{d}x\, x^2 \left[\frac{2}{9}(g_1^{\mathrm{p}} - g_1^{\mathrm{n}})(x,Q^2) + \frac{4}{3}(g_2^{\mathrm{p}} - g_2^{\mathrm{n}})(x,Q^2)\right] \\ &\quad - \frac{4}{27}\frac{1}{Q^2}(f_{2\mathrm{u}} - f_{2\mathrm{d}}) \end{aligned} \tag{5.247}$$

with

$$2f_{2q}s^\sigma = \langle \mathrm{p}\, s\, | g\, \bar{q}\, \tilde{G}^{\sigma\lambda} \gamma_\lambda\, q\, | \mathrm{p}\, s \rangle \quad , \quad q = u, d \quad . \tag{5.248}$$

The f_{2q} can be calculated, for example, by QCD sum rule techniques and turn out to be small. Also the mass correction is small, basically because of the factor x^2 which suppresses the small x contributions. Thus the Q^2 dependence of the Bjorken sum rule is exceptionally weak and well under control. It is especially noteworthy that no anomalous dimension, i.e., no factor of the form $[\alpha(Q^2)\,/\,\alpha(Q_0^2)]^{-d/2b}$ occurs on the right-hand side of (5.247). This fact has a simple physical reason. The main part of the Q^2 evolution of unpolarized structure functions is due to

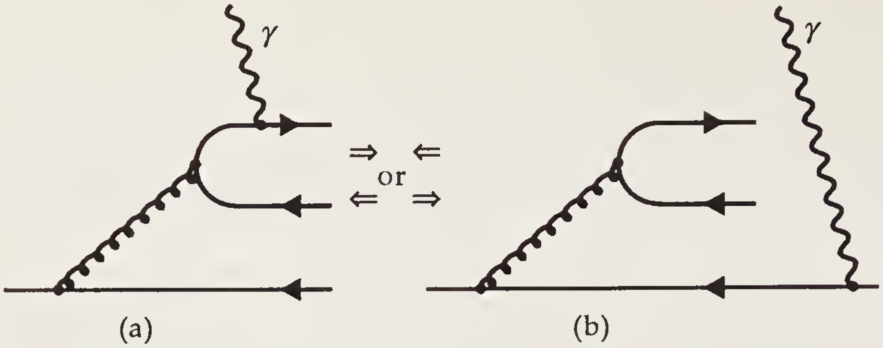


Fig. 5.16a,b. Graphs contributing to the Q^2 dependence of the zeroth moment of the unpolarized structure functions. For the spin-dependent structure functions only **(b)** contributes, since the quark–antiquark pairs produced are unpolarized

the many quark–antiquark pairs that contribute at high Q^2 (see Fig. 5.16). Because the vector coupling to gluons conserves helicity, these quark pairs are, however, predominantly unpolarized. In fact a double spin flip is needed to obtain a polarized q$\bar{\text{q}}$ pair, and except for very small values of x this is completely suppressed.

Consequently the zeroth moment of g_1^p, g_1^n, g_2^p, g_2^n should only show a very mild Q^2 dependence. (Actually there is good reason to believe that $\int g_2(x)\,\text{d}x \equiv 0$.) Now, assuming that the strange quarks are unpolarized, which is actually a very controversal assumption, we can obtain the Ellis–Jaffe sum rule from (5.247). From the definition of $g_1(x)$ we have, assuming strict isospin symmetry at the quark level, which is also controversial,

$$\begin{aligned} \int_0^1 \text{d}x\; g_1^\text{p}(x) &= \frac{1}{2}\left(\frac{4}{9}\Delta u + \frac{1}{9}\Delta d\right) \quad , \\ \int_0^1 \text{d}x\; g_1^\text{n}(x) &= \frac{1}{2}\left(\frac{4}{9}\Delta d + \frac{1}{9}\Delta u\right) \quad , \end{aligned} \tag{5.249}$$

where Δu, Δd are the fraction of the proton spin carried by the u and d quark.

The Bjorken sum rule implies that

$$\begin{gathered} \int_0^1 \text{d}x\; g_1^\text{p}(x) - \int_0^1 \text{d}x\; g_1^\text{n}(x) = \frac{1}{6}(\Delta u - \Delta d) = \frac{1}{6}\frac{g_A}{g_V} \quad , \\ \Delta u = \Delta d + \frac{g_A}{g_V} \quad , \end{gathered} \tag{5.250}$$

which is already one constraint. Still another constraint comes from the coupling of s and u quarks in hyperon decays. To understand this we have to review a little group theory. The general axial vectorial flavor SU(3) matrix element $\langle B|S^5_{j\sigma}|B'\rangle$ can be analyzed with the equivalent of the Wigner–Eckhardt theorem. Here B, B' are baryon states from the flavor octet, σ is the Lorentz index, and $j = 1, \ldots, 8$ is the index of the SU(3) generator. We find that

$$
\begin{aligned}
\langle \mathrm{p}|S^5_{j\sigma}|\mathrm{p}\rangle = c\cdot n_\sigma \Bigg[& \left(\begin{array}{cc|c} 8 & 8 & 8_1 \\ I=\frac{1}{2}\;\; I_3=-\frac{1}{2} & I=\frac{1}{2}\;\; I_3=\frac{1}{2} & I_j\;\; -I_{3j} \\ Y=-1 & Y=1 & -Y_j \end{array}\right) A_1 \\
& + \left(\begin{array}{cc|c} 8 & 8 & 8_2 \\ I=\frac{1}{2}\;\; I_3=-\frac{1}{2} & I=\frac{1}{2}\;\; I_3=\frac{1}{2} & I_j\;\; -I_{3j} \\ Y=-1 & Y=1 & -Y_j \end{array}\right) A_2 \Bigg] \\
& \times \left(\begin{array}{cc|c} 8 & 8 & \\ I_j\;\; -I_{3j} & I_j\;\; I_{3j} & 1 \\ -Y_{-j} & Y_j & 0 \end{array}\right) \quad .
\end{aligned}
\tag{5.251}
$$

The isoscalar factors are listed.[3] A peculiarity of the SU(3) group is the appearance of two unequivalent octet representations. This is related to the fact that three octets can be coupled either by the symmetric d_{abc} or the antisymmetric f_{abc} structure constants. It leads to the definition of two constants, D and F:

$$
\langle N|S^5_{3\sigma}|N\rangle = c\cdot n_\sigma \left(\frac{\sqrt{30}}{40}A_1 + \frac{\sqrt{6}}{24}A_2\right) =: c\cdot n_\sigma (D+F) \quad ,
$$

$$
D = \frac{\sqrt{30}}{40}A_1 \quad , \quad F = \frac{\sqrt{6}}{24}A_2 \quad . \tag{5.252}
$$

Every axial-vector matrix element coupling two baryon states from the octet is thus proportional to a specific combination of D and F and can therefore be used to measure F/D. However, the combined analysis of all the hyperon decays described by different groups gave conflicting results. A conservative estimate is $F/D = 0.55$–0.60. From (5.252) we read off

$$
\Delta u - \Delta d = F + D \quad . \tag{5.253}
$$

Similarly by inserting the SU(3) isoscalar factors we get

$$
\langle N|S^5_{8\sigma}|N\rangle = c{\cdot}n_\sigma \left(-\frac{1}{4\sqrt{10}}A_1 + \frac{\sqrt{2}}{8}A_2\right) = c{\cdot}n_\sigma\left(-\frac{1}{\sqrt{3}}D + \sqrt{3}F\right) \tag{5.254}
$$

$$
\Rightarrow \frac{1}{\sqrt{3}}(\Delta u + \Delta d - 2\Delta s) = \sqrt{3}F - \frac{1}{\sqrt{3}}D \quad . \tag{5.255}
$$

Assuming again that $\Delta s = 0$ we obtain another independent combination,

$$
\frac{\Delta u + \Delta d}{\Delta u - \Delta d} = \frac{3F - D}{F + D} = \frac{3F/D - 1}{F/D + 1} \quad , \tag{5.256}
$$

which together with (5.251) allows us to determine $\int\limits_0^1 g_1^{\mathrm{p}}(x)\,\mathrm{d}x$. A complete analysis along these lines gives

---

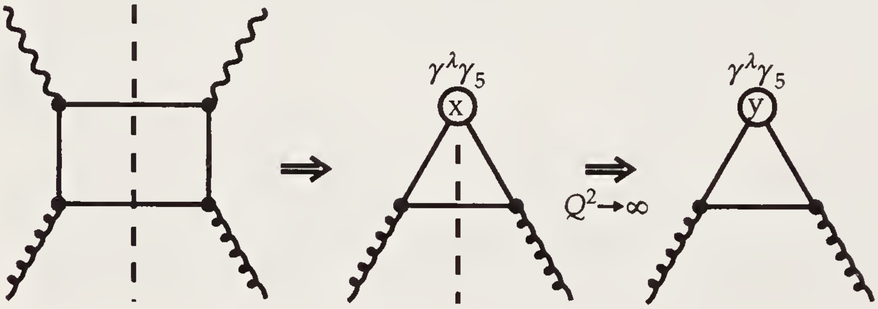
[3] See J. J. de Swart: Rev. Mod. Phys. **35**, 916 (1963).

$$\begin{aligned}\int\limits_0^1 \mathrm{d}x\, g_1^{\mathrm{p}}(x,Q^2) = \frac{g_A}{g_V}&\left[\frac{1}{18}\,\frac{3F/D+1}{F/D+1}\left(1-\frac{\alpha_s}{\pi}\right)+\frac{1}{9}\,\frac{3F/D-1}{F/D+1}\left(1-\frac{\alpha_s}{\pi}\right)\right]\\ &+\frac{M^2}{Q^2}\int\limits_0^1 \mathrm{d}x\, x^2\left[\frac{2}{9}\,g_1^{\mathrm{p}}(x,Q^2)+\frac{4}{3}\,g_2^{\mathrm{p}}(x,Q^2)\right]\\ &-\frac{4}{9}\,\frac{1}{Q^2}\left(\frac{4}{9}f_{2\mathrm{u}}+\frac{1}{9}f_{2\mathrm{d}}+\frac{1}{9}f_{2\mathrm{s}}\right)\\ &\approx 0.172\pm 0.009 \quad \text{for } Q^2 = 5\ \mathrm{GeV}^2 \quad . \end{aligned} \tag{5.257}$$

This is the Ellis–Jaffe sum rule. As mentioned several times already, its validity is a matter of dispute. We have presented it here to illustrate the potential usefulness of flavor SU(3) for the analysis of deep inelastic scattering cross sections.

Next let us discuss the role of the anomaly for the isosinglet axial-vector current. To illustrate the point consider first the perturbative graph in Fig. 5.17.

Fig. 5.17. The perturbative anomalous gluon contribution to g_1



It is obvious that for $Q^2 \to \infty$ one indeed obtains the triangle anomaly, implying pointlike photon–gluon coupling. The basic problem of this interpretation is also obvious from Fig. 5.17, namely that there is no unambiguous way to separate the quark contribution and the anomalous gluonic contribution. It is unclear whether the quarks of the fermion line should be absorbed into Δu, Δd, etc. or into a contribution of the structure

$$\Delta g_1^{\mathrm{p}}(x) = \left\langle \frac{Q_q^2}{2} \right\rangle \int\limits_0^1 \frac{\mathrm{d}z}{z}\, A\left(\frac{x}{z}\right)\, \Delta G(z) \quad . \tag{5.258}$$

In other words, the Δq's appearing in the analysis we have presented so far should be split up according to

$$\Delta q = \Delta\tilde{q} - \frac{\alpha_s}{2\pi}\Delta G \cdot c \quad . \tag{5.259}$$

It turns out that, according to variety of reasoning such a separation is very problematic.

1. Performing the perturbative calculation for the graph in Fig. 5.17 we get constants c that depend critically on the chosen infrared regulators indicating that we are not looking at a infrared save quantity.

2. From the operator-product-expansion point of view,we find that there is no gauge-independent local definition of ΔG.
3. A so-called large gauge transformation, i.e., a gauge transformation with a nonzero topological quantum number, shifts contributions from $\Delta\tilde{q}$ to ΔG. A detailed discussion of this rather complicated issue is beyond the framework of this book.

On the other hand, there exists a practical argument for the decomposition (5.259). If we construct a phenomenological model for Δq, we will most probably miss the highly virtual quark components from Fig. 5.17. Therefore it might be easier to model $\Delta\tilde{q}$ and ΔG instead.

We do not want to discuss these still very much disputed questions further. Instead we shall show in the next chapter how one can calculate g_1 and g_2 from the MIT bag model.

5.6 The Spin-Dependent Structure Functions in the MIT Bag Model

To illustrate the properties of structure functions and the working of GLAP evolution we present in this section a very specific example, namely the spin-dependent structure functions as they are calculated in the framework of the MIT bag model.[4]

As discussed in Sect. 3.3 the quark wave function in the MIT bag model reads for massless quarks

$$\Psi_m(x) = \frac{N}{\sqrt{4\pi}} \begin{pmatrix} j_0(ER)\chi_{\kappa m} \\ \mathrm{i} j_1(ER)\frac{\boldsymbol{\sigma}\cdot\boldsymbol{r}}{|\boldsymbol{r}|}\chi_{\kappa m} \end{pmatrix} \exp(-\mathrm{i}Et) \tag{5.260}$$

$$\text{with} \qquad N^2 = \frac{ER}{2R^3(ER-1)j_0^2(ER)} \quad . \tag{5.261}$$

The corresponding field operator reads

$$\hat{\Psi}(x) = \sum_{m=\pm 1/2} \left\{\hat{b}_\alpha(m)\Psi_m(x) + \hat{d}_\alpha^\dagger(m)\Psi_m^*(x)\right\} \quad . \tag{5.262}$$

A major problem of the MIT model and all other models which lack Lorentz invariance is that the hadronic scattering tensor $W_{\mu\nu}$ depends explicitly on the two coordinates $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$. To cope with this problem a modified form of the scattering tensor is used. Into the usual expression

$$W_{\mu\nu} = \frac{1}{4\pi}\int \mathrm{d}^4x \; \exp(-\mathrm{i}q\cdot x)\langle p|[j_\mu(x),j_\nu(0)]|p\rangle \tag{5.263}$$

with the covariantly normalized proton state

$$\langle p|p'\rangle = (2\pi)^3 2E\delta(\boldsymbol{p}-\boldsymbol{p}') \tag{5.264}$$

---

[4] We follow the reasoning given in the article R.L. Jaffe: Phys. Rev. **D11**, 74(1953) for the unpolarized structure functions and adapt all steps to the polarized case.

we substitute target states at rest denoted by $|T\rangle$ and normalized according to

$$\begin{aligned} |T\rangle &:= [(2\pi)^3 2E\delta^3(0)]^{-1/2}|p\rangle \\ \langle T|T\rangle &= 1 \quad . \end{aligned} \tag{5.265}$$

Their SU(6) spin-flavor structure has the well known form

$$\begin{aligned} |T\rangle = \frac{1}{3\sqrt{2}}[&2\hat{u}(\uparrow)\hat{u}(\uparrow)\hat{d}(\downarrow) - \hat{u}(\uparrow)\hat{u}(\downarrow)\hat{d}(\uparrow) - \hat{u}(\downarrow)\hat{u}(\uparrow)\hat{d}(\uparrow) \\ &- \hat{u}(\uparrow)\hat{d}(\uparrow)\hat{u}(\downarrow) + 2\hat{u}(\uparrow)\hat{d}(\downarrow)\hat{u}(\uparrow) - \hat{u}(\downarrow)\hat{d}(\uparrow)\hat{u}(\uparrow) \\ &- \hat{d}(\uparrow)\hat{u}(\uparrow)\hat{u}(\downarrow) - \hat{d}(\uparrow)\hat{u}(\downarrow)\hat{u}(\uparrow) + 2\hat{d}(\downarrow)\hat{u}(\uparrow)\hat{u}(\uparrow)]|0\rangle \quad . \end{aligned} \tag{5.266}$$

We thus get

$$W_{\mu\nu} = (2\pi)^2 E\delta^3(0) \int \mathrm{d}^4x \; \exp(-\mathrm{i}qx) \, \langle T|[j_\mu(x), j_\nu(0)]|T\rangle \Big|_{E=p^0=M} \quad . \tag{5.267}$$

Introducing a second coordinate this is equivalent to

$$W_{\mu\nu} = \frac{E}{4\pi^2\delta(0)} \int \mathrm{d}^4x_1 \int \mathrm{d}^4x_2 \; \exp(-\mathrm{i}q(x_1 - x_2)) \, \langle T|[j_\mu(x_1), j_\nu(x_2)]|T\rangle \quad , \tag{5.268}$$

where we have used the standard equivalence of $(2\pi\delta(0))^4$ with the total time-space volume $T \cdot V$. The expression (5.268) does not depend on the average time $T := 1/2(t_1 + t_2)$ which can be integrated out, leaving:

$$\begin{aligned} W_{\mu\nu} = \frac{E}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}t \int &d^3x_1 \, \mathrm{d}^3x_2 \; \exp\left(-\mathrm{i}q^0 t + \mathrm{i}\boldsymbol{q} \cdot (\boldsymbol{x}_1 - \boldsymbol{x}_2)\right) \\ &\times \langle T|[j_\mu(\boldsymbol{x}_1, t), j_\nu(\boldsymbol{x}_2, 0)]|T\rangle \quad . \end{aligned} \tag{5.269}$$

For the nucleon $E = M = 3E_1(\text{quark}) = 6.12/R$. The limitations of the bag model come into play if $|\boldsymbol{x}_1 - \boldsymbol{x}_2| \sim R$, implying $M|t_1 - t_2| \approx M|\boldsymbol{x}_1 - \boldsymbol{x}_2| \approx 2MR_0$. The corresponding Bjorken x value is

$$x \leq \frac{1/(2R)}{M} = \frac{1}{12.24} = 0.08 \quad . \tag{5.270}$$

Thus, even if one tries to correct for the static boundary condition the model remains intrinsically bad for $x < 0.1$. We can only hope to get a reasonably good fit to the data for larger x values. Actually, it will turn out that the situation is not quite so bad. The bag model is a constituent quark model and thus can only apply for small Q^2. By evolving the bag structure functions to much larger Q^2 they become much more realistic. Inserting $j_\mu(\boldsymbol{x}, t) = \bar{\Psi}(\boldsymbol{x}, t)_1\gamma_\mu\Psi(\boldsymbol{x}, t)$ we obtain in the usual manner (see (5.122)–(5.135))

$$\begin{aligned}
W_{\mu\nu} = {} & \frac{M}{(2\pi)^4} \sum_{\alpha=\mathrm{u,d,s}} Q_\alpha^2 \int \mathrm{d}^4k\, k^\rho \cdot \mathrm{sgn}(k^0)\delta(k^2) \int_{-\infty}^{\infty} \mathrm{d}t \int_{\mathrm{bag}} \mathrm{d}^3x_1 \int_{\mathrm{bag}} \mathrm{d}^3x_2 \\
& \times \exp\left(\mathrm{i}(k^0+q^0)t - \mathrm{i}(\boldsymbol{k}+\boldsymbol{q})\cdot(\boldsymbol{x}_1-\boldsymbol{x}_2)\right) \qquad (5.271) \\
& \times \left[s_{\mu\rho\nu\sigma} \left\langle T \left| \hat{\bar{\Psi}}_\alpha(\boldsymbol{x}_1,t)\gamma^\sigma \hat{\Psi}_\alpha(\boldsymbol{x}_2,0) - \hat{\bar{\Psi}}_\alpha(\boldsymbol{x}_2,0)\gamma^\sigma \hat{\Psi}_\alpha(\boldsymbol{x}_1,t) \right| T \right\rangle \right. \\
& \left. -\mathrm{i}\varepsilon_{\mu\rho\nu\sigma} \left\langle T \left| \hat{\bar{\Psi}}_\alpha(\boldsymbol{x}_1,t)\gamma^\sigma\gamma_5 \hat{\Psi}_\alpha(\boldsymbol{x}_2,0) + \hat{\bar{\Psi}}_\alpha(\boldsymbol{x}_2,0)\gamma^\sigma \hat{\Psi}_\alpha(\boldsymbol{x}_1,t) \right| T \right\rangle \right] \quad .
\end{aligned}$$

The spin-dependent structure functions describe the antisymmetric part of $W_{\mu\nu}$, called $W_{\mu\nu}^A$

$$W_{\mu\nu}^A = \mathrm{i}\varepsilon_{\mu\nu\lambda\sigma}\frac{q^\lambda}{\nu}\left[g_1(x)s^\sigma + g_2(x)\left(s^\sigma - p^\sigma\frac{q\cdot s}{\nu}\right)\right] \qquad (5.272)$$

with $p^2 = M^2$, $s^2 = -M^2$, $\nu = P\cdot q$, $q^2 = -Q^2$. s^σ is the polarization vector of the nucleon. Choosing the target rest frame and the z axis in the direction of the spin implies

$$P_\mu = (M,0,0,0) \quad , \quad s^\mu = (0,0,0,M) \quad , \qquad (5.273)$$

$$\begin{aligned}
q^\mu = \left(\frac{\nu}{M},0,0,\sqrt{Q^2+\left(\frac{\nu}{M}\right)^2}\right) & \approx \left(\frac{\nu}{M},0,0,\frac{\nu}{M}+Mx\right) \qquad (5.274) \\
& \approx \left(\frac{\nu}{M},0,0,\frac{\nu}{M}\right) \quad ,
\end{aligned}$$

$$\begin{aligned}
W_{\mu\nu}^A & = \mathrm{i}\varepsilon_{\mu\nu03}\frac{1}{M}\left[M g_1(x) + g_2(x)\left(M - 0 + 0 - M\frac{\nu M}{M\nu}\right)\right] \\
& = \mathrm{i}\varepsilon_{\mu\nu03}\, g_1(x) \quad . \qquad (5.275)
\end{aligned}$$

From (5.271) it thus follows that

$$\begin{aligned}
g_1(x) = {} & \frac{M}{(2\pi)^4} \sum_\alpha Q_\alpha^2 \int \mathrm{d}^4k\, \mathrm{sgn}(k^0)\delta(k^2) \int_{-\infty}^{\infty} \mathrm{d}t \int_{\mathrm{bag}} \mathrm{d}^3x_1 \int_{\mathrm{bag}} \mathrm{d}^3x_2 \\
& \times \exp\left(\mathrm{i}(q^0+k^0)t - \mathrm{i}(\boldsymbol{k}+\boldsymbol{q})(\boldsymbol{x}_1-\boldsymbol{x}_2)\right) \\
& \times \left\langle T \left| \hat{\bar{\Psi}}_\alpha(\boldsymbol{x}_1,t)\left(\gamma^3k^0 - \gamma^0k^3\right)\gamma_5\hat{\Psi}_\alpha(\boldsymbol{x}_2,0) - [(\boldsymbol{x}_1,t)\leftrightarrow(\boldsymbol{x}_2,0)] \right| T \right\rangle . \\
& \qquad (5.276)
\end{aligned}$$

This is the general formula from which one can calculate $g_1(x)$ for any model. Next we insert for $\hat{\Psi}_\alpha$ the standard decomposition into eigenstates $\Psi_{\alpha,m}$ and creation and annihilation operators $\hat{b}_{\alpha,m}, \hat{d}_{\alpha,m}$. Neglecting quark masses we write furthermore

$$\Psi_{\alpha,m}(\boldsymbol{x},t) = \frac{N}{\sqrt{4\pi}}\Psi_m(\boldsymbol{x})\exp\left(-\mathrm{i}E\cdot t\right) \quad , \qquad (5.277)$$

with $\Psi_m(\boldsymbol{x})$ from (5.260), so we use the fact that all flavors are described by the same wave function. Also in the bag model only u and d quarks contribute:

$$
\begin{aligned}
g_1(x) = {} & \frac{MN^2}{2(2\pi)^5} \sum_{\alpha=\mathrm{u,d}} Q_\alpha^2 \sum_m \langle T|b^+_{\alpha m} b_{\alpha m}|T\rangle \\
& \times \int \mathrm{d}^4k \ \mathrm{sgn}(k^0)\delta(k^2) \int\limits_{-\infty}^{\infty} \mathrm{d}t \int\limits_{\mathrm{bag}} \mathrm{d}^3x_1 \\
& \times \int\limits_{\mathrm{bag}} \mathrm{d}^3x_2 \ \exp\left(\mathrm{i}(q^0+k^0)t - \mathrm{i}(\boldsymbol{k}+\boldsymbol{q})(\boldsymbol{x}_1-\boldsymbol{x}_2)\right) \\
& \times \left\{\bar{\Psi}_m(\boldsymbol{x}_1)\left(\gamma^3k^0-\gamma^0k^3\right)\gamma_5\Psi_m(\boldsymbol{x}_2)\exp(\mathrm{i}Et)\right. \\
& \left. - \bar{\Psi}_m(\boldsymbol{x}_2)\left(\gamma^3k^0-\gamma^0k^3\right)\gamma_5\Psi_m(\boldsymbol{x}_1)\exp(-\mathrm{i}Et)\right\} \quad . \qquad (5.278)
\end{aligned}
$$

The t-integration leads to a δ-function which allows the k^0-integral to be performed. Furthermore the α and m sum is easy to do as only the lowest m-state ($\kappa = -1$) contributes. The proton flavor decomposition is

$$
\begin{aligned}
|T\rangle = \frac{1}{\sqrt{18}} & \left[2u^\uparrow u^\uparrow d^\downarrow - u^\uparrow u^\downarrow d^\uparrow - u^\downarrow u^\uparrow d^\uparrow - u^\uparrow d^\uparrow u^\downarrow + 2u^\uparrow d^\downarrow u^\uparrow \right. \\
& \left. - u^\downarrow d^\uparrow u^\uparrow - d^\uparrow u^\uparrow u^\downarrow - d^\uparrow u^\downarrow u^\uparrow + 2d^\downarrow u^\uparrow u^\uparrow\right]
\end{aligned}
\qquad (5.279)
$$

and we use the fact that the result must be proportional to $\mathrm{sgn}(\mu)$ as the spin direction enters. This factor is thus introduced in the remaining equations.

$$
\begin{aligned}
& \sum_{\alpha m} Q_\alpha^2 \langle T|b^+_{\alpha m} b_{\alpha m}|T\rangle \mathrm{sgn}(m) \\
& = \frac{4}{9}\cdot\frac{1}{18}[4\cdot 2+0+0]\cdot 3 \\
& \quad + \frac{1}{9}\cdot\frac{1}{18}[-4+1+1]\cdot 3 = \frac{32-2}{9\cdot 6} = \frac{5}{9}
\end{aligned}
\qquad (5.280)
$$

for the proton and

$$
\frac{1}{9}\frac{1}{18}[4\cdot 2+0+0]\cdot 3 + \frac{4}{9}\frac{1}{18}[-4+1+1]\cdot 3 = 0 \qquad (5.281)
$$

for the neutron. The fact that in the bag model $g_1^{\mathrm{n}}(x) \equiv 0$ is a typical problem of this model. We shall keep this shortcoming in mind and calculate $g_1^{\mathrm{p}}(x)$. We shall also omit from now on the index m of Ψ_m.

$$
\begin{aligned}
g_1^{\mathrm{p}}(x) = {} & \frac{5MN^2}{18(2\pi)^5} \int \mathrm{d}k \ d\Omega_k \mathrm{sgn}(k^0)\frac{k^2}{2k}[\delta(k^0+k)+\delta(k^0-k)] \\
& \times \int\limits_{\mathrm{bag}} \mathrm{d}^3x_1 \int\limits_{\mathrm{bag}} \mathrm{d}^3x_2 \ \exp\left(-\mathrm{i}(\boldsymbol{q}+\boldsymbol{k})(\boldsymbol{x}_1-\boldsymbol{x}_2)\right) \\
& \times \mathrm{sgn}(\mu)\left\{\bar{\Psi}(\boldsymbol{x}_1)\left[\gamma^3k^0-\gamma^0\cos\theta_k\, k\right]\gamma_5\Psi(\boldsymbol{x}_2)\Big|_{k^0=-q^0-E}\right. \\
& \left. + \bar{\Psi}(\boldsymbol{x}_2)\left[\gamma^3k^0-\gamma^0\cos\theta_k\, k\right]\gamma_5\Psi(\boldsymbol{x}_1)\Big|_{k^0=-q^0+E}\right\} \quad . \qquad (5.282)
\end{aligned}
$$

The values for $k^0 = -q^0 \pm E$ have to be inserted in the whole expression. As $q^0 = \nu/M \gg E$, k^0 is in both cases negative, implying $\mathrm{sgn}(k^0) = -1$ and $k = -k^0$:

$$
\begin{aligned}
g_1^{\mathrm{p}}(x) = \frac{5MN^2}{36(2\pi)^4} \int \mathrm{d}\cos\theta_k \; k^0 \int_{\text{bag}} \mathrm{d}^3x_1 \int_{\text{bag}} \mathrm{d}^3x_2 \; \exp\left(-\mathrm{i}(\boldsymbol{q}+\boldsymbol{k})(\boldsymbol{x}_1-\boldsymbol{x}_2)\right) \\
\times \operatorname{sgn}(\mu) \cdot k^0 \left\{ \bar{\Psi}(\boldsymbol{x}_1)\left(\gamma^3+\gamma^0\cos\theta_k\right)\gamma_5\Psi(\boldsymbol{x}_2)\big|_{k^0=q^0+E} \right. \\
\left. + \; \bar{\Psi}(\boldsymbol{x}_2)\left(\gamma^3+\gamma^0\cos\theta_k\right)\gamma_5\Psi(\boldsymbol{x}_1)\big|_{k^0=q^0-E} \right\} \quad . \qquad (5.283)
\end{aligned}
$$

Inserting (5.260) we find

$$
\begin{aligned}
A &= \bar{\Psi}(\boldsymbol{x}_1)\left(\gamma^3+\gamma^0\cos\theta_k\right)\gamma_5\Psi(\boldsymbol{x}_2)\operatorname{sgn}(\mu) \\
&= \operatorname{sgn}(\mu)\left(j_0(Ex_1)\chi^+_{\kappa\mu}(\Omega_1), -\mathrm{i}j_1(Ex_1)\chi^+_{-\kappa\mu}(\Omega_1)\right) \\
&\quad \times \begin{pmatrix} 1\!\!1 & 0 \\ 0 & -1\!\!1 \end{pmatrix} \begin{pmatrix} \cos\theta_k \cdot 1\!\!1 & \sigma^3 \\ -\sigma^3 & -\cos\theta_k \cdot 1\!\!1 \end{pmatrix} \\
&\quad \times \begin{pmatrix} 0 & 1\!\!1 \\ 1\!\!1 & 0 \end{pmatrix} \begin{pmatrix} j_0(Ex_2)\chi_{\kappa\mu}(\Omega_2) \\ \mathrm{i}j_1(Ex_2)\chi_{-\kappa\mu}(\Omega_2) \end{pmatrix} \quad . \qquad (5.284)
\end{aligned}
$$

We use the well known relation

$$
\boldsymbol{\sigma} \cdot \frac{\boldsymbol{x}}{|\boldsymbol{x}|}\chi_{\kappa\mu}(\Omega) = -\chi_{-\kappa\mu}(\Omega) \qquad (5.285)
$$

and define $\sigma(1)$ as $-\boldsymbol{\sigma}\cdot\boldsymbol{x}_1/|\boldsymbol{x}_1|$ etc.

$$
\begin{aligned}
A &= \operatorname{sgn}(\mu)\left(j_0(Ex_1)\chi^+_{\kappa\mu}(\Omega_1), -\mathrm{i}j_1(Ex_1)\chi^+_{\kappa\mu}(\Omega_1)\sigma(1)\right) \\
&\quad \times \begin{pmatrix} \sigma^3 & \cos\theta_k \\ \cos\theta_k & \sigma^3 \end{pmatrix} \begin{pmatrix} j_0(Ex_2)\chi_{\kappa\mu}(\Omega_2) \\ \mathrm{i}j_1(Ex_2)\sigma(2)\chi_{\kappa\mu}(\Omega_2) \end{pmatrix} \\
&= \operatorname{sgn}(\mu)\left[j_0(Ex_1)j_0(Ex_2)\chi^+_{\kappa\mu}(\Omega_1)\sigma^3\chi_{\kappa\mu}(\Omega_2)\right. \\
&\quad - \mathrm{i}j_1(Ex_1)j_0(Ex_2)\chi^+_{\kappa\mu}(\Omega_1)\sigma(1)\cos\theta_k\chi_{\kappa\mu}(\Omega_2) \\
&\quad + \mathrm{i}j_0(Ex_1)j_1(Ex_2)\chi^+_{\kappa\mu}(\Omega_1)\cos\theta_k\sigma(2)\chi_{\kappa\mu}(\Omega_2) \\
&\quad \left. + j_1(Ex_1)j_1(Ex_2)\chi^+_{\kappa\mu}(\Omega_1)\sigma(1)\sigma^3\sigma(2)\chi_{\kappa\mu}(\Omega_2)\right] \quad . \qquad (5.286)
\end{aligned}
$$

For the $1s_{1/2}$ state we have

$$
\chi_{\kappa\mu}(\Omega_1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{for } \mu = +\frac{1}{2} \quad , \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{for } \mu = -\frac{1}{2} \quad , \qquad (5.287)
$$

thus

$$
\chi^+_{\kappa\mu}(\Omega_1)\sigma^3\chi_{\kappa\mu}(\Omega_2) = \operatorname{sgn}(\mu) \qquad (5.288)
$$

$$
\begin{aligned}
\chi^+_{\kappa\mu}\sigma(1)\,\cos\theta\chi_{\kappa\mu} &= \begin{cases} \frac{(x_1)^3}{|\boldsymbol{x}_1|} & \text{for } \mu = \frac{1}{2} \\ -\frac{(x_1)^3}{|\boldsymbol{x}_1|} & \text{for } \mu = -\frac{1}{2} \end{cases} \\
&= \operatorname{sgn}(\mu)\cos\theta_1 \qquad (5.289)
\end{aligned}
$$

$$
\begin{aligned}
\chi^+_{\kappa\mu}\sigma(1)\sigma^3\sigma(2)\chi_{\kappa\mu} &= \chi^+_{\kappa\mu}\begin{pmatrix} (x_1)^3 & (x_1)^1 - \mathrm{i}(x_1)^2 \\ (x_1)^1 + \mathrm{i}(x_1)^2 & -(x_1)^3 \end{pmatrix} \\
&\quad \times \begin{pmatrix} (x_2)^3 & (x_2)^1 - \mathrm{i}(x_2)^2 \\ -(x_2)^1 - \mathrm{i}(x_2)^2 & (x_2)^3 \end{pmatrix} \chi_{\kappa\mu}\frac{1}{|\boldsymbol{x}_1||\boldsymbol{x}_2|} \quad . \qquad (5.290)
\end{aligned}
$$

For $\mu = 1/2$ this gives

$$\frac{1}{|\boldsymbol{x}_1||\boldsymbol{x}_2|}\left((x_1)^3(x_2)^3 - (x_1)^1(x_2)^1 - (x_1)^2(x_2)^2 + \mathrm{i}(x_1)^2(x_2)^1 - \mathrm{i}(x_1)^1(x_2)^2\right) \quad , \tag{5.291}$$

and for $\mu = -1/2$

$$-\frac{1}{|\boldsymbol{x}_1||\boldsymbol{x}_2|}\left((x_1)^3(x_2)^3 - (x_1)^1(x_2)^1 - (x_1)^2(x_2)^2 + \mathrm{i}(x_1)^1(x_2)^2 - \mathrm{i}(x_1)^2(x_2)^1\right) \quad , \tag{5.292}$$

which can be combined to give

$$\mathrm{sgn}(\mu)\left(2\cos\theta_1\cos\theta_2 - \frac{\boldsymbol{x}_1\cdot\boldsymbol{x}_2}{|\boldsymbol{x}_1||\boldsymbol{x}_2|}\right) + \mathrm{i}\frac{(x_1)^1(x_2)^2 - (x_1)^2(x_2)^1}{|\boldsymbol{x}_1||\boldsymbol{x}_2|} \quad . \tag{5.293}$$

Upon integrating over Ω_1 and Ω_2 the second term in (5.293) vanishes. This is easy to see. If one chooses in (5.283) $(\boldsymbol{q}+\boldsymbol{k})$ to lie in the 3 and 1 direction the expression must be invariant under $(x_1)^2 \longrightarrow -(x_1)^2$ or $(x_2)^2 \longrightarrow -(x_2)^2$. This ensures the vanishing of the mixed term in (5.293). Thus we have found

$$\begin{aligned} A &= j_0(Er_1)j_0(Er_2) - \mathrm{i}j_1(Er_1)j_0(Er_2)\cos\theta_1 + \mathrm{i}j_1(Er_2)j_0(Er_1)\cos\theta_2 \\ &\quad + j_1(Er_1)j_1(Er_2)\left(2\cos\theta_1\cos\theta_2 - \frac{\boldsymbol{x}_1\cdot\boldsymbol{x}_2}{|\boldsymbol{x}_1||\boldsymbol{x}_2|}\right) \quad . \end{aligned} \tag{5.294}$$

The factor $\mathrm{sgn}(\mu)$ cancelled in fact, as claimed above:

$$\begin{aligned} g_1^{\mathrm{p}}(x) &= \frac{5MN^2}{36(2\pi)^4}\int_{-1}^{1}\mathrm{d}\cos\theta_k \int \mathrm{d}\boldsymbol{x}_1\,\mathrm{d}\boldsymbol{x}_2\,\exp\left(-\mathrm{i}\boldsymbol{p}\cdot(\boldsymbol{x}_1-\boldsymbol{x}_2)\right)\cdot k^2 \\ &\quad \times \left\{A(1,2)|_{k=q^0+E} + A(2,1)|_{k=q^0-E}\right\} \quad . \end{aligned} \tag{5.295}$$

We have introduced the momentum of the hit quark $\boldsymbol{p} = \boldsymbol{q} + \boldsymbol{k}$:

$$E^2 + \boldsymbol{p}^2 = m_q^2 \ll 1\ \mathrm{GeV}^2 \quad . \tag{5.296}$$

The angular integrals we need are

$$\begin{aligned} \int \mathrm{d}\Omega_1\,\exp(-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}_1) &= \int \mathrm{d}\Omega_1\,4\pi\sum_{lm}(-\mathrm{i})^l j_l(|\boldsymbol{p}||\boldsymbol{x}_1|) \\ &\qquad \times Y_{lm}(\Omega_p)Y_{lm}(\Omega_1) = 4\pi j_0(|\boldsymbol{p}||\boldsymbol{x}_1|) \end{aligned} \tag{5.297}$$

and

$$\begin{aligned} &\int \mathrm{d}\Omega_1\,\exp(-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}_1)\cos\theta_1 \\ &\quad = \int \mathrm{d}\Omega_1\,4\pi\sum_{lm}(-\mathrm{i})^l j_l(|\boldsymbol{p}||\boldsymbol{x}_1|)Y_{lm}(\Omega_p)Y_{lm}(\Omega_1)Y_{10}(\Omega_1)\sqrt{4\pi} \\ &\quad = -4\pi\mathrm{i}j_1(px_1)\cos(\Omega_p) \end{aligned} \tag{5.298}$$

$$\int \mathrm{d}\Omega_1\,\exp(-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}_1)\cos\theta_1\cos\varphi = -4\pi\mathrm{i}j_1(px_1)\cos(\Omega_p)\cos(\varphi_p) \tag{5.299}$$

$$
\begin{aligned}
g_1(x) = {} & \frac{5MN^2}{36(2\pi)^4}(4\pi)^2 \int \mathrm{d}\cos\theta_k \int_0^R \mathrm{d}x_1\, x_1^2 \int_0^R \mathrm{d}x_2\, x_2^2 \cdot k^2 \\
& \times \Bigg\{ \Bigg[j_0(Ex_1) j_0(Ex_2) j_0(px_1) j_0(px_2) \\
& - \cos\theta_\mathrm{p} \cos\theta_k j_0(Ex_1) j_1(Ex_2) j_0(px_1) j_1(px_2) \\
& - \cos\theta_\mathrm{p} \cos\theta_k j_1(Ex_1) j_0(Ex_2) j_0(px_2) j_1(px_1) \\
& + j_1(Ex_1) j_1(Ex_2) \Bigg(2 j_1(px_1) j_1(px_2) \cos\theta_\mathrm{p} \cos\theta_\mathrm{p} \\
& - j_1(px_1) j_1(px_2) \left(\cos\theta_\mathrm{p}^2 + \sin\theta_\mathrm{p}^2 \cos^2\varphi + \sin^2\theta_\mathrm{p} \sin^2\varphi \right) \Bigg) \Bigg]_{k=q^0+E} \\
& + \Big[\dots \Big]_{k=q^0-E} \Bigg\}_{p=q+k} \quad . \qquad (5.300)
\end{aligned}
$$

The x_1, x_2 integration can now be done analytically with

$$
\begin{aligned}
t_l(a,b) &= \int_0^1 \mathrm{d}x\; x^2 j_l(ax) j_l(bx) \\
&= \frac{1}{a^2-b^2} \left[b j_l(a) j_{l-1}(b) - a j_l(b) j_{l-1}(a) \right] \qquad (5.301)
\end{aligned}
$$

$$
\begin{aligned}
g_1(x) = {} & \frac{5MN^2}{9(2\pi)^2} \int \mathrm{d}\cos\theta_k\; k^2 \Big[\Big(t_0^2(ER,pR) - 2\cos\theta_\mathrm{p} \cos\theta_k t_0(ER,pR) \\
& \times t_1(ER,pR) + t_1^2(ER,pR)(2\cos^2\theta_\mathrm{p} - 1) \Big) \Big|_{k=q^0+E} \\
& + (\dots)\big|_{k=q^0-E} \Big] \quad . \qquad (5.302)
\end{aligned}
$$

(5.302) is basically the MIT bag model result for $g_1^\mathrm{p}(x)$. The only part still missing is the relation between θ_p and θ_k. With it everything will be defined:

$$\cos\theta_\mathrm{p} = \frac{p_3}{|\boldsymbol{p}|} = \frac{k\cos\theta_k + q_3}{|\boldsymbol{k}+\boldsymbol{q}|} \qquad (5.303)$$

$$p^2 = |\boldsymbol{p}|^2 = q_3^2 + k^2 + 2q_3 k \cos\theta_k \qquad (5.304)$$

$$\cos\theta_k = \frac{p^2 - q_3^2 - k^2}{2q_3 k} \quad . \qquad (5.305)$$

The last equation implies

$$p^2 > p_\mathrm{min}^2 = (q_3 - k)^2 \quad . \qquad (5.306)$$

And with $k = q_0 \pm E$ this means

$$\left(p_\mathrm{min}^\pm\right)^2 = (q_3 - q_0 \mp E)^2 = (Mx \mp E)^2 \quad . \qquad (5.307)$$

Inserting (5.312) into (5.310) gives

$$\begin{aligned}\cos\theta_{\mathrm{p}} &= \frac{\left(p^2-q_3^2-k^2\right)+2q_3^2}{2q_3p} = \frac{p^2+(q_3+k)(q_3-k)}{2q_3p}\\ &= \frac{p^2+(q_3+q_0\pm E)(xM\mp E)}{2q_3p} \approx \frac{p^2+2q_3\left(p_{\min}^{\pm}\right)}{2q_3p}\\ \cos\theta_k\cos\theta_{\mathrm{p}} &= \frac{\left(p^2-q_3^2-k^2\right)\left(p^2+\left(q_3^2-k^2\right)\right)}{4q_3^2kp}\\ &= \frac{p^4-2p^2k^2-\left(q_3^2+k^2\right)\left(q_3^2-k^2\right)}{4q_3^2kp} \quad .\end{aligned} \tag{5.308}$$

The quark momentum $\boldsymbol{p}$ is in principle unknown, but at a proton at rest the probability of finding a parton with a momentum in the multi GeV range should be exponentially small. This fact is also contained in (5.302) and (5.303). $t(ER,pR)$ and thus $g_1(x)$ decrease sharply with increasing p. Thus we only get a sizeable contribution if $p \sim p_{\min} \ll q_0,\ k,\ q_3$, implying $k \approx q_3$:

$$\cos\theta_{\mathrm{p}} \approx \frac{p_{\min}^{\pm}}{p} \tag{5.309}$$

$$\begin{aligned}\cos\theta_k\cos\theta_{\mathrm{p}} &\approx -\frac{q_3^2-k^2}{2kp} \approx -\frac{q_3^2-q_0^2\mp 2p_0E}{2q_0p}\\ &\approx \frac{2q_3xM\pm 2q_0E}{2q_0p} = \frac{xM\pm p}{p} \quad .\end{aligned} \tag{5.310}$$

With

$$\frac{\mathrm{d}\cos\theta_k}{\mathrm{d}p} = \frac{2p}{2q_3k} \sim \frac{p}{k^2} \quad , \tag{5.311}$$

(5.302) simplifies to

$$\begin{aligned}g_1^{\mathrm{p}}(x) &= \frac{5MER}{2R^3(ER-1)j_0^2(ER)\cdot 9\cdot(2\pi)^2}\left\{\int_{p_{\min}^{+}}^{\infty}\mathrm{d}p\, p\right.\\ &\quad\times\left[t_0^2(ER,pR)-2t_0(ER,pR)t_1(ER,pR)\frac{p_{\min}^{+}}{p}\right.\\ &\quad\left.\left.+\,t_1^2(ER,pR)\left(2\left(\frac{p_{\min}^{+}}{p}\right)^2-1\right)\right]+p_{\min}^{+}\to p_{\min}^{-}\right\}\\ p_{\min}^{\pm} &= |E\pm Mx| \quad .\end{aligned} \tag{5.312}$$

The result is shown in Fig. 5.18 (remember $g_1^{\mathrm{n}}(x)\equiv 0$).

It is obviously very far from being physical. However, one can argue that this would be the correct result only at small scales and that one thus should evolve it with the GLAP equations. As these equations have the very nice property of producing acceptable gluon and sea-quark distributions if one evolves from sufficiently small scales one thus obtains an acceptable result after evolution from $\mu^2 = 0.2\ \mathrm{GeV}^2$ to $\mu^2 = 10\ \mathrm{GeV}^2$, see Fig. 5.19.

While this agreement does not look too bad it should be realized that this is only due to the GLAP evolution down to unjusifyable small values uf μ^2. To

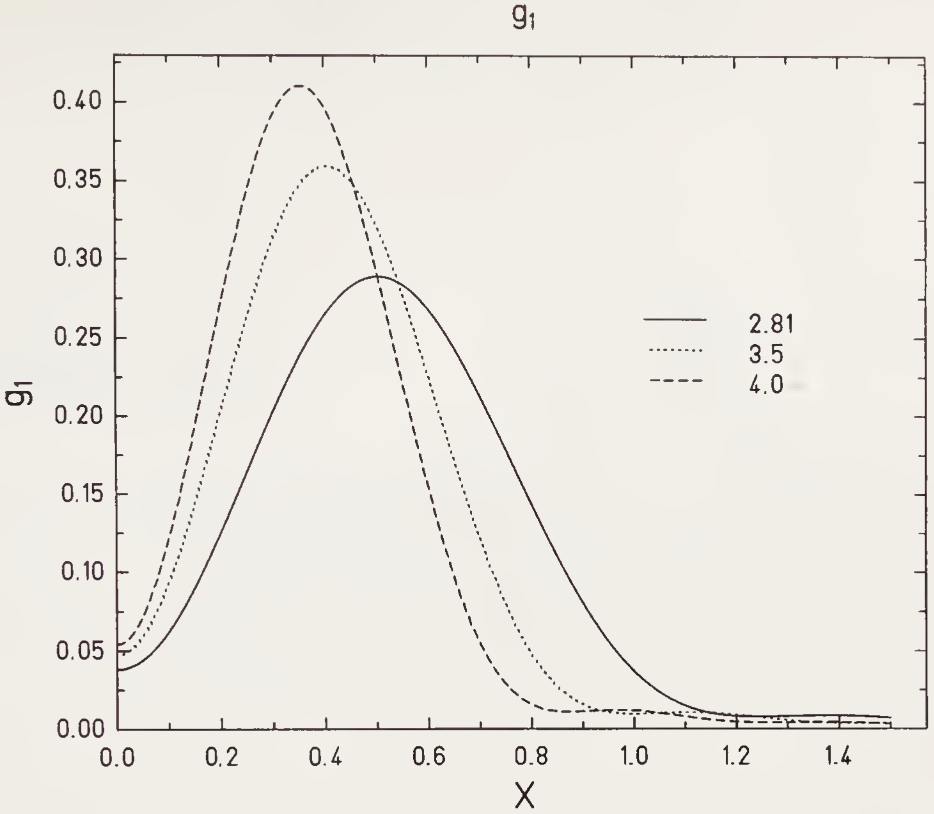


Fig. 5.18. MIT-bag-model predictions for $g_1^p(x)$ without GLAP evolution for various values of MR (solid line $MR = 2.81\omega$, dotted line $MR = 3.5\omega$, dashed line $MR = 4.0\omega$)

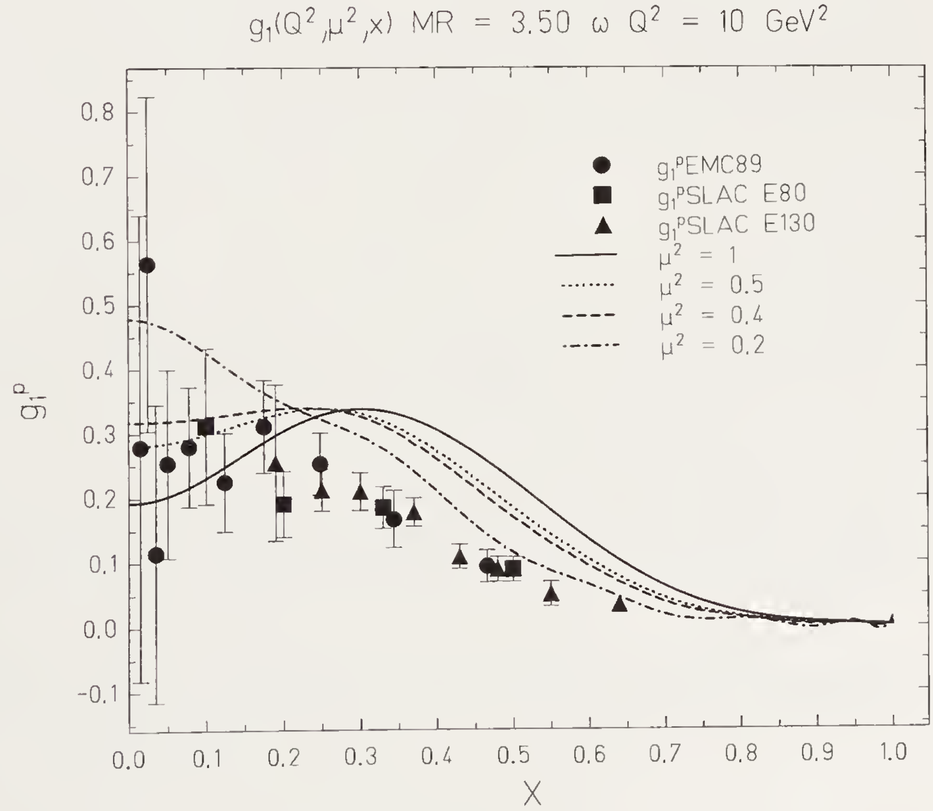
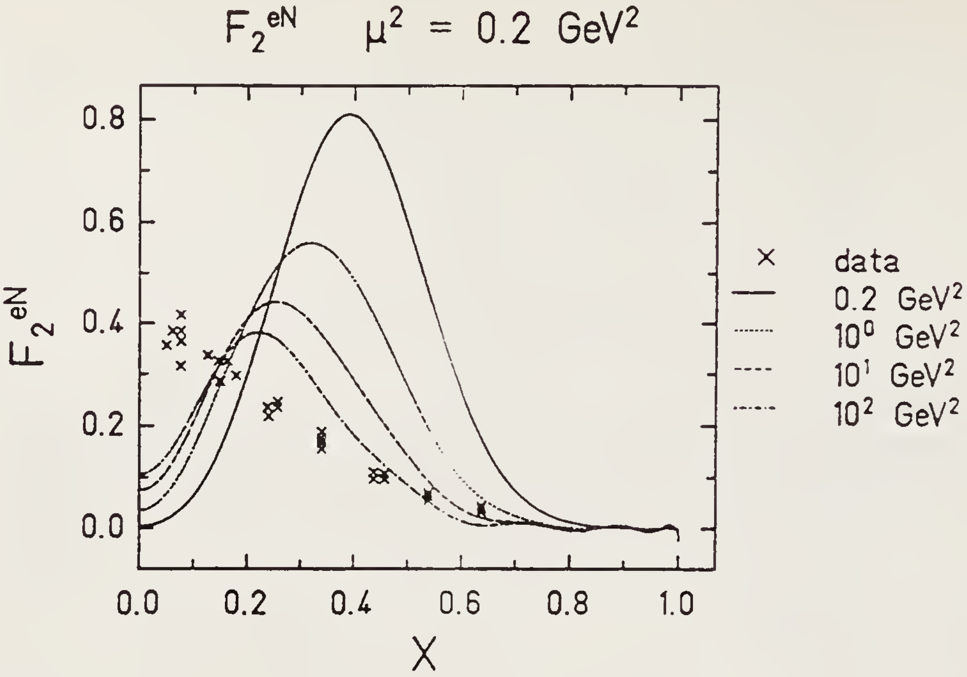


Fig. 5.19. MIT-bag-model prediction for $g_1^p(x)$ after GLAP-evolution from various starting values of μ^2(solid line $\mu^2 = 1$ GeV^2, dotted line $\mu^2 = 0.5$ GeV^2, dashed line $\mu^2 = 0.4$ GeV^2, dashed-dotted line $\mu^2 = 0.2$ GeV^2) up to 10 GeV^2. $MR = 3.5\omega$ was used

see how strongly this evolution softens the distribution, we show in Fig. 5.20 the correspondigly evolved result for our simple modell of Example 3.9.

Fig. 5.20. GLAP evolution of $F_2(x,Q^2)$ as obtained from the simple model discussed in Example 3.9. For very large values of Q^2 the agreement becomes acceptible



6. Perturbative QCD II: The Drell–Yan Process and the Gribov–Levin–Ryskin Equation

6.1 The Drell–Yan Process

Deep inelastic scattering is investigated by shooting electrons or muons at different fixed targets or by colliding them with a proton beam. From these experiments only information about nucleon structure functions can be extracted, while little is learned about the inner structure of pions, for example. There are, however, different options for high-energy collisions. Electrons can be made to collide with positrons (as is done at Stanford, DESY, and CERN), and protons can be made to collide with protons, pions, kaons, or antiprotons. The latter is done at sites such as Fermilab, where secondary beams of pions, antiprotons, etc. are created and collide with a fixed target.

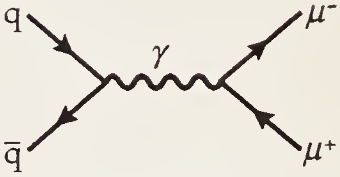


Fig. 6.1. The Drell–Yan process

To determine the internal structure of hadrons in these reactions, it is again preferable to consider reactions as hard as possible. For such reactions there is hope that perturbation theory will yield good results. To obtain information about the inner structure of hadrons, these must be in the initial channel. We therefore consider the collision of two hadrons. The process most similar to deep inelastic scattering in these reactions is the annihilation of a quark and an antiquark, each deriving from a different hadron, into a lepton pair (see Fig. 6.1). This is called the Drell–Yan process.

Bear in mind that the creation of hadrons in e^+e^- scattering is described by the very same, time-reversed graph. While quark annihilation into leptons tells us about the initial momentum distribution of the quarks inside the hadrons involved, e^+e^- annihilation is used to investigate how full-blown hadrons are created from just two dissociating quarks, that is, how the momentum originally carried by the quark–antiquark pair that has been created is distributed between real and virtual constituents of all created hadrons. This is mainly a dynamical problem in which nonperturbative effects are crucial. Its description in the framework of QCD is of the utmost difficulty and has consequently not yet proceeded beyond phenomenological models. Keywords characterizing this area of research are “hadronic strings” and “jet physics”.

We shall first address the Drell–Yan process of Fig. 6.1. By carrying over our experience of deep inelastic scattering to the Drell–Yan process, we expect the cross section to behave like

$$d\sigma(\mathrm{DY}) = \left[q_a(x_a)\bar{q}_b(x_b) + q_b(x_b)\bar{q}_a(x_a)\right]\hat{\sigma}(\mathrm{q} + \bar{\mathrm{q}} \rightarrow \gamma^* \rightarrow \mu^+\mu^-)\, dx_a\, dx_b \ . \quad (6.1)$$

Here a designates the first and b the second hadron. Consequently $q_a(x_a)$ gives the probability of finding a quark of flavor q carrying the momentum fraction x_a, in the first hadron. This is based on the expectation that at sufficiently high energies all color interactions will be suppressed by the large denominator of a propagator. However, this argument has to be treated with great caution, since there are also soft processes that cannot be neglected (see below). Another problematic point is the kinematic difference between Drell–Yan and deep inelastic scattering. While in deep inelastic scattering the photon momentum is $q^2 = -Q^2 < 0$ with a large Q^2, now $q^2 = M^2 > 0$ holds with M the invariant mass of the lepton pair. An extended formal analysis has shown that such formal continuation is possible and (6.1) can be justified. Basically Q^2 is substituted by $-M^2$ in all formulas. This, however, quite substantially changes all logarithmic terms: $\ln(Q^2/m) \to \ln(-M^2/m^2) = \mathrm{i}\pi + \ln(M^2/m^2)$. The running coupling constant still has its usual form with $\alpha(Q^2) \to \alpha(M^2)$, but in practical calculations the perturbative expansion does not converge well owing to the term proportional to $\mathrm{i}\pi$. It must be conceded, therefore, that the analysis of Drell–Yan reactions is afflicted with far more uncertainties than deep inelastic scattering. All this will be discussed in more detail below.

First we shall evaluate (6.1) further while skipping the more involved questions. In doing this, it is customary to define some new quantities. Let $s = (P_a + P_b)^2$ be the invariant mass of the colliding hadrons. The momenta of the partons participating in the Drell–Yan process are $p_a = x_a P_a$, $p_b = x_b P_b$. Neglecting hadron and quark masses, the invariant mass of the lepton pair is then

$$M^2 = (p_a + p_b)^2 = 2 p_a \cdot p_b = 2 x_a x_b \, P_a \cdot P_b = x_a x_b (P_a + P_b)^2 = x_a x_b s \ . \tag{6.2}$$

It is furthermore standard to change the variables x_a and x_a to

$$\tau = x_a x_b = \frac{M^2}{s} \quad , \quad y = \frac{1}{2} \ln \left(\frac{x_a}{x_b} \right) \quad . \tag{6.3}$$

Keeping in mind that for the energy and longitudinal momentum of the lepton pair

$$E = x_a E_a + x_b E_b \quad , \quad P_L = x_a E_a - x_b E_b$$

is valid, we obtain in the center-of-momentum system of the two hadrons $E_a = E_b = \sqrt{s}/2$

$$y = \frac{1}{2} \ln \left(\frac{E + P_L}{E - P_L} \right) \quad .$$

Thus y is the rapidity of the lepton pair in the center-of-momentum system. To rewrite (6.1) in the new variables, we need the functional determinant

$$\frac{\partial(\tau, y)}{\partial(x_a, x_b)} = 1 \quad . \tag{6.4}$$

Finally we must know the elementary cross section $\mathrm{q} + \bar{\mathrm{q}} \to \mu^+ + \mu^-$. Calculating this is very easy and will be done in Problem 6.1. We obtain

$$\sigma(\mathrm{q} + \bar{\mathrm{q}} \to \mu^+ + \mu^-) = \frac{1}{3} \frac{4\pi\alpha^2}{3M^2} \quad . \tag{6.5}$$

The additional factor 1/3 stems from averaging over quark colors, since the structure functions of the nucleon are already known.

The Drell–Yan cross section for quark–antiquark annihilation is thus to lowest order

$$\frac{\mathrm{d}^2\sigma_{\mathrm{DY}}}{\mathrm{d}\tau\mathrm{d}y}=\frac{4\pi\alpha^2}{9M^2}\sum_q e_q^2\left[q_a(x_a)\bar{q}_b(x_b)+q_b(x_b)\bar{q}_a(x_a)\right] \tag{6.6}$$

or with

$$\frac{\mathrm{d}y}{\mathrm{d}x_a}=\frac{1}{2x_a}\quad,\quad M^2=\tau s\quad,$$

$$\sigma_{\mathrm{DY}}=\frac{4\pi\alpha^2}{9}\int\limits_0^1\mathrm{d}\tau\int\limits_\tau^1\frac{\mathrm{d}x_a}{2x_a}\frac{1}{\tau s}\sum_q e_q^2\left[q_a(x_a)\bar{q}_b\left(\frac{\tau}{x_a}\right)+q_b\left(\frac{\tau}{x_a}\right)\bar{q}_a(x_a)\right]\quad. \tag{6.7}$$

To calculate Drell–Yan cross sections one has in addition to calculate the Compton process $q+G\rightarrow\gamma^*+q\rightarrow\mu^+\mu^-+q$, which is usually even more important. We shall not do this, since all the problems encountered are exactly the same for both processes. We shall simply state the corresponding result for the Compton process at the very end.

Equation (6.7) and the corresponding Compton cross section yield unique prediction for lepton-pair production in high-energy proton–proton collisions. When this was measured, it turned out that the experimental cross section exceeded the prediction by a factor of about two. This missing factor, i.e., the quotient of the measured cross section over the calculated one, is termed the “K factor”. Surprisingly it is independent of x_a and x_b, i.e., a true constant. Its source can, in the framework of perturbative QCD, only lie in higher-order corrections. Since such

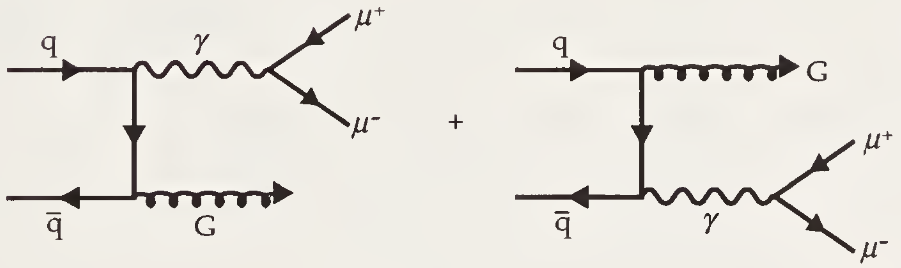


Fig. 6.2. First order gluonic corrections to the Drell–Yan process

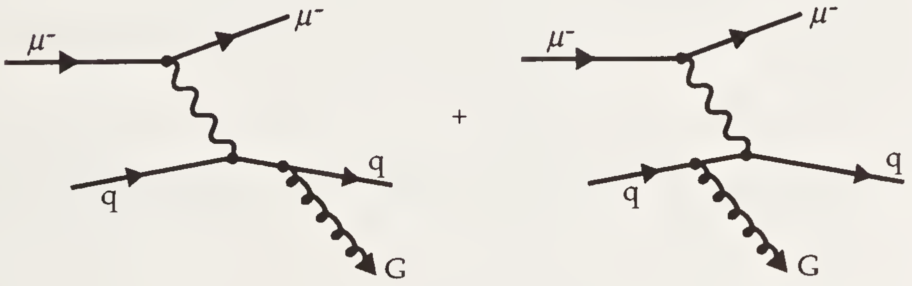


Fig. 6.3 First order gluonic corrections the muon–quark scattering

higher-order perturbative corrections are of the same order as the lowest-order contribution, convergence of the perturbation series is exceedingly questionable.

To gain a deeper understanding of the problems in Drell-Yan reactions, in the following we shall discuss the calculation of some corrections to the annihilation graph $\mathrm{q}+\bar{\mathrm{q}} \to \gamma^*$. The relevant processes are displayed in Fig. 6.2. However, we can avoid much work by using some results from Chap. 5. The graphs in Fig. 6.3 differ from those in Fig. 6.2 only by the exchange of the Mandelstam variables s and u. But the graphs in Fig. 6.3 are just those we calculated in Chap. 5, namely the term with $\mu = \mu'$. Accordingly one gets for the processes in Fig. 6.3 the result (see (5.10))

$$F^2 = S^\mu_\mu = -8\left(\frac{s}{t}+\frac{t}{s}+\frac{2M^2u}{st}\right) \quad . \tag{6.8}$$

Here we have replaced $-Q^2$ by M^2. We must then, according to the "crossing symmetry", exchange s and u:

$$F^2 = 8\left(\frac{u}{t}+\frac{t}{u}+\frac{2M^2s}{ut}\right) \quad . \tag{6.9}$$

Here an additional minus sign had to be taken into account. This phase is related to the definition of particle states (we replaced a quark by an antiquark). We shall not discuss this any further, since it is immediately obvious that F^2 must be positive. Now the combinatorial factors remain to be determined. The different charges give $e^2e_q^2g^2$, spin–averaging yields 1/4, and color averaging

$$\sum_a \frac{1}{9}\sum_{c,c'}\left(\frac{\lambda^a}{2}\right)_{cc'}\left(\frac{\lambda^a}{2}\right)_{c'c} = \frac{8}{9}\frac{1}{2} = \frac{4}{9} \quad .$$

We therefore have

$$\mathrm{d}\sigma' = \frac{1}{2s}\frac{\mathrm{d}^3q_1}{(2\pi)^2 s}e^2e_q^2g^2\frac{8}{9}\delta\left(q_1^0+q_2^0-p_1^0-p_2^0\right)\left(\frac{u}{t}+\frac{t}{u}+\frac{2M^2s}{ut}\right) \quad . \tag{6.10}$$

With $\mathrm{d}t = \mathrm{d}\Omega|\boldsymbol{q}|^2/\pi$ (Exercise 2.6, (6)), we obtain

$$\begin{aligned}\mathrm{d}\sigma' &= \frac{1}{2s}\frac{\mathrm{d}t}{8\pi s}\frac{8}{9}\alpha\alpha_s e_q^2\, 16\pi^2\left(\frac{u}{t}+\frac{t}{u}+\frac{2M^2s}{ut}\right)\\ &= \frac{\alpha\alpha_s e_q^2\pi}{s^2}\frac{8}{9}\left(\frac{u}{t}+\frac{t}{u}+\frac{2M^2s}{ut}\right)\mathrm{d}t \quad .\end{aligned} \tag{6.11}$$

It should be noted that the relation between the Mandelstam variables connects u, s, and t:

$$u = M^2 - s - t \quad .$$

The additional contribution to the cross section is thus, at given s, a function of t only. We find that

$$M^2 - s \le t \le 0$$

holds and that the integral diverges at the upper bound of the integration, i.e., for $t \to 0$. The appearance of infrared divergences is typical for all calculations of perturbative QCD. Its reason is that in the framework of the parton model all small dimensional quantities like quark masses and gluon virtualities are neglected. These quantities normally inhibit the appearance of infrared divergences. The simplest solution to the problem would therefore be to introduce finite masses and virtualities for all particles. But then the results would exhibit a strong dependence on these parameters and thus be more or less meaningless. The only chance of getting reliable predictions lies in the hope that the infrared divergences of different contributions cancel. This is indeed what happens.

At this point it is useful to recall QED bremsstrahlung, where similar problems surface owing to the vanishing mass of the photon. It turned out there that the infrared divergence canceled with the diverging radiative corrections. This effect, known as the Bloch–Nordsieck theorem, is of basic relevance for the consistency of QED, and even more so of QCD. Its physical background is related to the fact that massless particles with arbitrarily small energy, e.g., photons of infinitely long wavelengths, are, strictly speaking, unphysical since they cannot be detected by any means. The transition from some state to the same state with an additional undetectable photon is not well defined. In perturbation theory, however, all states can be classified by their occupation numbers and such states can be strictly distinguished. Since this distinction is unphysical, it can very well lead to spurious divergences in different contributions that cancel each other.

Next we face the question of how a radiative correction whose amplitude is proportional to g^2 can cancel an amplitude that is of first order in the coupling. This is explained by noting that all radiative correction graphs interfere with the lowest-order graph since they have the same initial and final states. Fig. 6.4 depicts this.

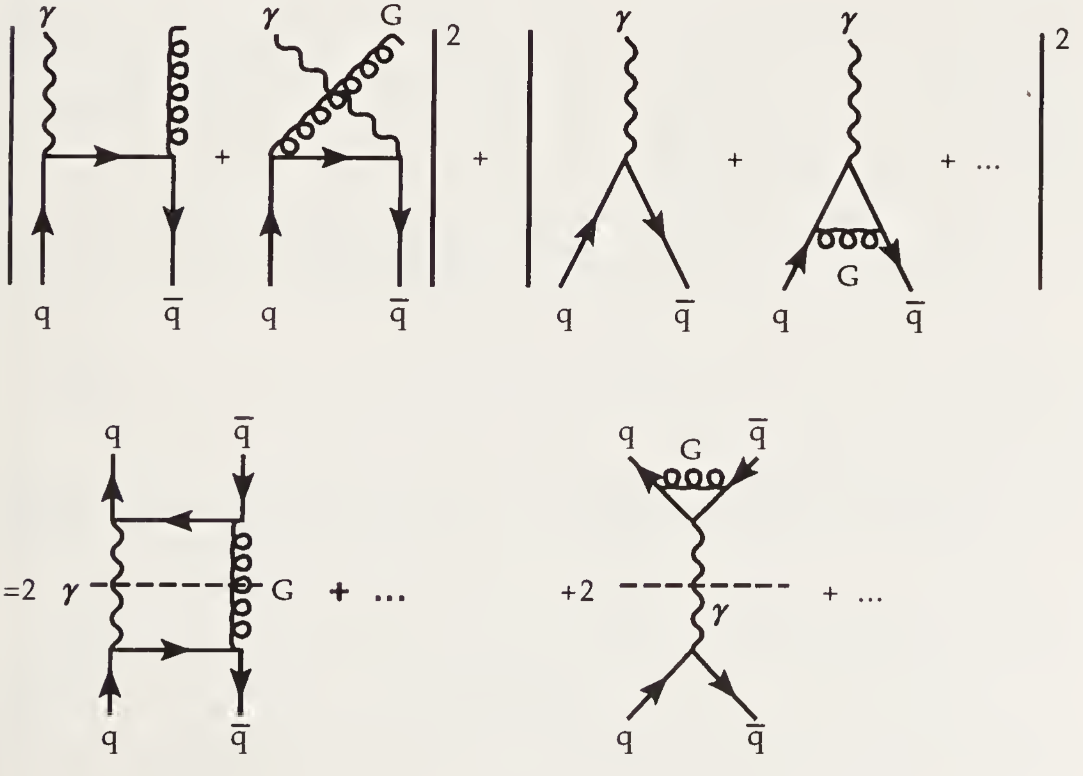


Fig. 6.4. All graphs leading to the same final state interfere with one another. Therefore both photon + gluon graphs and radiative corrections contribute to the order α_s. The Bloch–Nordsieck theorem states that infrared divergencies of these contributions cancel

The validity of the Bloch–Nordsieck theorem is of great relevance for the whole of perturbative QCD. Nonetheless a general complete proof seems still to be missing and is clearly confronted with problems for the Drell–Yan process in particular. We are unable to explore this theoretical question in more detail here, but the Bloch–Nordsieck theorem holds for the lowest-order term, investigated in the following, as will be shown.

The procedure to show that two divergences cancel is as follows. First a regulator is introduced to render the contributions finite. Then, after summing the contibutions, we send the regulator to zero or infinity depending on the nature of the divergences (infrared or ultraviolet).

In our case we choose a finite gluon mass m_G as a regulator. Then the calculation of the graphs in Fig. 6.2 must be repeated with a finite gluon mass. We shall skip this and give the final result only. (Usually such calculations are performed with computer algebra programs such as REDUCE). The result is

$$\begin{aligned} &\mathrm{d}\sigma'(\mathrm{q}+\bar{\mathrm{q}}\to\gamma^*+G)\\ &\quad=\frac{\alpha\alpha_s e_q^2\pi}{s^2}\frac{8}{9}\left[\frac{u}{t}+\frac{t}{u}+\frac{2(M^2+m_G^2)s}{ut}-M^2m_G^2\left(\frac{1}{t^2}+\frac{1}{u^2}\right)\right]\mathrm{d}t \end{aligned}\tag{6.12}$$

with

$$u=M^2+m_G^2-s-t$$

and

$$\mathrm{d}^2\sigma''_{\mathrm{DY}}(\mathrm{q}+\bar{\mathrm{q}}\to\mu^++\mu^-+G)=\frac{\alpha}{3\pi M^2}\,\mathrm{d}\sigma'(\mathrm{q}+\bar{\mathrm{q}}\to\gamma^*+G)\,\mathrm{d}M^2 \quad .$$

Obviously the changes in the cross section are minor. In particular, the troublesome terms with $1/t$ are unchanged, and an even more troublesome term in $1/t^2$ has appeared. The crucial question is thus: What are the bounds of the integral for finite m_G. We choose the center-of-momentum frame to calculate these bounds. Here $\boldsymbol{k}_\gamma=-\boldsymbol{k}_G=\boldsymbol{k}$, and thus

$$s=(E_\gamma+E_G)^2=k^2+M^2+2\sqrt{k^2+M^2}\sqrt{k^2+m_G^2}+k^2+m_G^2 \quad , \tag{6.13}$$

$$(s-M^2-m_G^2-2k^2)^2=4(k^2+M^2)(k^2+m_G^2) \quad . \tag{6.14}$$

From this, k is calculated:

$$k=\sqrt{\frac{(s-M^2-m_G^2)^2-4M^2m_G^2}{4s}} \quad . \tag{6.15}$$

Bounds for t arc calculated from

$$t=(p_q-k_\gamma)^2=M^2-2(E_\gamma E_q-E_q k\cos\theta) \quad , \tag{6.16}$$

$$t_{\mathrm{min/max}}=M^2-2E_\gamma E_q\mp 2E_q k \quad . \tag{6.17}$$

We put $2E_q = \sqrt{s}$ and we use (6.15):

$$\begin{aligned} t_{\mathrm{min/max}} &= M^2 - \sqrt{\frac{s}{4s}}\left(\sqrt{(s-M^2-m_G^2)^2-4M^2m_G^2+4sM^2}\right. \\ &\qquad \left.\pm\sqrt{(s-M^2-m_G^2)^2-4M^2m_G^2}\right) \\ &= M^2-\frac{1}{2}\left(\sqrt{(s+M^2-m_G^2)^2}\pm\sqrt{(s-M^2-m_G^2)^2-4M^2m_G^2}\right) \\ &= -\frac{1}{2}\left((s-M^2-m_G^2)\pm\sqrt{(s-M^2-m_G^2)^2-4M^2m_G^2}\right) \quad . \qquad (6.18) \end{aligned}$$

Obviously $t_{\max}$ is now nonzero, so the integral is finite. As claimed above, $t_{\max}$ goes to zero as $m_G \to 0$. This result can be rewritten by intoducing the quantities $a = M^2/s$ and $b = m_G^2/M^2$:

$$t_{\mathrm{min/max}} = -\frac{M^2}{2a}\left(1-a-ab\pm\sqrt{(1-a)^2+ba(ba-2-2a)}\right) \quad . \qquad (6.19)$$

The integral over t is explicitly calculated in Problem 6.2, leading to

$$\begin{aligned} \sigma' = \frac{8\pi\alpha\alpha_s e_q^2}{9s}&\left[2\frac{1+B^2/s^2}{1-B/s}\ln\left(\frac{s-B+\sqrt{(s-B)^2-4M^2m_G^2}}{s-B-\sqrt{(s-B)^2-4M^2m_G^2}}\right)\right. \\ &\left.-4\sqrt{\left(1-\frac{B}{s}\right)^2-4\frac{M^2m_G^2}{s^2}}\,\right] \qquad (6.20) \end{aligned}$$

with

$$B = M^2 + m_G^2 \quad .$$

To get the total Drell–Yan contribution we have to substitute $\tau s = x_a x_b s$ for s, where s is now the total four-momentum squared of the colliding hadrons. We also have to insert the relation between $\sigma(\mathrm{q}+\bar{\mathrm{q}} \to \gamma^*)$ and $\mathrm{d}\sigma/\mathrm{d}M^2(\mathrm{q}+\bar{\mathrm{q}} \to \gamma^* \to \mu^+\mu^-)$, which is derived in Problem 6.3. For negligible lepton masses this is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}M^2}(\mathrm{q}+\bar{\mathrm{q}} \to \mu^+\mu^-) = \sigma(\mathrm{q}+\bar{\mathrm{q}} \to \gamma^*)\frac{\alpha}{3\pi M^2}\mathrm{d}M^2 \quad . \qquad (6.21)$$

In our problem, i.e., the first order gluon correction, M^2 corresponds to the invariant mass of the γ^* and gluon. We therefore replace it by τs. Thus the correction to the annihilation part of the Drell–Yan cross section is

$$\begin{aligned} \frac{\mathrm{d}\Delta\sigma_{\mathrm{DY}}}{\mathrm{d}M^2} &= \frac{\alpha}{3\pi}\int \mathrm{d}x_b \int \mathrm{d}x_a \frac{\sigma'(s\to\tau s)}{\tau s}\sum_q [q_a(x_a)\bar{q}_b(x_b)+\bar{q}_a(x_a)q_b(x_b)] \\ &= \frac{\alpha}{3\pi}\int \mathrm{d}\tau \int \frac{\mathrm{d}x_a}{x_a}\frac{\sigma'(\tau s)}{\tau s}\sum_q [q_a(x_a)\bar{q}_b(x_b)+\bar{q}_a(x_a)q_b(x_b)] \quad . \qquad (6.22) \end{aligned}$$

We still have to specify the integration boundaries. Obviously it holds that

$$\tau s \geq (M + m_G)^2 \approx M^2 \left(1 + 2\frac{m_G}{M}\right) \quad . \tag{6.23}$$

On the other hand τs can be very large. We write

$$\tau s \leq M^2 N \tag{6.24}$$

and later on set $1/N \approx 0$. Because $q_a(x_a)$, $\bar{q}_b(x_b)$, and so on are smooth functions of x_a and x_b, we can neglect the small shifts in x_a and x_b induced by the difference between τs and M^2, i.e., we can neglect the τ dependence of the $x_a x_b$:

$$\begin{aligned}
\frac{\mathrm{d}\Delta\sigma_{\mathrm{DY}}}{\mathrm{d}M^2} = {} & \frac{\alpha}{3\pi}\int\limits_0^1 \frac{\mathrm{d}x_a}{x_a} \sum_q \left[q_a(x_a)\bar{q}_b(x_b) + \bar{q}_a(x_a)q_b(x_b)\right] \\
& \times \int\limits_{M^2(1+2\frac{m_G}{M})/s}^{M^2N/s} \mathrm{d}\tau \, \frac{8\pi\alpha\alpha_s e_q^2}{9(\tau s)^2} \left[2\frac{1+\left(\frac{M^2+m_G^2}{\tau s}\right)^2}{1-\frac{M^2+m_G^2}{\tau s}} \right. \\
& \times \ln\left(\frac{1-\frac{M^2+m_G^2}{s\tau} + \sqrt{\left(1-\frac{M^2+m_G^2}{s\tau}\right)^2 - 4\frac{M^2m_G^2}{s^2\tau^2}}}{1-\frac{M^2+m_G^2}{s\tau} - \sqrt{\left(1-\frac{M^2+m_G^2}{s\tau}\right)^2 - 4\frac{M^2m_G^2}{s^2\tau^2}}} \right) \\
& \left. -4\sqrt{\left(1-\frac{M^2+m_G^2}{s\tau}\right)^2 - 4\frac{M^2m_G^2}{s^2\tau^2}} \right] \quad .
\end{aligned} \tag{6.25}$$

The τ integral is expanded in $\beta = m_0^2/M^2$. To this end we substitute $\tau = M^2/sr$:

$$\begin{aligned}
I = {} & \frac{8\pi\alpha\alpha_s e_q^2}{9M^2} \int\limits_{\frac{1}{N}\sim 0}^{1-2\sqrt{\beta}} \mathrm{d}r \left[2\frac{1+r^2(1+\beta)^2}{1-r(1+\beta)} \right. \\
& \times \ln\left(\frac{1-r(1+\beta)+\sqrt{[1-r(1+\beta)]^2 - 4\beta r^2}}{1-r(1+\beta)-\sqrt{[1-r(1+\beta)]^2 - 4\beta r^2}} \right) \\
& \left. - 4\sqrt{[1-r(1+\beta)]^2 - 4\beta r^2} \right] \quad .
\end{aligned} \tag{6.26}$$

We study the limit $\beta \to \infty$. We shall now take this limit, and in doing so we shall make sure that we keep all the constant terms as well as those proportional to logarithms of β. Powers of β are, however, neglected.

We make the substitution $r \to r/(1+\beta)$. The factors $1/(1+\beta)$ lead only to corrections proportional to β, which we neglect. The leading term, which we have to take care to treat correctly, is that proportional to $2\sqrt{\beta}$.

$$
\begin{aligned}
I = \frac{8\pi\alpha\alpha_s e_q^2}{9M^2} \int\limits_0^{1-2\sqrt{\beta}} \mathrm{d}r \Bigg[& 2\frac{1+r^2}{1-r} \ln\left(\frac{2(1-r)}{2\beta r^2/(1-r)}\right) \\
& - 2\frac{1+r^2}{1-r} \ln\left(\frac{2(1-r)}{1-r+\sqrt{(1-r)^2-4\beta r^2}}\right) \\
& + 2\frac{1+r^2}{1-r} \ln\left(\frac{2\beta r^2}{(1-r)^2\left[1-\sqrt{1-4\beta r^2/(1-r)^2}\right]}\right) \\
& - 4\sqrt{(1-r)^2-4\beta r^2}\Bigg] \quad . \qquad (6.27)
\end{aligned}
$$

In the terms on the right-hand side it is tempting to neglect $4\beta r^2$. This would be incorrect, however, since it is of the same order as $(1-r)^2$ at the upper integration boundary. To treat this carefully we have split up the logarithmic term into three parts. The first contains the approximation valid for $(1-r)^2 > 4\beta r^2$. The other two terms contain the ratio between the approximate expressions and the exact ones. We investigate these first:

$$
\begin{aligned}
I_1 = 2 \int\limits_0^{1-2\sqrt{\beta}} \mathrm{d}r \frac{1+r^2}{1-r} \Bigg[& \ln\left(\frac{2\beta r^2}{(1-r)^2\left[1-\sqrt{1-4\beta r^2/(1-r)^2}\right]}\right) \\
& - \ln\left(\frac{2(1-r)}{(1-r)+\sqrt{(1-r)^2-4\beta r^2}}\right)\Bigg] \quad . \qquad (6.28)
\end{aligned}
$$

Obviously these terms vanish unless $r \approx 1$. Thus we set $r^2 = 1$ except in $(1-r)$. Next we substitute $1-r=t$:

$$
I_1 = 4 \int\limits_{2\sqrt{\beta}}^{1} \mathrm{d}t \, \frac{1}{t} \left[\ln\left(\frac{2\beta}{t^2\left(1-\sqrt{1-4\beta/t^2}\right)}\right) - \ln\left(\frac{2}{1+\sqrt{1-4\beta/t^2}}\right)\right] \quad . \qquad (6.29)
$$

Now we substitute $t = 2\sqrt{\beta}/u$, $\mathrm{d}t = -2\sqrt{\beta}\,\mathrm{d}u/u^2 = -t\,\mathrm{d}u/u$:

$$
I_1 = -4 \int\limits_1^{2\sqrt{\beta}} \frac{\mathrm{d}u}{u} \left[\ln\left(\frac{u^2}{2\left(1-\sqrt{1-u^2}\right)}\right) - \ln\left(\frac{2}{1+\sqrt{1-u^2}}\right)\right] \quad . \qquad (6.30)
$$

In this expression it is now safe to let $\sqrt{\beta}$ go to zero. We are left with finite integrals only. To do these we further substitute $u = \sqrt{1-z^2}$, $\mathrm{d}u = -z\,\mathrm{d}z/u$:

$$
\begin{aligned}
I_1 &= 4\int_0^1 \mathrm{d}z \left[\ln\left(\frac{1-z^2}{2(1-z)}\right) - \ln\left(\frac{2}{1+z}\right)\right]\frac{z}{1+z^2} \\
&= 2\int_0^1 \mathrm{d}z\; 2\ln\left[\frac{1}{2}(1+z)\right]\left(\frac{1}{1-z}-\frac{1}{1+z}\right) \\
&= -2\ln^2\left(\frac{1+z}{2}\right)\Big|_0^1 + 4\int_0^1 \mathrm{d}z\,\frac{1}{1-z}\ln\left(\frac{1+z}{2}\right) \quad .
\end{aligned}
\tag{6.31}
$$

We finally substitute $z = 2u - 1$ and use the fact that

$$
-\int_{1-x}^{1} \mathrm{d}t\,\frac{\ln(t)}{1-t} = \mathrm{Li}_2(x) \quad , \quad \mathrm{Li}_2\left(\frac{1}{2}\right) = \frac{\pi^2}{12} - \frac{1}{2}\ln^2(2)
\tag{6.32}
$$

Here $\mathrm{Li}_2(x) = \sum_{n=1}^{\infty} x^2/n^2$, $|x| < 1$ is the dilogarithm, also called Struve function.

$$
\begin{aligned}
I_1 &= 2\ln^2(2) + 8\int_{1/2}^{1} \mathrm{d}u\,\frac{1}{2(1-u)}\ln(u) \\
&= 2\ln^2(2) - 4\left(\frac{\pi^2}{12} - \frac{1}{2}\ln^2(2)\right) = -\frac{\pi^2}{3} + 4\ln^2(2) \quad .
\end{aligned}
\tag{6.33}
$$

The last integral in (6.27) is trivial, since one can directly set $\beta = 0$:

$$
I_2 = -4\int_0^{1-2\sqrt{\beta}} \mathrm{d}r\,\sqrt{(1-r)^2 - 4\beta r^2} \to -4\int_0^1 \mathrm{d}r\,(1-r) = -2 \quad .
\tag{6.34}
$$

The remaining integral leads to an expression found in good integration tables:

$$
\begin{aligned}
I_3 &= -2\int_0^{1-2\sqrt{\beta}} \mathrm{d}r\,\frac{1+r^2}{1-r}\left[\ln(\beta) + 2\ln(r) - 2\ln(1-r)\right] \\
&= -2\ln(\beta)\int_0^{1-2\sqrt{\beta}} \mathrm{d}r\,\frac{(r-1)^2 + 2(r-1) + 2}{1-r} \\
&\quad - 4\int_0^{1-2\sqrt{\beta}} \mathrm{d}r\,\frac{1+r^2}{1-r}\ln(r) + 4\int_0^{1-2\sqrt{\beta}} \mathrm{d}r\,\frac{1+r^2}{1-r}\ln(1-r) \\
&= -2\ln(\beta)\left[\frac{1}{2} - 2 - 2\ln\left(2\sqrt{\beta}\right) + O\left(\sqrt{\beta}\right)\right] \\
&\quad - 4\left(-\frac{\pi^2}{3} + \frac{5}{4}\right) + 4\left[-\ln^2\left(2\sqrt{\beta}\right) + \frac{7}{4}\right]
\end{aligned}
$$

$$
\begin{aligned}
&= 2 - 4\ln^2\left(2\sqrt{\beta}\right) + \frac{4}{3}\pi^2 + 3\ln(\beta) + 4\ln(\beta)\ln\left(2\sqrt{\beta}\right) \\
&= \frac{4}{3}\pi^2 + 2 + 3\ln(\beta) + 4\ln\left(2\sqrt{\beta}\right)\left[\ln(\beta) - \ln\left(2\sqrt{\beta}\right)\right] \\
&= \frac{4}{3}\pi^2 + 2 + 3\ln(\beta) + 4\left[\ln\left(\sqrt{\beta}\right) + \ln(2)\right]\left[\ln\left(\sqrt{\beta}\right) - \ln(2)\right] \\
&= \frac{4}{3}\pi^2 + 2 + 3\ln(\beta) + 4\left[\ln^2\left(\sqrt{\beta}\right) - \ln^2(2)\right] \\
&= \ln^2(\beta) + 3\ln(\beta) + \frac{4}{3}\pi^2 + 2 - 4\ln^2(2) \quad . \qquad (6.35)
\end{aligned}
$$

Finally we put the results (6.33), (6.34) and (6.35) together to get

$$
I = \left(\ln^2(\beta) + 3\ln(\beta) + \pi^2\right)\frac{8\pi\alpha\alpha_s e_q^2}{9M} \quad , \qquad (6.36)
$$

or

$$
\begin{aligned}
\frac{d\Delta\sigma_{DY}}{dM^2} = \frac{\alpha}{3\pi}\int_0^1 \frac{dx_a}{x_a}\sum_q \left[q_a(x_a)\bar{q}_b(x_b) + \bar{q}_a(x_a)q_b(x_b)\right] \\
\times \frac{8\pi\alpha\alpha_s e_q^2}{9M^2}\left[\ln^2(\beta) + 3\ln(\beta) + \pi^2\right] \quad . \qquad (6.37)
\end{aligned}
$$

We hope that this explicit example shows how the logarithms associated with the infrared divergences can be isolated and where one has to be careful to avoid errors. We shall next discuss how the Block–Nordsieck theorem works for this specific example, i.e., how the logarithmic terms in (6.37) are cancelled by those of the radiative corrections. However, we shall give only an overview of the explicit calculatioon of the radiative corrections, since it contains nothing new.

According to Fig. 6.4 the graphs we have to calculate are those contributing to the vertex correction to first order (see Fig. 6.5). If we introduce a momentum cutoff Λ, after the introduction of Feynman parameters the two relevant integrals are

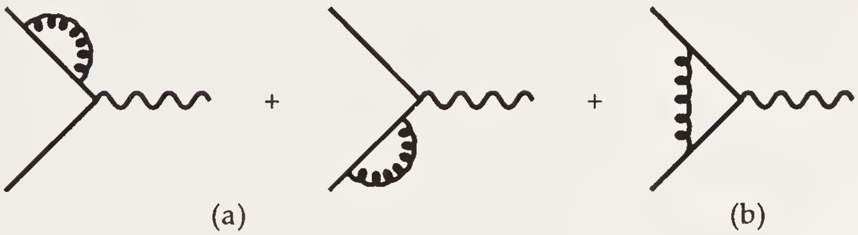


Fig. 6.5a,b. First-order radiative vertex corrections

$$
\begin{aligned}
\Sigma(p) &= -g_s^2\int_{m_g^2}^{\Lambda^2} ds \int_0^1 dx \int \frac{d^4k}{(2\pi)^4}\frac{2(1-x)[-2(1-x)]}{\left[k^2 + p^2x(1-x) - s(1-x)\right]^3} \\
&= \ldots = -\frac{\alpha_s}{4\pi}\ln\left(\frac{\Lambda^2}{m_g^2}\right) \qquad (6.38)
\end{aligned}
$$

and

$$\alpha \to \alpha\left(1 + \frac{2\alpha_s}{3\pi}\int\limits_{m_g^2}^{\Lambda^2} \mathrm{d}s \int\limits_0^1 \mathrm{d}y \int\limits_0^1 \mathrm{d}x \left\{\frac{x(1-x)q^2\left[2+2x^2y(1-y)-2x\right]}{\left[-y(1-y)x^2q^2+s(1-x)\right]^2}\right.\right.$$

$$\left.\left. + \frac{2x(1-x)}{\left[-y(1-y)x^2q^2+s(1-x)\right]}\right\}\right)$$

$$= \alpha\left\{1 + \frac{2\alpha_s}{3\pi}\left[-\ln^2\left(\frac{m_g^2}{-q^2}\right) - 3\ln\left(\frac{m_g^2}{-q^2}\right) - \frac{7}{2} - \frac{2\pi^2}{3}\right]\right\} \quad . \tag{6.39}$$

In contrast to deep inelastic scattering the four-momentum squared $\left(q^2 = M^2\right)$ is now positive:

$$\ln\left(\frac{m_g^2}{-q^2}\right) = \mathrm{Re}\left[\ln\left(-\frac{m_g^2}{M^2}\right)\right] = \mathrm{Re}\left[\ln(\beta) - \mathrm{i}\pi\right] = \ln(\beta) \tag{6.40}$$

$$\ln^2\left(\frac{m_g^2}{-q^2}\right) = \mathrm{Re}\left[\ln^2\left(-\frac{m_g^2}{M^2}\right)\right] = \mathrm{Re}\left[\ln^2(\beta) - 2\mathrm{i}\pi\ln(\beta) - \pi^2\right]$$

$$= \ln^2(\beta) - \pi^2 \quad , \tag{6.41}$$

so the correction to α can be written as

$$\alpha \to \alpha\left\{1 + \frac{2\alpha_s}{3\pi}\left[-\ln^2(\beta) - 3\ln(\beta) - \frac{7}{2} + \frac{\pi^2}{3}\right]\right\} \quad . \tag{6.42}$$

The corresponding correction to the Drell–Yan cross section is

$$\frac{\mathrm{d}\Delta\sigma_{\mathrm{DY}}(\mathrm{rad.-corr.})}{\mathrm{d}M^2} = \frac{\alpha}{3\pi}\int\limits_0^1 \frac{\mathrm{d}x_a}{x_a}\sum_q \left[q_a(x_a)\bar{q}_b(x_b) + \bar{q}_a(x_a)q_b(x_b)\right]$$

$$\times \frac{8\pi\alpha\alpha_s e_q^2}{9M^2}\left(-\ln^2(\beta) - 3\ln(\beta) - \frac{7}{2} + \frac{\pi^2}{3}\right) \quad . \tag{6.43}$$

Adding this result to (6.37) gives the final result:

$$\frac{\mathrm{d}\Delta\sigma_{\mathrm{DY}}(\mathrm{total})}{\mathrm{d}M^2} = \frac{2\alpha_s}{3\pi}\int\limits_0^1 \frac{\mathrm{d}x_a}{x_a}\sum_q \left[q_a(x_a)\bar{q}_b(x_b) + \bar{q}_a(x_a)q_b(x_b)\right]$$

$$\times \frac{4\pi\alpha^2 e_q^2}{9M^2}\left(\frac{4}{3}\pi^2 - \frac{7}{2}\right) \quad , \tag{6.44}$$

which implies that the Drell–Yan cross section is just multiplied by a constant factor, which can be identified with the K factor

$$K(\text{1st order}) = 1 + \frac{2\alpha_s}{3\pi}\left(\frac{4\pi^2}{3} - \frac{7}{2}\right) = 1 + 2.05\,\alpha_s \quad . \tag{6.45}$$

If we insert $\alpha_s \approx 0.3$, we get $K \approx 1.6$, which is already a good step towards the experimental value $K \approx 2$. Thus it can be hoped that the perturbative expression will indeed converge to the correct K factor. One can continue by pursuing this tedious calculation order by order. Instead we wish to address a much more fundamental approach, namely that it is often possible to sum up certain radiative corrections exactly, i.e., to all orders in the coupling constant. Such techniques are called "resummation techniques" and are a central issue of current research in QCD. We shall sketch the relevant arguments for a comparably simple problem. The result will be that K is proportional to $\exp[2\pi\alpha_s(M^2)/3]$, implying that

$$\begin{aligned} K &= \exp\left[\frac{2}{3}\pi\alpha_s\left(M^2\right)\right]\left[1-\frac{2}{3}\pi\alpha_s+\frac{2\alpha_s}{3\pi}\left(\frac{4\pi^2}{3}-\frac{7}{2}\right)\right] \\ &= \exp\left[\frac{2}{3}\pi\alpha_s\left(M^2\right)\right]\left[1-\alpha_s\left(\frac{2\pi}{3}-\frac{8\pi}{9}+\frac{7}{3\pi}\right)\right] \\ &= \exp\left[\frac{2}{3}\pi\alpha_s\left(M^2\right)\right]\left(1-0.0446\,\alpha_s\right) \quad . \end{aligned} \tag{6.46}$$

This suggests an excellent convergence of the perturbative expansion *after* the resummed exponential factor is isolated. Furthermore the numerical result for $\alpha_s \approx 0.25-0.35$,

$$\begin{aligned} K(\text{1st order + resummation}) &= \exp\left[\frac{2}{3}\pi\alpha_s\left(M^2\right)\right]\left(1-0.0446\,\alpha_s\right) \\ &= 1.7-2.0 \quad , \end{aligned} \tag{6.47}$$

agrees well with the empirical values. In fact (6.47) is further improved by also taking into account the second order and serves then to determine $\alpha_s(M^2)$ or Λ_{QCD}. The starting point of the argument is as follows. The terms absorbed into the exponent are the π^2 terms in (6.43) and (6.44). According to (6.40) and (6.41) these arise from the analytic continuation of the logarithm to negative arguments. Thus it is sufficient to keep only the logarithmic terms to high orders, to sum them up, and to continue them then to negative arguments. This procedure is called "leading logarithm-approximation" (LLA) and is the standard procedure for resummation. Next we shall discuss the correction due to gluon emission in the production of a quark–antiquark pair by a massive photon. The resummed corrections for the Drell–Yan process can then be obtained from the result by analytic continuation. The calculation can be found in Exercise 6.4. Here we simply give the result for the decay rate Γ:

$$\frac{\mathrm{d}^2\Gamma}{\mathrm{d}x_1\mathrm{d}x_2} = 3\alpha e_q^2 M\frac{32\alpha_s}{3\pi}\frac{x_1^2+x_2^2}{(1-2x_1)(1-2x_2)} \quad , \tag{6.48}$$

where x_1 and x_2 are the energy fractions carried by the quark and antiquark. The gluon energy fraction is $x_3 = 1-x_1-x_2$. We treat the decay of the massive photon in its rest frame, i.e., $k_\gamma = (M,0,0,0)$. Obviously this expression diverges if x_1 or x_2 approaches 1/2. This is obvious from Fig. 6.6 for the case $x_1 \longrightarrow 1/2$. The intermediate propagator becomes on-mass-shell, if

$$\begin{aligned} (k-p_1)^2 &= (M(1-x_1),0,0,-x_1M)^2 = M^2((1-x_1)^2-x_1^2) \\ &= M^2(1-2x_1) = 0 \quad . \end{aligned} \tag{6.49}$$

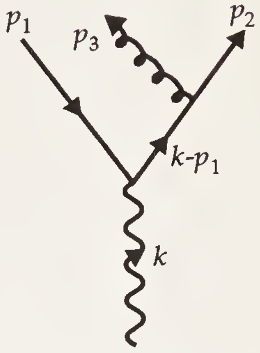


Fig. 6.6. Illustration of the origin of divergences in (6.48)

For a finite gluon energy only x_1 or x_2 can be equal to 1/2. Thus (6.48) has two leading, logarithmically diverging contributions: one for $x_1 \to 1/2$ and one for $x_2 \to 1/2$. We change to the coordinates

$$\begin{aligned} t &= (p_2+p_3)^2 = (k-p_1)^2 = M^2(1-2x_1) \\ z &= \left.\frac{E_2}{E_2+E_3}\right|_{x_1 \to \frac{1}{2}} = \frac{E_2}{\frac{1}{2}M} = 2x_2 \qquad (6.50) \\ 4M^2 &\frac{\mathrm{d}^2}{\mathrm{d}x_1\,\mathrm{d}x_2} \Rightarrow \frac{\mathrm{d}^2}{\mathrm{d}t\,\mathrm{d}z} \qquad (6.51) \\ \frac{\mathrm{d}^2\Gamma}{\mathrm{d}t\,\mathrm{d}z} &= 3\alpha e_q^2\,\frac{2\alpha_s}{3\pi}\,\frac{1+z^2}{t(1-z)} \\ &= 3\alpha e_q^2\,\frac{\alpha_s}{2\pi t}\left(\frac{4}{3}\,\frac{1+z^2}{1-z}\right) \qquad (6.52) \end{aligned}$$

The new variable z has obviously the meaning of the fraction of the energy of the original quark state carried by the final quark state after gluon emission. Therefore it is not suprising that we recover in (6.51) again the quark splitting function $P_{qq}(z)$ known from our derivation of the GLAP-equations. As usual the virtual corrrection adds effectively the '+' prescription to the bracket in (6.52). Let us now turn to multi-gluon emission. An important point to know is that for the specific gauge we used in Problem 6.4 only a single amplitude contributes, namely S_{22}. The t-integration leads to a logarithmic divergence. With (6.50) we get

$$\begin{aligned} S_{11} &= 8c\frac{1-2x_1}{1-x_2} = 8c\frac{t}{M^2(1-z)} && \text{no logarithmic divergence,} \\ S_{22} &= 8c\frac{(1+z^2)M^2}{t(1-z)} && \text{logarithmic divergence,} \\ S_{12} &= -16c\frac{1}{1-2x_2} = -\frac{16c}{1-z} && \text{no logarithmic divergence,} \\ c &= 4g^2e^2e_{\mathrm{q}}^2 \quad . \end{aligned} \qquad (6.53)$$

Thus for the leading term the two-gluon emission probability is just the product of two one-gluon emission probabilities. There are no interference effects. This fact is intimately connected to the form of the GLAP-equations. If interference effects were important, one would not get such a simple equation on the level of distribution functions, i.e. probabilities. Therefore in leading logarithmic approximation one simply gets

$$\frac{\mathrm{d}^n\Gamma}{\mathrm{d}z_1\ldots\mathrm{d}z_n\,\mathrm{d}t_1\ldots\mathrm{d}t_n} = 3\alpha e_q^2\left(\frac{\alpha_s(t_1)}{2\pi t_1}P_{qq}(z_1)\right)\ldots\left(\frac{\alpha_s(t_n)}{2\pi t_n}P_{qq}(z_n)\right) \quad . \qquad (6.54)$$

We define the t_i such that $t_1 \geq t_2 \geq \ldots \geq t_n$. Note that for our specific gauge only those graphs in which the quark lines couple to gluons contribute in the LLA. Graphs in which the antiquark couples to the gluons give no leading-log contribution. This fact has no deep physical meaning but is exclusively due to the specific gauge used. Equation (12) in Problem 6.4 contains p_2/x_2 and not p_1/x_1. This asymmetry generates the asymmetry in the contributing processes. Now let us define $S(\Theta)$ to be the probability that after n-gluon emissions the outgoing quark still has a scattering angle $\theta = p_\perp/p_\parallel < \Theta$, with $p_\parallel$ and $p_\perp$ being defined with respect to the jet axis, and the original quark momentum, respectively. For the following we need the relation between t and the scattering angle θ. With

$$p_2 = \left(z(1-x_1)M, k_\perp, \sqrt{z^2(1-x_1)^2M^2 - k_\perp^2}\right) \quad , \tag{6.55}$$

$$p_3 = \left((1-z)(1-x_1)M, -k_\perp, \sqrt{(1-z)^2(1-x_1)^2M^2 - k_\perp^2}\right) \tag{6.56}$$

we find that [using $k_\perp^2 \ll (1-z)^2(1-x_1)^2M^2 \sim (1-z_1)^2M^2/4$]

$$\begin{aligned} t = 2\, p_2 \cdot p_3 = 2 \Bigg\{ & z(1-z)(1-x_1)^2M^2 + k_\perp^2 - \left[z(1-x_1)M - \frac{k_\perp^2}{2z(1-x_1)M}\right] \\ & \times \left[(1-z)(1-x_1)M - \frac{k_\perp^2}{2(1-z)(1-x_1)M}\right]\Bigg\} \\ \approx {} & 2\left(k_\perp^2 + \frac{k_\perp^2(1-z)}{2z} + \frac{k_\perp^2 z}{2(1-z)}\right) \approx \frac{k_\perp^2}{(1-z)} \quad . \end{aligned} \tag{6.57}$$

We are interested in typical bremsstrahlung processes, i.e., $z \approx 1$. For $z \to 0$, multigluon emission plays no role. The angle is related to $k_\perp$ by

$$\vartheta = \frac{k_\perp}{z(1-x_1)M} \approx \frac{2k_\perp}{M} < \Theta \tag{6.58}$$

$$\Rightarrow k_\perp^2 < \frac{\Theta^2}{4}M^2 \quad . \tag{6.59}$$

Remember that the leading-logarithm approximation (LLA) we are investigating is only valid for $x_1 \approx 1/2$. As a consequence of (6.58) we find that

$$(1-z) < \frac{\Theta^2M^2}{4t} \quad . \tag{6.60}$$

Instead of calculating $S(\Theta)$ we calculate the conjugate probability

$$\begin{aligned} S_1(\Theta) = 1 - T_1(\Theta) &= 1 - \int\limits_{\frac{\Theta^2M^2}{4}}^{M^2} \mathrm{d}t \int\limits_0^{1-\frac{\Theta^2M^2}{4t}} \mathrm{d}z\, \frac{2\alpha_s}{3\pi}\frac{1+z^2}{t(1-z)} \\ &\approx 1 - \frac{4\alpha_s}{3\pi}\int\limits_{\frac{\Theta^2M^2}{4}}^{M^2} \mathrm{d}t \int\limits_0^{1-\frac{\Theta^2M^2}{4t}} \mathrm{d}z\, \frac{1}{t(1-z)} \\ &= 1 - \frac{4\alpha_s}{3\pi}\int\limits_{\frac{\Theta^2M^2}{4}}^{M^2} \mathrm{d}t\, \frac{1}{t}\left[-\ln\left(\frac{\Theta^2M^2}{4t}\right)\right] \end{aligned}$$

$$= 1 - \frac{4\alpha_s}{3\pi}\left[-\ln\left(\frac{\Theta^2}{4}\right)\ln(t) + \frac{1}{2}\ln^2\left(\frac{t}{M^2}\right)\right]_{\frac{\Theta^2 M^2}{4}}^{M^2}$$

$$= 1 - \frac{4\alpha_s}{3\pi}\left[\ln^2\left(\frac{\Theta^2}{4}\right) - \frac{1}{2}\ln^2\left(\frac{\Theta^2}{4}\right)\right]$$

$$= 1 - \frac{2\alpha_s}{3\pi}\ln^2\left(\frac{\Theta^2}{4}\right) \quad . \tag{6.61}$$

For $T(\Theta)$ one does not have to treat the subtleties of the radiative corrections canceling the infrared divergencies for $t \longrightarrow 0, z \longrightarrow 1$. For finite Θ the integrals for $T_1(\Theta)$ are finite. The probability is strongly peaked toward $\theta \ll \Theta$. Therefore we can approximate

$$\frac{S_{n+1}(\Theta)}{S_n(\Theta)} \sim 1 - T_1(\Theta) \quad , \tag{6.62}$$

which results in

$$S_n(\Theta) \sim \left[1 - T_1(\Theta)\right]^n \quad . \tag{6.63}$$

We still have to correct for the fact that the t_j were ordered according to

$$t_1 \geq t_2 \geq t_3 \ldots \geq t_n \quad , \tag{6.64}$$

so that only one of the $n!$ permutations contained in (6.62) is chosen:

$$S_n(\Theta) \sim \frac{1}{n!}\left[1 - T_1(\Theta)\right]^n \quad . \tag{6.65}$$

With (6.64) we can now easily sum the gluon emission to all orders:

$$S(\Theta) \sim \sum_n S_n(\Theta) = \mathrm{e}^{1-T_1(\Theta)} = \mathrm{e} \times \mathrm{e}^{-\frac{2}{3}\frac{\alpha_s}{\pi}\ln^2\left(\frac{\Theta^2}{4}\right)} \quad . \tag{6.66}$$

Obviously this still lacks the proper normalization factor. For $T_1(\Theta) \equiv 0$, $S(\Theta)$ must clearly be equal to one. Thus the missing normalisation factor is e^{-1} and the final expression reads

$$S(\Theta) = \mathrm{e}^{-\frac{2}{3}\frac{\alpha_s}{\pi}\ln^2\left(\frac{\Theta^2}{4}\right)} = \mathrm{e}^{-\frac{2}{3}\frac{\alpha_s}{\pi}\ln^2\left(\frac{k_{\perp\mathrm{min}}}{M}\right)^2} \quad . \tag{6.67}$$

This expression has the structure of a form factor and is usually interpreted as such. It is then called "Sudakov form factor". Clearly (6.66) suppresses the emission of collinear soft gluons, rendering the cross sections finite. The Sudakov form factor is probably the best-known example of resummation in QCD and is of great phenomenological importance in describing jet formation correctly. As we shall discuss at the end of this section the renormalization group equation imposes the condition that the exponential must always be of the form (6.66). The only problem remaining is to find the correct analytic continuation for the given dynamics. In our case this is relatively easy. Figure 6.7 compares the graph for which we have calculated (6.65) with the gluon corrections to the Drell–Yan cross section.

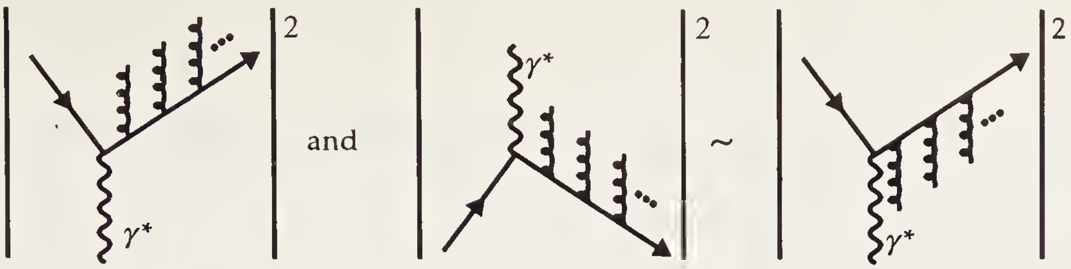


Fig. 6.7. Comparison of the gluonic corrections for two Drell–Yan processes

Obviously the outgoing gluons just have to be substituted by incoming ones. This corresponds to the analytic continuation to $z \geq 1$ rather than $z \leq 1$. According to (6.59) this is equivalent to the continuation $\Theta^2 \longrightarrow -\Theta^2$, which in turn implies that

$$\begin{aligned} &\exp\left\{-\frac{2}{3}\frac{\alpha_s}{\pi}\,\mathrm{Re}\left[\ln^2\left(\frac{\Theta^2}{4}\right)\right]\right\} \Rightarrow \exp\left\{-\frac{2}{3}\frac{\alpha_s}{\pi}\left[-\pi^2+\ln^2\left(\frac{\Theta^2}{4}\right)\right]\right\} \\ &\qquad \Rightarrow \exp\left(\frac{2}{3}\alpha_s\,\pi\right)\exp\left[-\frac{2}{3}\frac{\alpha_s}{\pi}\ln^2\left(\frac{\Theta^2}{4}\right)\right] \quad . \qquad (6.68) \end{aligned}$$

As we have integrated over t respectively Θ the second factor is contained in the α_s corrections in the last bracket in (6.46). The remaining exponential factor is indeed $\exp(2\pi\alpha_s/3)$ as stated in (6.46).

The appearence of this factor is a typical property of higher order corrections and the standard result of resummation. We also wish to analyze it in a different way starting from the renormalization group equation, for a general form factor with $q^2 < 0$:

$$\left(\frac{\partial}{\partial\tau}+\beta\frac{\partial}{\partial g}-\gamma\right)F\left[\tau,q^2,\alpha\right]=0 \quad . \qquad (6.69)$$

Here τ is the logarithm of the renormalization scale $\tau = -\ln(\mu)$. Note that τ is defined with a minus sign. The general solution to (6.69) is

$$F\left[\tau,q^2,\alpha\right]=\tilde{F}\left[q^2,\bar{\alpha}(\tau,\alpha)\right]\exp\left(\int_0^\tau d\vartheta\;\gamma\left[q^2,\bar{\alpha}(\vartheta,\alpha)\right]\right) \quad , \qquad (6.70)$$

with the running coupling constant

$$\bar{\alpha}(\vartheta)=\frac{\alpha}{1-\frac{\alpha}{\pi}\frac{11N_c-2N_f}{6}\vartheta} \qquad (6.71)$$

fulfilling

$$\begin{aligned} \left(\frac{\partial}{\partial\tau}+\beta\frac{\partial}{\partial g}\right)\bar{\alpha}(\tau) &= \left(\frac{\partial}{\partial\tau}+\beta\frac{2g}{4\pi}\frac{\partial}{\partial\alpha}\right)\bar{\alpha}(\tau) \\ &= \frac{\alpha\frac{\alpha}{\pi}\frac{33-2N_f}{6}}{\left(1-\frac{\alpha}{\pi}\frac{33-2N_f}{6}\tau\right)^2}+\frac{2g}{4\pi}\left[-\left(11-\frac{2}{3}N_f\right)\frac{g^2}{(4\pi)^2}\right] \\ &\quad \times\frac{1}{\left(1+\frac{\alpha}{\pi}\frac{33-2N_f}{6}\tau\right)^2} \end{aligned}$$

$$= \frac{\frac{\alpha^2}{2\pi}\left(11-\frac{2}{3}N_f\right)}{\left(1-\frac{\alpha}{2\pi}\left(11-\frac{2}{3}N_f\right)\tau\right)^2}(1-1)=0 \tag{6.72}$$

where we have used the explicit form of the β function from (5.181). The trick is now to expand (6.70) in orders of τ

$$\begin{aligned} & F\left[0,q^2,\alpha\right]+\tau F'\left[0,q^2,\alpha\right]+O\left(\alpha^2\right) \\ & \quad = F\left[0,q^2,\alpha\right]\left(1+\tau\gamma\left[q^2,\bar{\alpha}(\tau,\alpha)\right]+\ldots\right) \\ & \Rightarrow \gamma\left[q^2,\bar{\alpha}(\tau,\alpha)\right] = \frac{F'[0,q^2,\alpha]}{F[0,q^2,\alpha]} \quad . \end{aligned} \tag{6.73}$$

Thus the anomalous dimensions can be obtained by calculating perturbatively the vertex corrections and then dividing the coefficient of $\ln(\mu)=\tau$ by the constant term. As the logarithmic divergent term must have one order of α_s more than the constant, it follows that

$$\gamma[q,\alpha_s]=\alpha_s f(q^2) \tag{6.74}$$

and

$$\gamma\left[q,\bar{\alpha}_s(\vartheta,\alpha)\right]=\bar{\alpha}_s(\vartheta,\alpha)f(q^2)\,. \tag{6.75}$$

To continue this argument one would have to calculate $f(q^2)$ explicitly, which we do not want to do again. The point we wish to make is that from (6.70) it is already clear that any form factor must have an exponential factor. The "Sudakov form factor" is consequently strongly connected to the existence of anomalous dimensions in QCD.

To complete this chapter let us add a few brief remarks on the phenomenology of the Drell–Yan process.

As discussed in Sect. 4.2, the parton model predicts quite simple behavior for the total cross section:

$$R=\frac{\sigma\left(\mathrm{e^+e^-}\to \mathrm{q\bar{q}}\right)}{\sigma\left(\mathrm{e^+e^-}\to\mu^+\mu^-\right)}=\begin{cases} 2 & E_{\mathrm{cm}}>2\ \mathrm{GeV} \\ 10/3 & E_{\mathrm{cm}}>4\ \mathrm{GeV} \\ 11/3 & E_{\mathrm{cm}}>10\ \mathrm{GeV} \end{cases} \quad . \tag{6.76}$$

We have just discussed the K factor that describes the resummed QCD corrections to this process. We sketch in Fig. 6.8 how these corrections improve the agreement with the data. The QCD corrections increase R, i.e., the K factor is always larger than 1. The change in the theoretical prediction for R has been included in Fig. 6.8 (at $\Lambda=2$ GeV). To obtain a truly precise prediction, QED radiative corrections, higher QCD corrections, mass corrections, and so on must also be included. These increase R somewhat more, in particular for small E_{cm}.

Figure 6.8 shows that the parton–model predictions are in general slightly too low while the QCD-corrected results clearly gives a better description of the data. In this comparison one must take into account that in some energy regions the presence of resonances can induce large variations of R.

Choosing a special subgroup of the Drell–Yan processes, one can perform a far more specific test of QCD. This has been done most successfully for 3-jet events, which are characterized by the property that most of the energy is carried

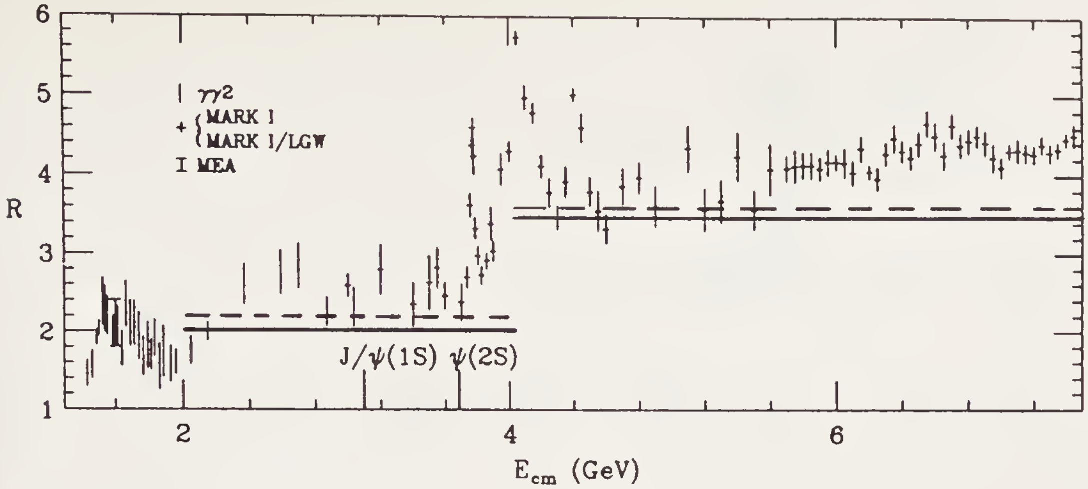
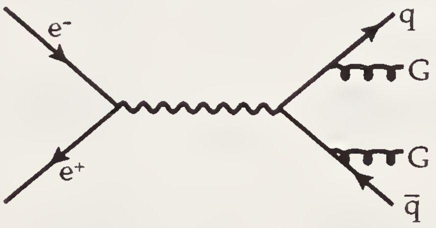
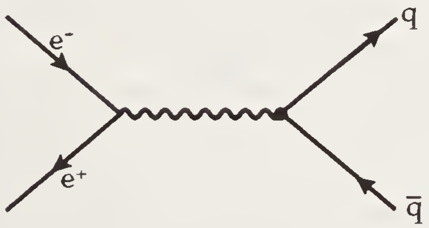


Fig. 6.8. The prediction of the parton model for the Drell–Yan process including QCD corrections. The data are taken from Review of Particle Properties, Phys. Rev. **D45** (1992)

by hadrons, leading to three narrow and clearly distinguished angular regions. The simplest process that can lead to a 3-jet event is the gluon bremsstrahlung whose graph is shown in Fig. 6.9. However, not all bremsstrahlung events are identified as 3-jet events, but only those in which the angle between gluon jet and quark jets is sufficiently large. Conversely, the graph



also contributes to 3-jet events, as long as one gluon jet is not resolved from the others. Finally the simplest Drell–Yan process,



contains, with a certain statistical weight, events with unbalanced momentum distributions generated by the nonperturbative process of hadronization. A calculation of the 3-jet probability and a determination of the strong coupling constant α_s is therefore an extensive and model-dependent venture.

A typical calculation goes like this:

1. One calculates elementary QCD processes in perturbation theory. To order α_s^2, these are the graphs of Fig. 6.10, for example.
2. One calculates hadronization using one or more of the special computer codes developed for this purpose. These programs are based on some basic assump-

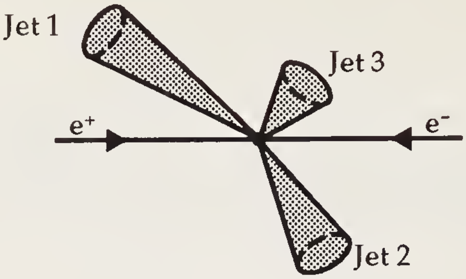


Fig. 6.9. A typical 3-jet event in the laboratory system

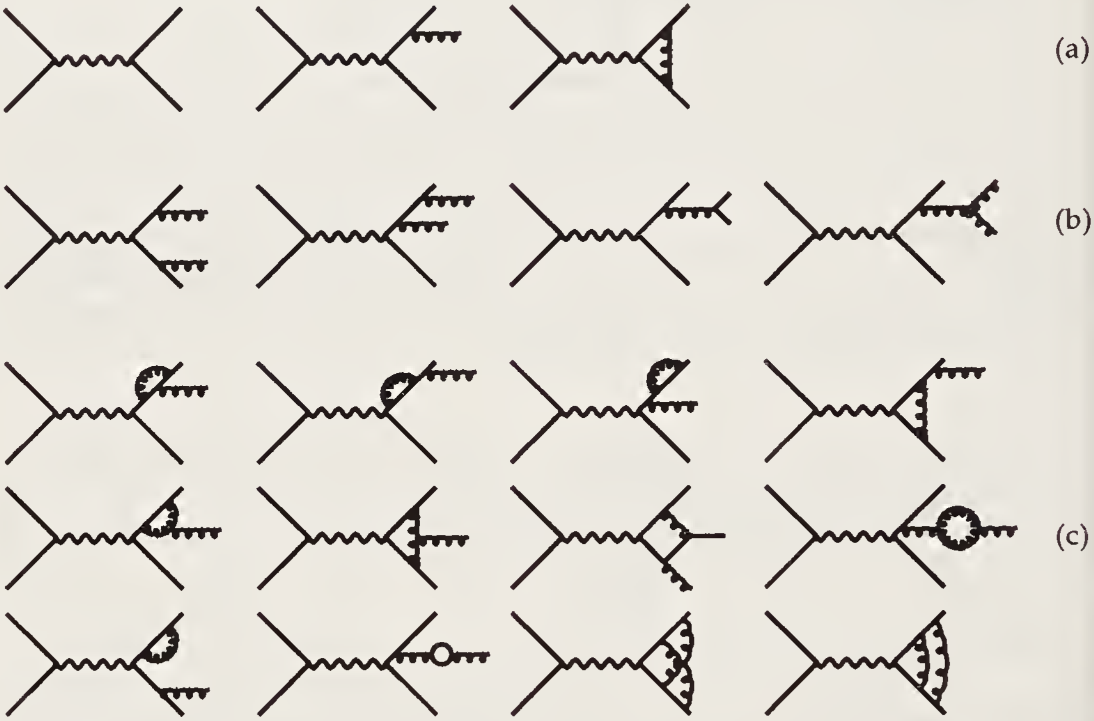


Fig. 6.10a–c. Elementary graphs of the Drell–Yan process up to order α_s^2

tions about the creation and decay of color strings between separating quarks (see Fig. 6.11). A typical code of this type depends on a substantial number of parameters and gives good fits for a large number of processes.

3. Finally one determines by the experimentally used definition of 3-jet events the corresponding contribution to this class of events. Here also experimental sensitivities enter.

In this way the result of Fig. 6.12[1] was obtained. Here the energy asymmetry (whose exact definition is not important to us) is shown against the total center–of–mass energy. The two curves result from different hadronization programs. The coupling constant used was

$$\alpha_s = \frac{12\pi}{(33-2N_f)\ln\left(\frac{Q^2}{\Lambda^2}\right)}\left\{1-\frac{6(153-19N_f)}{(33-2N_f)^2}\frac{\ln\left[\ln\left(\frac{Q^2}{\Lambda^2}\right)\right]}{\ln\left(\frac{Q^2}{\Lambda^2}\right)}\right\} \tag{6.77}$$

with $N_f = 5$ and $\Lambda = 100$ MeV.

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[1] See M. Chen: Int. J. Mod. Phys. **A1**, 669 (1986).

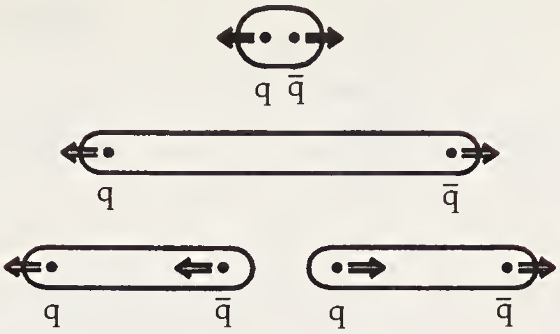


Fig. 6.11. Creation and breaking-up of a color-string

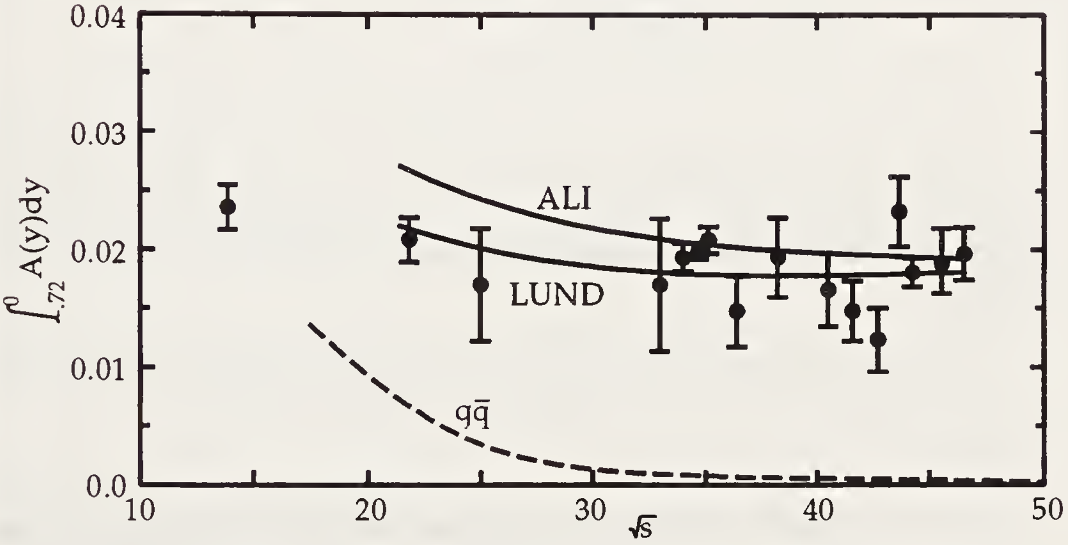


Fig. 6.12. The fit of experimental 3-jet events with Λ = 100 MeV. LUND and ALI denote the two hadronization routines used

Equation (6.75) is the correct form of the running coupling constant when two-loop processes are taken into account. As the Drell–Yan process is calculated to order α_s^2, the coupling constant must also be determined to this order, to be consistent.

Using this fit to determine Λ gives

$$\Lambda = 100 \pm 30 \begin{array}{c}+60\\-45\end{array} \text{ MeV} \quad \text{or} \quad \alpha_s(E_{\text{cm}} = 44 \text{ GeV}) = 0.12 \pm 0.02 \qquad (6.78)$$

Other analyses give somewhat different values for Λ. We do not want to enter this controversy here; we simply wanted to illustrate the procedure used.

In a similar way one attempts to describe semi–exclusive hadron–hadron scattering processes with hadrons in the final state. But here the uncertainties and model-dependence is even larger, so that frequently nothing more than rather general statements result. Such statements may concern, for example, the power of Q^2 with which a specific cross section decays. These processes are therefore less important as tests of QCD, with the exception of specific reactions like charmonium production (see Fig. 6.13).

The Drell–Yan process offers a number of possible ways to tests QCD. Although the results are quite convincing as a whole, their quality bears no comparison with tests of QED or of the Glashow–Salam–Weinberg model. Another problem is that there is no competing model to QCD at this time. One does not therefore

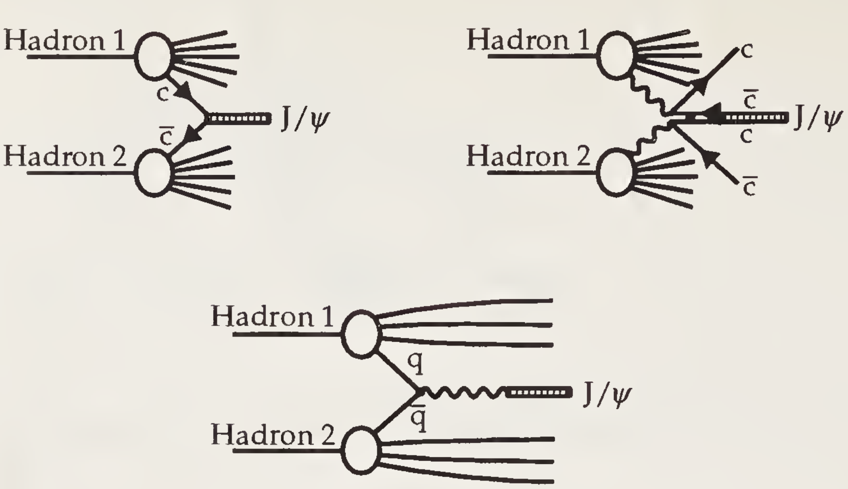


Fig. 6.13. Charmonium creation in hadron–hadron reactions

know how specific the predictions of QCD really are. There may be other models that describe scaling violations, 3-jet events, and so on equally well. Conversely many theoreticians consider the fact that there is only one theoretical candidate for describing quark–quark interactions to be the strongest argument for the correctness of QCD.

As a result of the analysis described here we conclude that Λ lies in a region between 100 and 200 MeV. (Λ depends on the renormalization procedure. The values given are for dimensional regularization ($\overline{MS}$), see Sect. 5.1).

EXERCISE

6.1 The Drell-Yan Cross-Section

Problem. Derive the elementary Drell–Yan cross section in (6.5).

Solution. According to the techniques given in Chap. 2, we get

$$d\sigma = \frac{e^4}{2M^2}\frac{d^3q_1}{(2\pi)^3 2E}\frac{d^3q_2}{(2\pi)^3}\frac{1}{4}\frac{F^2}{M^4}(2\pi)^4\delta^4(p_1+p_2-q_1-q_2) \quad . \tag{1}$$

Here we have simply neglected the quark and gluon masses, thus getting the flux factor to $1/M^2$. (M is the invariant mass of the two quarks.) Spin–averaging yields a factor of 1/4, and F^2 is calculated from

$$\begin{aligned} F^2 &= \mathrm{tr}(\not{p}_1\gamma_\mu\not{p}_2\gamma_\nu)\,\mathrm{tr}(\not{q}_1\gamma^\mu\not{q}_2\gamma^\nu) \\ &= 16(p_1^\mu p_2^\nu + p_2^\mu p_1^\nu - g^{\mu\nu}\,p_1\cdot p_2)(q_{1\mu}q_{2\nu}+q_{2\mu}q_{1\nu}-g_{\mu\nu}\,q_1\cdot q_2) \\ &= 32(q_1\cdot p_1\,q_2\cdot p_2 + q_1\cdot p_2\,q_2\cdot p_1) \quad . \end{aligned} \tag{2}$$

This is simplest in the center-of-momentum frame, where

$$d\sigma = \frac{e^4}{8M^2}\frac{d^3q_1}{(2\pi)^2M^6}\,F^2\,\delta(2q_1^0-2p_1^0) \quad . \tag{3}$$

Exercise 6.1.

With $p_1 = (E, 0, 0, E)$, $p_2 = (E, 0, 0, -E)$, $q_1 = (E, E\sin\theta\cos\phi, E\sin\theta\sin\phi, E\cos\theta)$ and $q_2 = (E, -E\sin\theta\cos\phi, -E\sin\theta\sin\phi, -E\cos\theta)$, it follows that

$$\begin{aligned} F^2 &= 32\left[E^4(1-\cos\theta)^2 + E^4(1+\cos\theta)^2\right] \\ &= 64E^4(1+\cos^2\theta) \quad , \end{aligned} \tag{4}$$

and therefore

$$\mathrm{d}\sigma = \frac{e^4}{8M^2}\frac{\mathrm{d}(\cos\theta)\,E^2}{4\pi M^6}\,64E^4(1+\cos^2\theta) \tag{5}$$

$$\sigma = \frac{e^4}{8M^2}\frac{1}{4\pi}\frac{8}{3} = \frac{4\pi\alpha^2}{3M^2} \quad . \tag{6}$$

This is the relation we used in (6.5).

EXERCISE

6.2 The One-Gluon Contribution to the Drell–Yan Cross-Section

Problem. Calculate the total Drell–Yan cross section for one-gluon emission, i.e., integrate (6.12) with bounds (6.17).

Solution. As s, u, and t are related by (introducing B for simplicity)

$$s + t + u = M^2 + m_G^2 = B \quad , \tag{1}$$

we have to choose which variables we want to treat as independent. We use t and s, thus the integral becomes

$$\begin{aligned} &\frac{8\pi\alpha_s\alpha e_q^2}{9s^2}\int\limits_{t_{\min}}^{t_{\max}} dt\left[\frac{B-s}{t} - 1 + \frac{t+s-B-s-B}{B-s-t}\right. \\ &\qquad \left. + \frac{2Bs}{t(B-s-t)} - M^2m_G^2\left(\frac{1}{t^2} + \frac{1}{(B-s-t)^2}\right)\right] \\ &= \frac{8\pi\alpha_s\alpha e_q^2}{9s^2}\left[(B-s)\ln t - 2t + (s-B)\ln(B-s-t)\right. \\ &\qquad \left. + \frac{2Bs}{B-s}\left[\ln(t) - \ln(B-s-t)\right] + M^2m_G^2\left(-\frac{1}{t} + \frac{1}{B-s-t}\right)\right]_{t_{\min}}^{t_{\max}} \end{aligned} \tag{2}$$

We now write

$$t_{\max/\min} = -\frac{1}{2}\left[s - B \mp \sqrt{(s-B)^2 - 4M^2m_G^2}\right] \quad , \tag{3}$$

Exercise 6.2.

$$\sigma' = \frac{8\pi\alpha_s\alpha e_q^2}{9s^2}\left[\frac{B^2+s^2}{B-s}\ln\left|\frac{t}{B-s-t}\right| - 2t + M^2 m_G^2\left(\frac{1}{B-s-t}-\frac{1}{t}\right)\right]_{t_{\min}}^{t_{\max}}$$

$$= \frac{8\pi\alpha_s\alpha e_q^2}{9s^2}\left[2\frac{B^2+s^2}{B-s}\ln\left(\frac{s-B-\sqrt{(s-B)^2-4M^2m_G^2}}{s-B+\sqrt{(s-B)^2-4M^2m_G^2}}\right)\right.$$

$$\left. - 2\sqrt{(s-B)^2-4M^2m_G^2} + M^2 m_G^2(-2)\frac{2\sqrt{(s-B)^2-4M^2m_G^2}}{(s-B)^2-(s-B)^2+4M^2m_G^2}\,2\right] \tag{4}$$

and obtain the final expression

$$\sigma' = \frac{8\pi\alpha\alpha_s e_q^2}{9s}\left[2\frac{1+B^2/s^2}{1-B/s}\ln\left(\frac{s-B+\sqrt{(s-B)^2-4M^2m_G^2}}{s-B-\sqrt{(s-B)^2-4M^2m_G^2}}\right)\right.$$

$$\left. - 4\sqrt{(s-B)^2-4M^2m_G^2}\right] \quad , \tag{5}$$

$$B = M^2 + m_G^2 \quad .$$

Obviously for $m_G \to 0$ the logarithmic divergence is recovered.

EXERCISE

6.3 The Drell-Yan Process as Decay of a Heavy Photon

Problem. Derive the relation between the cross sections for $q+\bar{q} \to \gamma^* \to \mu^+ + \mu^-$ and $q + \bar{q} \to \gamma^*$:

$$\frac{d\sigma}{dM^2}(q+\bar{q} \to \mu^+ + \mu^-) = \sigma(q+\bar{q} \to \gamma^*)\,\frac{\alpha}{3\pi M^2}\left(1-\frac{4m^2}{M^2}\right)^{3/2} dM^2 \quad .$$

Solution. The decay cross section for $q + \bar{q} \to \gamma^* \to \mu^+ + \mu^-$ is

$$d\sigma = \frac{Q_q^2 e^4}{M^4}\,\frac{1}{2p_0 2p_0'}\,W^{\mu\nu}L_{\mu\nu}(2\pi)^{4-6}\delta(p+p'-k-k')\,\frac{d^3k\,d^3k'}{4k_0k_0'} \quad , \tag{1}$$

where p, p', k, and k' are the quark, antiquark, μ^-, and μ^+ momenta. The virtual photon has the four-momentum q with $q^2 = M^2$. We now multiply the right-hand side of (1) by

$$\delta^4(q-k-k')\,\frac{d^3q}{2q_0}\,dM^2 = \delta^4(q-k-k')\,d^4q = 1 \quad , \tag{2}$$

Exercise 6.3.

giving

$$\begin{aligned}\mathrm{d}\sigma = {} & \frac{Q_q^2 e^4}{M^4}\,\frac{1}{16k_0k_0'p_0p_0'\,2q_0}\,(2\pi)^{-2}\,W^{\mu\nu}L_{\mu\nu} \\ & \times \delta^4(p+p'-q)\,\delta^4(q-k-k')\mathrm{d}^3k\,\mathrm{d}^3k'\,\mathrm{d}^3q\,\mathrm{d}M^2 \quad .\end{aligned} \tag{3}$$

We perform the k and k' integrations

$$\begin{aligned}I^{\mu\nu} &= \int \mathrm{d}^3k\,\mathrm{d}^3k'\,(2\pi)^{-2}L^{\mu\nu}\,\delta^4(q-k-k') \\ &= \frac{4}{(2\pi)^2}\int \mathrm{d}^3k\,\mathrm{d}^3k'\,\left[k^\mu k'^\nu + k^\nu k'^\mu - g^{\mu\nu}(k\cdot k'+m^2)\right]\delta^4(q-k-k') \quad .\end{aligned} \tag{4}$$

We then evaluate this expression in the rest frame of the virtual photon, $q = (M,0,0,0)$, $k^0 = M/2$:

$$\begin{aligned}I^{\mu\nu} &= \frac{1}{\pi^2}\int \mathrm{d}^3k\,\mathrm{d}^3k'\,\big[-2k^\mu k^\nu + k^\mu q^\nu \\ &\quad + q^\mu k^\nu - g^{\mu\nu}(-k^2 + k\cdot q + m^2)\big]\delta(q_0-2k_0)\,\delta^3(\boldsymbol{k}-\boldsymbol{k}') \\ &= \frac{1}{\pi^2}\int \mathrm{d}^3k\,\big[2k^\mu k^0\delta^{\nu 0} + 2k^\nu k^0\delta^{\mu 0} - 2k^\mu k^\nu \\ &\quad - g^{\mu\nu}(k^0q^0 - 2m^2)\big]\delta(q^0-2k^0) \quad .\end{aligned} \tag{5}$$

Because the integral over just one component of $\boldsymbol{k}$ vanishes, this expression is diagonal. For $\mu = \nu = 0$ we get

$$I^{00} = \frac{1}{\pi^2}\,4\pi\int \mathrm{d}k\,k^2\left[2(k^0)^2 - 2(k^0)^2\right]\delta\left(q^0 - 2\sqrt{\boldsymbol{k}^2+m^2}\right) = 0 \quad . \tag{6}$$

This is also required by current conservation:

$$q_\nu I^{\mu\nu} = q_\mu I^{\mu\nu} = 0 \quad . \tag{7}$$

For $\mu = \nu = j$ we find that

$$I^{jj} = \frac{1}{\pi^2}\int \mathrm{d}^3k\left[-2(k^j)^2 + \left(\frac{M^2}{2} - 2m^2\right)\right]\delta\left(q^0 - 2\sqrt{\boldsymbol{k}^2+m^2}\right) \quad . \tag{8}$$

We substitute $k^jk^j \to 1/3\,\boldsymbol{k}^2$ and get

$$\begin{aligned}I^{jj} &= \frac{4}{\pi}\int \mathrm{d}k\,k^2\left(-\frac{2}{3}\boldsymbol{k}^2 + 2\boldsymbol{k}^2\right)\delta\left(M - 2\sqrt{\boldsymbol{k}^2+m^2}\right) \\ &= \frac{4}{\pi}\,\frac{4}{3}\left(\frac{M^2}{4} - m^2\right)\frac{\sqrt{\frac{M^2}{4}-m^2}\,\frac{M}{2}}{2} \\ &= \frac{2}{3\pi}\,\frac{M^4}{4}\left(1 - \frac{4m^2}{M^2}\right)^{3/2} \quad .\end{aligned} \tag{9}$$

As $I^{\mu\nu}$ must be proportional to $g^{\mu\nu} - q^\mu q^\nu/q^2$, owing to current conservation, it is easy to obtain (note that $g_{\mu\nu}I^{\mu\nu} = I^{00} - 3I^{jj}$)

Exercise 6.3.

$$I^{\mu\nu} = \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) \frac{M^4}{6\pi} \left(1 - \frac{4m^2}{M^2}\right)^{3/2} \quad . \tag{10}$$

Substituting this result back into (3) gives

$$\begin{aligned} \mathrm{d}\sigma = {} & \frac{Q_q^2 e^4}{M^4} \frac{1}{4M^2 p_0 p_0' 2q_0} W_{\mu\nu}\, \delta^4(p + p' - q) \\ & \times \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) \frac{M^4}{6\pi} \left(1 - \frac{4m^2}{M^2}\right)^{3/2} \mathrm{d}^3 q \, \mathrm{d}M^2 \quad . \end{aligned} \tag{11}$$

To bring this into the form of $\mathrm{d}\sigma(\mathrm{q} + \bar{\mathrm{q}} \rightarrow \gamma^*)$, we substitute

$$\left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) = \sum_\varepsilon \varepsilon^\mu \varepsilon^{*\nu} \tag{12}$$

and get

$$\begin{aligned} \mathrm{d}\sigma = {} & \frac{e^2}{6\pi M^2} \left[Q_q^2 e^2 W_{\mu\nu} \sum_\varepsilon \varepsilon^\mu \varepsilon^{*\nu} (2\pi)^{4-3} \delta^4(p + p' - q) \frac{d^3 q}{4p_0 p_0' \cdot 2q_0} \right] \\ & \times \mathrm{d}M^2 \frac{1}{2\pi} \left(1 - \frac{4m^2}{M^2}\right)^{3/2} \\ = {} & \frac{\alpha}{3\pi M^2} \left(1 - \frac{4m^2}{M^2}\right)^{3/2} \sigma(\mathrm{q} + \bar{\mathrm{q}} \rightarrow \gamma^*) \, \mathrm{d}M^2 \quad . \end{aligned} \tag{13}$$

This completes our proof.

EXERCISE

6.4 Heavy Gluon Decay Into Quark, Antiquark and Gluon

Problem. Derive (6.48).

Solution. The differential decay rate for a one body decay in its rest frame is simply:

$$\begin{aligned} \mathrm{d}\Gamma = {} & \frac{1}{2M} |M_1 + M_2|^2 \frac{\mathrm{d}^3 p_1}{(2\pi)^3 2E_1} \frac{\mathrm{d}^3 p_2}{(2\pi)^3 2E_2} \frac{\mathrm{d}^3 p_3}{(2\pi)^3 2E_3} \\ & \times (2\pi)^4 \, \delta^4(k - p_1 - p_2 - p_3) \end{aligned} \tag{1}$$

with

$$M_1 = \bar{u}(p_2, s_2) \left(-\mathrm{i} g \gamma_\nu \frac{\lambda^a}{2}\right) \frac{\mathrm{i}(\not{p}_2 + \not{p}_3)}{(p_2 + p_3)^2} (-\mathrm{i}\, e\, e_q\, \gamma_\mu)\, v(p_1, s_1)\, \varepsilon_\mu\, \tilde{\varepsilon}_\nu \tag{2}$$

and

$$M_2 = -\bar{u}(p_2, s_2) (-\mathrm{i}\, e\, e_q\, \gamma_\mu) \frac{\mathrm{i}(\not{p}_1 + \not{p}_3)}{(p_1 + p_3)^2} \left(-\mathrm{i} g \gamma_\nu \frac{\lambda^a}{2}\right) v(p_1, s_1)\, \varepsilon_\mu\, \tilde{\varepsilon}_\nu \quad . \tag{3}$$

Exercise 6.4.

The factor $1/2M$ in (1) is just the photon field normalization. In the photon rest system all four-products are rather simple for zero masses.

$$\begin{aligned} 2p_1\cdot p_3 &= (p_1+p_3)^2 = (k_\gamma - p_2)^2 = k_\gamma^2 - 2\,k_\gamma\cdot p_2 \\ &= M^2 - 2M^2x_2 = M^2(1-2x_2) \\ 2p_2\cdot p_3 &= M^2(1-2x_1) \\ 2p_1\cdot p_2 &= M^2(1-2x_3) \quad , \end{aligned} \tag{4}$$

where x_i is the energy fraction carried by the parton i. Due to (4) $|M_1+M_2|^2$ depends only on x_1, x_2 and $x_3 = 1-x_1-x_2$, such that the phase space integral is very easy.

$$\begin{aligned} &\int \frac{\mathrm{d}^3p_1}{(2\pi)^3 2E_1}\,\frac{\mathrm{d}^3p_2}{(2\pi)^3 2E_2}\,\frac{\mathrm{d}^3p_3}{(2\pi)^3 2E_3}(2\pi)^4\,\delta^4(k-p_1-p_2-p_3)\cdots \\ &= \int \frac{\mathrm{d}^3p_1\,\mathrm{d}^3p_2}{2^5\pi^5 2^3 x_1x_2(1-x_1-x_2)M^3}\delta\left(M - x_1M - x_2M - \sqrt{(\boldsymbol{p}_1-\boldsymbol{p}_2)^2}\right)\cdots \\ &= \int \frac{4\pi E_1^2\,\mathrm{d}E_1\;2\pi E_2^2\,\mathrm{d}E_2\,\mathrm{d}\cos(\theta_{12})}{2^8\pi^5 x_1x_2(1-x_1-x_2)M^3} \\ &\quad\times\delta\left(M(1-x_1-x_2) - M\sqrt{2x_1x_2\cos(\theta_{12})}\right)\cdots \\ &= \int \frac{x_1^2M^3x_2^2M^3\,\mathrm{d}x_1\,\mathrm{d}x_2}{2^5\pi^3x_1x_2(1-x_1-x_2)M^3}\,\frac{\sqrt{2x_1x_2\cos(\theta_{12})}}{Mx_1x_2}\bigg|_{\sqrt{2x_1x_2\cos(\theta_{12})}=1-x_1-x_2}\cdots \\ &= \int \frac{M^2\,\mathrm{d}x_1\,\mathrm{d}x_2}{2^5\pi^3}\cdots \end{aligned} \tag{5}$$

such that the decay rate reads

$$\mathrm{d}\Gamma = \frac{M}{64\pi^3}\,\mathrm{d}x_1\,\mathrm{d}x_2\,|M_1+M_2|^2 \quad . \tag{6}$$

We now make an special choice for the polarization vectors of the gluon. The reason for this will become clear later

$$\sum_\varepsilon \varepsilon_\mu\varepsilon^*_{\mu'} = -g_{\mu\mu'} + \frac{n_\mu p_{3\mu'} + n_{\mu'}p_{3\mu}}{n\cdot p_3} = \Sigma_{\mu\mu'} \tag{7}$$

$$\sum_{\tilde\varepsilon} \tilde\varepsilon_\nu\tilde\varepsilon^*_{\nu'} = -g_{\nu\nu'} \quad . \tag{8}$$

The choice (7) is equivalent to choosing the axial gauge $n\cdot A = 0$. n_μ is an arbitrary four-vector with $n^2 = 0$ and $n\cdot\varepsilon = 0$. We get

$$\begin{aligned} S_{11} &= \sum_{\text{spins}} M_1M_1^* \\ &= -g^2e^2e_q^2\,\mathrm{tr}\left\{\frac{\lambda^a\lambda^a}{4}\right\}\Sigma^{\mu\mu'}\,\mathrm{tr}\left\{\not{p}_2\gamma_\nu\frac{\not{p}_1+\not{p}_3}{(p_1+p_3)^2}\gamma_\mu\not{p}_1\gamma_{\mu'}\frac{\not{p}_1+\not{p}_3}{(p_1+p_3)^2}\gamma^\nu\right\} , \end{aligned} \tag{9}$$

Exercise 6.4.

$$\begin{aligned}S_{12} &= -\sum_{\text{spins}}(M_1M_2^* + M_2M_1^*)\\ &= 4\,g^2e^2e_q^2\,\Sigma^{\mu\mu'}\left[\mathrm{tr}\left\{\not{p}_2\gamma_\nu\frac{\not{p}_1+\not{p}_3}{(p_1+p_3)^2}\gamma_\mu\not{p}_1\gamma^\nu\frac{\not{p}_2+\not{p}_3}{(p_2+p_3)^2}\gamma_{\mu'}\right\}\right.\\ &\quad\left.+\,\mathrm{tr}\left\{\not{p}_2\gamma_\mu\frac{\not{p}_2+\not{p}_3}{(p_2+p_3)^2}\gamma_\nu\not{p}_1\gamma_{\mu'}\frac{\not{p}_1+\not{p}_3}{(p_1+p_3)^2}\gamma_\nu\right\}\right] \quad , \qquad (10)\\ S_{22} &= \sum_{\text{spins}} M_2M_2^*\\ &= -4\,g^2e^2e_q^2\Sigma^{\mu\mu'}\mathrm{tr}\left\{\not{p}_2\gamma_\mu\frac{\not{p}_2+\not{p}_3}{(p_2+p_3)^2}\,\gamma_\nu\not{p}_1\gamma^\nu\frac{\not{p}_2+\not{p}_3}{(p_2+p_3)^2}\gamma_{\mu'}\right\} \quad . \qquad (11)\end{aligned}$$

We shall use the gauge

$$n_\mu = \left(k_\gamma - \frac{p_2}{2x_2}\right)_\mu \quad , \qquad (12)$$

$$n^2 = M^2 - 2\frac{x_2M^2}{2x_2} + 0 = 0 \quad . \qquad (13)$$

This choice will result in a specially simple result and will thus be justified only a posteriori. From

$$n_\mu\Sigma^{\mu\mu'} = n_{\mu'}\Sigma^{\mu\mu'} = 0 \quad \text{and} \quad p_{3\mu}\Sigma^{\mu\mu'} = 0 \qquad (14)$$

we find that

$$\left(p_1+p_2+p_3-\frac{p_2}{2x_2}\right)_\mu\,\Sigma^{\mu\mu'} = \left(p_1 - p_2\frac{1-2x_2}{2x_2}\right)_\mu\,\Sigma^{\mu\mu'} = 0 \quad . \qquad (15)$$

This allows us to substitute all momentum vectors occurring in the trace according to

$$\begin{aligned}&(p_3)_{\mu'}\,,\ (p_3)_{\mu'} \to 0\\ &\qquad (p_1)_\mu \to (p_2)_\mu\frac{1-2x_2}{2x_2}\\ &\qquad (p_1)_{\mu'} \to (p_2)_{\mu'}\frac{1-2x_2}{2x_2} \quad ,\end{aligned} \qquad (16)$$

which simplifies the calculation substantially. We start with S_{12} and introduce the notation $c = 4g^2e^2e_q^2$.

$$\begin{aligned}S_{11} &= \frac{2c\,\Sigma^{\mu\mu'}}{M^4(1-2x_2)^2}\mathrm{tr}\{\not{p}_2\,(\not{p}_1+\not{p}_3)\,\gamma_\mu\,\not{p}_1\,\gamma_{\mu'}\,(\not{p}_1+\not{p}_3)\}\\ &= \frac{2c\,\Sigma^{\mu\mu'}}{M^4(1-2x_2)^2}\Big[M^2(1-2x_3+1-2x_1)\mathrm{tr}\{\gamma_\mu\,\not{p}_1\,\gamma_{\mu'}\,(\not{p}_1+\not{p}_3)\}\\ &\quad - 2\,p_1\cdot p_3\,\mathrm{tr}\{\not{p}_2\,\gamma_\mu\,\not{p}_1\gamma_{\mu'}\}\Big]\\ &= \frac{2c\,\Sigma^{\mu\mu'}}{M^2(1-2x_2)^2}\Big[2(1-x_3-x_1)4\big(p_{1\mu}\,(p_1+p_3)_{\mu'} + p_{1\mu'}\,(p_1+p_3)_\mu\\ &\quad - g_{\mu\mu'}\,p_1\cdot p_3\big) - \frac{2\,p_1\cdot p_3}{M^2}\,4\big(p_{2\mu}\,p_{1\mu'} + p_{2\mu'}\,p_{1\mu} - g_{\mu\mu'}\,p_2\cdot p_1\big)\Big] \quad . \qquad (17)\end{aligned}$$

Exercise 6.4.

With (14) this reads

$$S_{11} = \frac{2c\,\Sigma^{\mu\mu'}}{M^2(1-2x_2)^2}\Big[16x_2 p_{1\mu}p_{1\mu'} - 8(1-2x_2)p_{2\mu'}p_{1\mu}$$

$$- 4x_2(1-2x_2)M^2 g_{\mu\mu'} + 2(1-2x_2)M^2(1-2x_3)g_{\mu\mu'}\Big] \quad . \tag{18}$$

With (16) and

$$\Sigma^{\mu\mu'} g_{\mu\mu'} = -4 + 2 = -2 \tag{19}$$

this gives

$$\begin{aligned} S_{11} &= \frac{2c}{1-2x_2}(-2)(-4x_2 + 2 - 4x_3) \\ &= -\frac{4c}{1-2x_2}(-2 + 4x_1) \\ &= 8c\,\frac{1-2x_1}{1-2x_2} \quad . \end{aligned} \tag{20}$$

Analogously S_{22} gives

$$S_{22} = \frac{2c\,\Sigma^{\mu\mu'}}{M^4(1-2x_1)^2}\mathrm{tr}\Big\{\not{p}_2\gamma_\mu(\not{p}_2 + \not{p}_3)\not{p}_1(\not{p}_2 + \not{p}_3)\gamma_{\mu'}\Big\} \quad , \tag{21}$$

which is the same as S_{11} with μ ans μ' interchanged and p_1 interchanged with p_2. Thus the analogy to (18) now reads

$$S_{22} = \frac{2c}{M^2(1-2x_1)^2}\Big\{\Big[16x_1 p_{2\mu}p_{2\mu'} - 8(1-2x_1)p_{1\mu'}p_{2\mu}\Big]\Sigma^{\mu\mu'}$$

$$+ 8x_1(1-2x_1)M^2 - 4(1-2x_1)M^2(1-2x_3)\Big\} \quad . \tag{22}$$

Again using (16) this simplifies substantially to

$$S_{22} = \frac{2c}{M^2(1-2x_1)^2}\left\{8p_{2\mu}p_{2\mu'}\left[2x_1 - \frac{(1-2x_2)(1-2x_1)}{2x_2}\right]\right.$$

$$\left. + 4M^2(1-2x_1)(1-2x_2)\right\} \quad . \tag{23}$$

With

$$\begin{aligned} \Sigma^{\mu\mu'} p_{2\mu}p_{2\mu'} &= 2\,\frac{n\cdot p_2\, 2p_3\cdot p_2}{2n\cdot p_3} \\ &= 2\,\frac{M^2x_2\, M^2(1-2x_1)}{M^2(1-2x_2)\frac{1-2x_3}{2x_2}} \\ &= M^2\,\frac{4x_2^2(1-2x_1)}{(1-2x_2)(1-2x_3)} \quad , \end{aligned} \tag{24}$$

Exercise 6.4.

$$
\begin{aligned}
2\,n\cdot p_3 &= 2\,p_1\cdot p_3-(1-2x_2)\frac{2\,p_2\cdot p_3}{2x_2}\\
&= M^2(1-2x_2)\left(1-\frac{1-2x_1}{2x_2}\right)\\
&= M^2\,\frac{1-2x_2}{2x_2}\,(2x_2+2x_1-1)\\
&= M^2\,\frac{(1-2x_2)(1-2x_3)}{2x_2}\quad ,
\end{aligned}
\tag{25}
$$

we obtain

$$
\begin{aligned}
S_{22} &= \frac{2c}{(1-2x_1)^2}\left\{8\frac{4x_2^2(1-2x_1)}{(1-2x_2)(1-2x_3)}\left[2x_1-\frac{(1-2x_2)(1-2x_1)}{2x_2}\right]\right.\\
&\quad\left.+\,4(1-2x_1)(1-2x_2)\right\}\\
&= \frac{8c}{(1-2x_1)(1-2x_2)(1-2x_3)}\left[16x_1\,x_2^2-4(1-2x_2)(1-2x_1)x_2\right.\\
&\quad\left.+\,(1-2x_2)^2(1-2x_3)\right]\\
&= \frac{8c}{(1-2x_1)(1-2x_2)(1-2x_3)}\left[16x_2^2-16x_2^3-16x_3x_2^2\right.\\
&\quad\left.+\,4x_2(1-2x_2)(1-2x_2-2x_3)+(1-2x_2)^2(1-2x_3)\right]\\
&= \frac{8c}{(1-2x_1)(1-2x_2)(1-2x_3)}\left[8x_2^2-16x_3\,x_2^2+4x_2(1-2x_2)(1-2x_3)\right.\\
&\quad\left.+\,(1-2x_2)^2(1-2x_3)\right]\\
&= \frac{8c}{(1-2x_1)(1-2x_2)}\left[8x_2^2+4x_2-8x_2^2+(1-2x_2)^2\right]\\
&= \frac{8c}{(1-2x_1)(1-2x_2)}\left(1+4x_2^2\right)\\
&= 8c\,\frac{1+4x_2^2}{(1-2x_1)(1-2x_2)}\quad .
\end{aligned}
\tag{26}
$$

Finally we calculate S_{12}.

$$
\begin{aligned}
S_{12} &= -\frac{2c\,\Sigma^{\mu\mu'}}{M^4(1-2x_1)(1-2x_2)}2\,\mathrm{tr}\left\{\not{p}_1\not{p}_2\gamma_{\mu'}\left(\not{p}_2+\not{p}_3\right)\left(\not{p}_1+\not{p}_3\right)\gamma_\mu\right\}\\
&= -\frac{4c\,\Sigma^{\mu\mu'}}{M^4(1-2x_1)(1-2x_2)}\mathrm{tr}\left\{\not{p}_1\not{p}_2\gamma_{\mu'}\left(\not{p}_2\not{p}_1+\not{p}_3\not{p}_1+\not{p}_2\not{p}_3\right)\gamma_\mu\right\}\\
&= -\frac{4c\,\Sigma^{\mu\mu'}}{M^4(1-2x_1)(1-2x_2)}\left[2\left(p_2\right)_{\mu'}\mathrm{tr}\{\not{p}_1\not{p}_2\not{p}_1\gamma_\mu\}\right.\\
&\quad\left.+\,2\left(p_1\right)_{\mu}\mathrm{tr}\{\not{p}_1\not{p}_2\gamma_{\mu'}\not{p}_3\}+2\left(p_2\right)_{\mu'}\mathrm{tr}\{\not{p}_1\not{p}_2\not{p}_3\gamma_\mu\}\right]\\
&= -\frac{4c\,\Sigma^{\mu\mu'}}{M^4(1-2x_1)(1-2x_2)}4\left[4\,p_{2\mu'}\,p_1\cdot p_2\,p_{1\mu}+2\,p_{1\mu}\,p_1\cdot p_3\,p_{2\mu'}\right.\\
&\quad\left.+\,2\,p_{1\mu}\,p_{1\mu'}\,p_2\cdot p_3+2\,p_{2\mu'}\,p_{1\mu}\,p_2\cdot p_3-2\,p_{2\mu'}\,p_1\cdot p_3\,p_{2\mu}\right]
\end{aligned}
$$

Exercise 6.4.

$$
\begin{aligned}
&= -\frac{16c\,\Sigma^{\mu\mu'}}{M^2(1-2x_1)(1-2x_2)} p_{2\mu}p_{2\mu'}\left[2\,\frac{(1-2x_2)(1-2x_3)}{2x_2} + \frac{(1-2x_2)^2}{2x_2}\right.\\
&\left. -(1-2x_1)\,\frac{(1-2x_2)^2}{(2x_2)^2} + (1-2x_1)\,\frac{1-2x_2}{2x_2} - (1-2x_2)\right]\\
&= -\frac{16c}{(1-2x_1)(1-2x_2)}\,\frac{4x_2^2(1-2x_1)}{(1-2x_2)(1-2x_3)}\,\frac{1-2x_2}{4x_2^2}\\
&\times\left[4x_2(1-2x_3) + 2x_2(1-2x_2) - (1-2x_1)(1-2x_2) + 2x_2(1-2x_1) - 4x_2^2\right]\\
&= -\frac{16c}{(1-2x_2)(1-2x_3)}\\
&\times\left[4x_2(1-2x_3) + (1-2x_2)(1-2x_3) + 2x_2(1-2x_1-2x_2)\right]\\
&= -\frac{16c}{(1-2x_2)(1-2x_3)}(1-2x_3)\big(4x_2+1-2x_2-2x_2\big)\\
&= -\frac{16c}{1-2x_2} \quad . \qquad (27)
\end{aligned}
$$

This completes our calculation:

$$
\begin{aligned}
\mathrm{d}\Gamma &= \frac{M}{64\pi^3}\,\mathrm{d}x_1\,\mathrm{d}x_2\,\big(S_{11}+S_{12}+S_{22}\big)\\
&= \frac{M}{64\pi^3}\,\mathrm{d}x_1\,\mathrm{d}x_2\,16\pi\alpha_s\,4\pi\alpha\,e_q^2\\
&\quad\times\left[8\,\frac{1-2x_1}{1-2x_2} + 8\,\frac{1+4x_2^2}{(1-2x_2)(1-2x_1)} - 16\,\frac{1}{1-2x_2}\right]\\
&= \frac{8M}{\pi}\alpha_s\alpha e_q^2\,\mathrm{d}x_1\,\mathrm{d}x_2\,\frac{-(1+2x_1)(1-2x_1)+1+4x_2^2}{(1-2x_2)(1-2x_1)}\\
&= 3\alpha e_q^2 M\,\frac{32}{3\pi}\,\alpha_s\,\frac{x_1^2+x_2^2}{(1-2x_2)(1-2x_1)}\,\mathrm{d}x_1\,\mathrm{d}x_2 \quad . \qquad (28)
\end{aligned}
$$

6.2 Small-x Physics and the Gribov–Levin–Ryskin Equation

In the previous section we acquainted ourselves with the leading log approximation, which enables us to resum all terms which contain factors $\ln(t)$ and are of lowest order in α_s. We have also seen with respect to the K-factor how powerful this technique is. We now turn, however, to a kinematical range in which the LLA is not sufficient, namely to deep inelastic scattering at very small x, more precisely, for x so small that e.g. $\alpha_s \ln(1/x)/\pi > 1$. With the completion of the HERA accelerator this region (down to $x \sim$ a few times 10^{-5}) became accessible for the first time, generating an intense theoretical effort to isolate, understand and resum the relevant graphs. In these reactions Q^2 is still rather large, so that QCD perturbation theory should be applicable. As one has to keep track of two logarithms, $\ln t$ and $\ln x$, the techniques we shall sketch in this section are called double leading logarithm approximation (DLLA). As we have discussed in Sect. 6.1, in leading

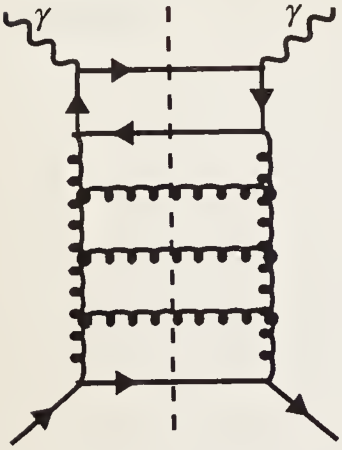


Fig. 6.14. Dominating graph in the double logarithmic approximation

logarithmic approximation only ladder graphs contribute. However, this still leaves two possibilities open. In the t-channel which is of interest to us here one can have either gluon or quark lines. It turns out, however, that gluon exchange in the t-channel dominates. This means that for e.g. deep-inelastic scattering the graph in Fig. 6.14 dominates over those in Fig. 6.15. One can show this by explicit calculation. Instead we would like to cite a result from Regge theory, which links the s-dependence of any scattering reaction to the spin of the exchanged particles. The leading-logarithmic behaviour for a ladder diagram like that shown in Fig. 6.16 is

$$\sigma \sim s^{s_1+s_2-1} \frac{(\ln s)^{n-1}}{(n-1)!} \quad . \tag{6.79}$$

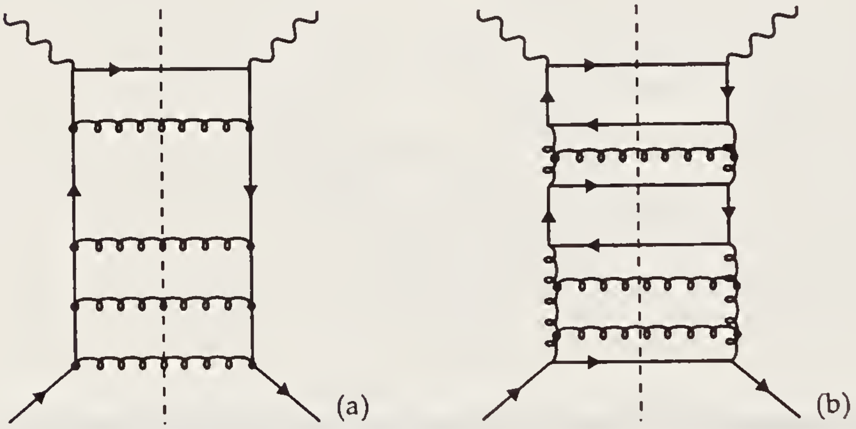


Fig. 6.15a,b. Sub-dominant graphs

Thus it is clearly advantageous that the particles at the sides have maximal spin. We shall discuss next the fact that in LLA the ladder decomposes in products of factors for each rung, such that the Regge argument applies to each part of the ladder and Fig. 6.15 really shows the dominant contribution. We shall now calculate this diagram explicitly, proceeding iteratively. This analysis was first performed by Gribov, Levin and Ryskin. Their work[2] is still the basis of the present day discussion. Our presentation follows their arguments. We start with just two exchange lines where q_μ and p_μ are the momenta of the entering photon and quark and $q'_\mu := q_\mu + xp_\mu$. The mass shell conditions imply

$$\begin{aligned}(k_{i-1}-k_i)^2 = 0 &= \left[(\alpha_{i-1}-\alpha_i)\,p_\mu + (\beta_{i-1}-\beta_i)\,q'_\mu + (k_{i-1,\perp}-k_{i,\perp})_\mu\right]^2 \\ &= (\alpha_{i-1}-\alpha_i)\,(\beta_{i-1}-\beta_i)\,2\,p\cdot q' + (k_{i-1,\perp}-k_{i,\perp})^2 \quad ,\end{aligned} \tag{6.80}$$

where we have used

$$\begin{aligned} &p\cdot k_{i\perp} = 0 \quad , \quad q\cdot k_{i\perp} = 0 \quad , \\ &p^2 = 0 \quad , \quad (q')^2 = q^2 + 2xq\cdot p = -Q^2 + 2x\nu = 0 \quad . \end{aligned} \tag{6.81}$$

Now the LLA implies that along the ladder the momentum changes continuously from $(k_0)_\mu = p_\mu$ to $(k_n)_\mu = q_\mu \sim q'_\mu$. (Remember that we are interested in the limit

---

[2] See Gribov, Levin and Ryskin: Phys. Rep. **100**, 1 (1983).

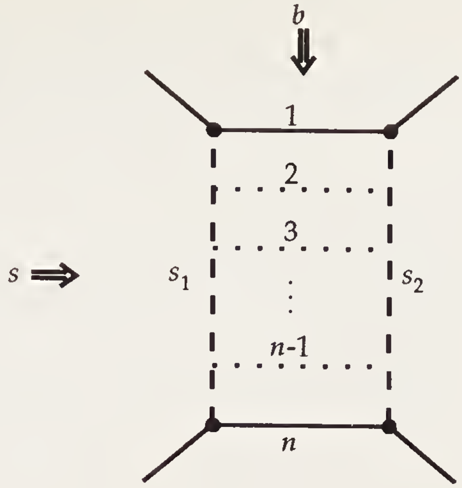


Fig. 6.16. Generalized ladder diagram

$x \to 0$, implying $q_\mu \sim q'_\mu$.) We saw that one gets a strict ordering of momenta, implying

$$\begin{aligned} &1 > \alpha_1 > \alpha_2 > \alpha_3 \ldots > \alpha_{n-1} > 0 \quad , \\ &0 < \beta_1 < \beta_2 < \beta_3 \ldots < \beta_{n-1} < 1 \quad . \end{aligned} \tag{6.82}$$

Furthermore p_μ and q_μ are assumed to be asymptotically large, implying that the α_i and β_i can become very small; (6.80) even shows that their product must be asymptotically small. This implies

$$\frac{\alpha_{i-1}}{\alpha_i} \gg 1 \quad , \quad \frac{\beta_{i-1}}{\beta_i} \ll 1 \tag{6.83}$$

such that (6.80) reads

$$\alpha_{i-1}\,\beta_i\,2\nu = -\left(k_{i-1,\perp} - k_{i,\perp}\right)^2 > 0 \quad . \tag{6.84}$$

On the other side we have

$$k_i^2 = 2\alpha_i\beta_i\nu + k_{i\perp}^2 = \frac{\alpha_i}{\alpha_{i-1}}\left(k_{i-1,\perp} - k_{i,\perp}\right)^2 + k_{i\perp}^2 \approx k_{i\perp}^2 \quad . \tag{6.85}$$

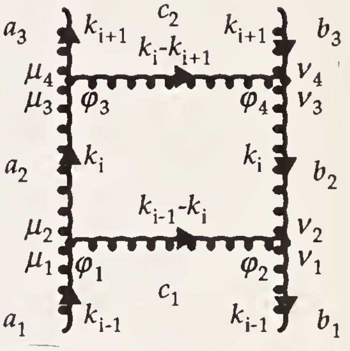


Fig. 6.17. Two-gluon-exchange subdiagram of Fig. 6.14

The loop integral reads

$$\begin{aligned} I = \int \frac{d^4k_i}{(2\pi)^4}\, &\Gamma^{a_1a_2c_1}_{\mu_1\mu_2\varrho_1}\left(k_{i-1}, -k_i, k_i - k_{i-1}\right)\Gamma^{b_1b_2c_1}_{\nu_1\nu_2\varrho_2}\left(-k_{i-1}, k_i, k_{i-1} - k_i\right) \\ &\times \Gamma^{a_2a_3c_2}_{\mu_3\mu_4\varrho_3}\left(k_i, -k_{i+1}, k_{i+1} - k_i\right)\Gamma^{b_2b_3c_2}_{\nu_3\nu_4\varrho_4}\left(-k_i, k_{i+1}, k_i - k_{i+1}\right) \\ &\times 2\pi\delta\left(\left(k_i - k_{i+1}\right)^2\right) 2\pi\delta\left(\left(k_{i-1} - k_i\right)^2\right)\left(\frac{1}{k_i^2}\right)^2 \\ &\times d^{\varrho_3\varrho_4}\left(k_i - k_{i+1}\right) d^{\varrho_1\varrho_2}\left(k_{i-1} - k_i\right) d^{\mu_2\mu_3}\left(k_i\right) d^{\nu_2\nu_3}\left(k_i\right) \delta^{a_1b_1} g^{\mu_1\nu_1} \quad , \end{aligned} \tag{6.86}$$

where $d^{\mu_2\mu_3}(k)$ denotes the nominator of the gluon propagator which is gauge dependent, e.g. $d^{\mu_2\mu_3} = g^{\mu_2\mu_3} - (k^{\mu_2}k^{\mu_3})/k^2$. Please remember that all the horizontal lines are cut, i.e. on-mass-shell. As the ladder is the square of a matrix element, its net colour content must be zero and it must be a scalar. This requirement is imposed by $\delta^{a_1 b_1} g_{\mu_1 \nu_1}$. To see which quantities are large and which are small in LLA it is useful to introduce the so called Sudakov variables in α_i and β_i

$$\begin{aligned}(k_i)_\mu &= \alpha_i \, p_\mu + \beta_i \, q'_\mu + k_{i\perp,\mu} \quad , \\ q'_\mu &= q_\mu + x p_\mu\end{aligned} \tag{6.87}$$

with all $k_{i\perp}^2$ being much smaller than ν. The two δ-functions in (6.86) are thus of the form

$$\delta\left(\alpha_{i-1} \, \beta_i \, 2\nu + \left(k_{i-1,\perp} - k_{i,\perp}\right)^2\right) \delta\left(\alpha_i \, p_{i+1} \, 2\nu + \left(k_{i-1,\perp} - k_{i+1,\perp}\right)^2\right) \quad , \tag{6.88}$$

and the integral reads

$$\int \mathrm{d}^4 k = \int \mathrm{d}\alpha \, \mathrm{d}\beta \, \mathrm{d}^2 k_\perp \left| \frac{\partial(\alpha, \beta)}{\partial(k_0, k_3)} \right|^{-1} \quad , \tag{6.89}$$

$$\alpha = \frac{k \cdot q}{p \cdot q} \quad , \quad \beta = \frac{k \cdot p}{p \cdot q} \quad , \tag{6.90}$$

$$\begin{aligned}\frac{\partial(\alpha, \beta)}{\partial(k_0, k_3)} &= \frac{1}{\nu^2} \begin{pmatrix} q_0 & -q_3 \\ p_0 & -p_3 \end{pmatrix} = \frac{1}{\nu^2} \left| q_0 p_3 - p_0 q_3 \right| \\ &= \frac{1}{\nu^2} \left| q_0 p_0 - q_3 p_3 \right| = \frac{1}{\nu} \quad ,\end{aligned} \tag{6.91}$$

$$s = 2\nu - Q^2 = 2\nu - 2x\nu \approx 2\nu \tag{6.92}$$

$$\Rightarrow \quad \int \mathrm{d}^4 k = \frac{1}{2} s \int \mathrm{d}\alpha \, \mathrm{d}\beta \, \mathrm{d}^2 k_\perp \quad . \tag{6.93}$$

Next we shall analyse the lowest exchanged gluon. Inserting the explicit form of the three gluon vertices we find

$$\begin{aligned}B_{\nu_2 \mu_2} &= \Gamma^{a_1 a_2 c_1}_{\mu_1 \mu_2 \varrho_1} \left(k_{i-1}, -k_i, k_i - k_{i-1}\right) \Gamma^{b_1 b_2 c_1}_{\nu_1 \nu_2 \varrho_2} \left(-k_{i-1}, k_i, k_{i-1} - k_i\right) \\ &\quad \times d^{\varrho_1 \varrho_2} \left(k_{i-1} - k_i\right) \delta^{a_1 b_1} g^{\mu_1 \nu_1} \\ &= -g^2 f^{a_1 a_2 c_1} f^{a_1 b_2 c_1} \left[\left(k_{i-1} + k_i\right)_{\varrho_1} g_{\mu_1 \mu_2} + \left(-2k_i + k_{i-1}\right)_{\mu_1} g_{\mu_2 \varrho_1} \right. \\ &\quad \left. + \left(k_i - 2k_{i-1}\right)_{\mu_2} g_{\varrho_1 \mu_1}\right] \left[-\left(k_{i-1} + k_i\right)_{\varrho_2} g^{\mu_1}_{\nu_2} + \left(2k_i - k_{i-1}\right)^{\mu_1} g_{\nu_2 \varrho_2} \right. \\ &\quad \left. + \left(2k_{i-1} - k_i\right)_{\nu_2} g^{\mu_1}_{\varrho_2}\right] d^{\varrho_1 \varrho_2} \left(k_{i-1} - k_i\right) \quad .\end{aligned} \tag{6.94}$$

As $d^{\varrho_1 \varrho_2}\left(k_{i-1} - k_i\right)$ belongs to an on-shell propagator we have

$$\left(k_i\right)_{\varrho_1} d^{\varrho_1 \varrho_2} \left(k_{i-1} - k_i\right) = \left(k_{i-1}\right)_{\varrho_1} d^{\varrho_1 \varrho_2} \left(k_{i-1} - k_i\right) \quad \text{etc.} \quad . \tag{6.95}$$

A choice of gauge in which the interference terms vanish in LLA is

$$d_{\alpha\beta}(k) = g_{\alpha\beta} - \frac{k_\alpha c_\beta + c_\alpha k_\beta}{c \cdot k} \quad , \quad c_\mu = q_\mu - x p_\mu \quad . \tag{6.96}$$

The expression for B simplifies drastically if we use this property and neglect all terms proportional to

$$\begin{aligned} k_i^2 \quad , \quad k_{i-1}^2 \quad \text{or} \quad k_i \cdot k_{i-1} &= \frac{1}{2}\left(k_i - k_{i-1}\right)^2 - \frac{1}{2}k_i^2 - \frac{1}{2}k_{i-1}^2 \\ &= -\frac{1}{2}k_i^2 - \frac{1}{2}k_{i-1}^2 \quad , \end{aligned} \tag{6.97}$$

$$k_{i\varrho_2}k_{i\varrho_1}d^{\varrho_1\varrho_2}\left(k_{i-1} - k_i\right) \approx 0 \quad \text{etc.} \quad . \tag{6.98}$$

We also use the relation

$$f^{a_1a_2c_1}f^{a_1b_2c_1} = 3\delta^{a_2b_2} \quad . \tag{6.99}$$

$$\begin{aligned} B_{\nu_2\mu_2} = &-3\,g^2\delta^{a_2b_2}\{-4\underbrace{k_{i\varrho_1}k_{i\varrho_2}}_{\longrightarrow 0}g_{\mu_2\nu_2} + \left(2k_i - k_{i-1}\right)_{\mu_2}2k_{i\varrho_1}g_{\varrho_2\nu_2} \\ &+ 2k_{i\varrho_1}\left(2k_{i-1} - k_i\right)_{\nu_2}g_{\varrho_2\mu_2} + \left(2k_i - k_{i-1}\right)_{\nu_2}2k_{i\varrho_2}g_{\varrho_1\mu_2} \\ &+ 0 - k_{i\varrho_2}\left(2k_{i-1} - k_i\right)_{\nu_2}g_{\mu_2\varrho_1} \\ &+ 2k_{i\varrho_2}\left(-k_i + 2k_{i-1}\right)_{\mu_2}g_{\varrho_1\nu_2} - k_{i\varrho_1}\left(2k_{i-1} - k_i\right)_{\mu_2}g_{\varrho_2\nu_2} \\ &- \left(k_i - 2k_{i-1}\right)_{\nu_2}\left(k_i - 2k_{i-1}\right)_{\mu_2}g_{\varrho_1\varrho_2}\}d^{\varrho_1\varrho_2}\left(k_{i-1} - k_i\right) \\ = &-3\,g^2\delta^{a_2b_2}\{k_{i\varrho_1}g_{\varrho_2\nu_2}\left(4k_i - 2k_{i-1} - 2k_{i-1} + k_i - 2k_i + 4k_{i-1}\right)_{\mu_2} \\ &+ k_{i\varrho_1}g_{\varrho_2\mu_2}\left(4k_{i-1} - 2k_i + 4k_i - 2k_{i-1} - 2k_{i-1} + k_i\right)_{\nu_2} \\ &- \left(k_i - 2k_{i-1}\right)_{\nu_2}\left(k_i - 2k_{i-1}\right)_{\mu_2}g_{\varrho_1\varrho_2}\}d^{\varrho_1\varrho_2}\left(k_{i-1} - k_i\right) \\ = &-3\,g^2\delta^{a_2b_2}\{k_{i\varrho_1}g_{\varrho_2\nu_2}k_{i\mu_2} + k_{i\varrho_1}g_{\varrho_2\mu_2}k_{i\nu_2} \\ &- \left(k_i - 2k_{i-1}\right)_{\nu_2}\left(k_i - 2k_{i-1}\right)_{\mu_2}g_{\varrho_1\varrho_2}\}d^{\varrho_1\varrho_2}\left(k_{i-1} - k_i\right) \quad . \end{aligned} \tag{6.100}$$

To simplify this expression further we assume that the gluon exchange we are considering is the lowest one. Then

$$k_{i-1,\mu} \approx \alpha_{i-1}p_\mu \quad , \tag{6.101}$$

and with $\alpha_{i-1} \gg \alpha_i$ we find

$$\begin{aligned} d^{\varrho_1\varrho_2}\left(k_{i-1} - k_i\right) &\approx d^{\varrho_1\varrho_2}\left(k_{i-1}\right) \quad , \\ k_{i\varrho_i}\,d^{\varrho_1\varrho_2}\left(k_{i-1} - k_i\right) &= k_{(i-1)\varrho_1}\,d^{\varrho_1\varrho_2}\left(k_{i-1} - k_i\right) \\ &\approx k_{(i-1)\varrho_1}\,d^{\varrho_1\varrho_2}\left(k_{i-1}\right) = 0 \end{aligned} \tag{6.102}$$

such that only

$$B_{\nu_2\mu_2} = 3\,g^2\delta^{a_2b_2}\,2\alpha_{i-1}p_{\nu_2}\,2\alpha_{i-1}p_{\mu_2}\,2 = 24\,g^2\alpha_{i-1}^2\,p_{\nu_2}\,p_{\mu_2} \tag{6.103}$$

survives. B is multiplied by the two gluon propagators of the ladder side, leading to

$$
\begin{aligned}
B_{\nu_2\mu_2} &= 24\,g^2\delta^{a_2b_2}\alpha_{i-1}^2\left(p_{\nu_3} - \frac{p\cdot k_i\,c_{\nu_3} + k_{i\nu_3}\,c\cdot p}{c\cdot k_i}\right) \\
&\quad\times\left(p_{\mu_3} - \frac{p\cdot k_i\,c_{\mu_3} + k_{i\mu_3}\,c\cdot p}{c\cdot k_i}\right) \\
&= 24\,g^2\delta^{a_2b_2}\alpha_{i-1}^2\left(p_{\nu_3} - \frac{\beta_i\,p\cdot q\,c_{\nu_3} + k_{i\nu_3}\,p\cdot q}{\alpha_i\,p\cdot q}\right) \\
&\quad\times\left(p_{\mu_3} - \frac{\beta_i\,p\cdot q\,c_{\mu_3} + k_{i\mu_3}\,p\cdot q}{\alpha_i\,p\cdot q}\right) \\
&= 24\,g^2\delta^{a_2b_2}\alpha_{i-1}^2\left(p_{\nu_3} - \frac{\beta_i}{\alpha_i}c_{\nu_3} - p_{\nu_3} - \frac{\beta_i}{\alpha_i}q_{\nu_3} - \frac{1}{\alpha_i}\left(k_{i\perp}\right)_{\nu_3}\right) \\
&\quad\times\left(\frac{\beta_i}{\alpha_i}p_{\mu_3} - \frac{1}{\alpha_i}\left(k_{i\perp}\right)_{\mu_3}\right) \\
&\approx 24\,g^2\delta^{a_2b_2}\frac{\alpha_{i-1}^2}{\alpha_i^2}\left(k_{i\perp}\right)_{\nu_3}\left(k_{i\perp}\right)_{\mu_3} \quad .
\end{aligned}
\tag{6.104}
$$

In the last step we have used the fact that

$$
-\,k_{i\perp} \gg \alpha_i\beta_i\,2p\cdot q \geq 2\beta_i\alpha_i|p_3q_3| \gg 2\big(\beta_iq_3\big)^2 \quad \text{as} \quad |\alpha_ip_3| \gg |\beta_iq_3| \quad .
\tag{6.105}
$$

Now we insert all our results into (6.86)

$$
\begin{aligned}
I &= \frac{24\,g^2}{(2\pi)^3}\int \mathrm{d}\alpha_i\,\mathrm{d}\beta_i\,\mathrm{d}^2k_{i\perp}\,\nu\,\Gamma^{a_2a_3c_2}_{\mu_3\mu_4\varrho_3}(k_i,-k_{i+1},k_{i+i}-k_i) \\
&\quad\times\Gamma^{b_2b_3c_2}_{\nu_3\nu_4\varrho_4}(-k_i,k_{i+1},k_i-k_{i+i})\,2\pi\,\delta\big(\alpha_{i-1}\beta_i\,2\nu + (k_{i-1\,\perp}-k_{i\perp})^2\big) \\
&\quad\times\delta\big(\alpha_i\beta_{i+1}\,2\nu + (k_{i+1\,\perp}-k_{i\perp})^2\big)\,\frac{\alpha_{i-1}^2}{\alpha_i^2}\,\big(k_{i\perp}\big)_{\nu_3}\big(k_{i\perp}\big)_{\mu_3}\delta^{a_2b_2} \\
&\quad\times d^{\varrho_3\varrho_4}(k_i-k_{i+1})\,\frac{1}{k_i^4} \quad .
\end{aligned}
\tag{6.106}
$$

We have seen that the combination $\Gamma\Gamma d$ depends in our approximation only on the momentum with the larger index, in this case k_{i+1} and not k_i. Therefore we can replace k_i by k_{i-1}. Furthermore we can perform the β_i integral.

$$
\begin{aligned}
I &= \frac{24\,g^2}{(2\pi)^3}\int \mathrm{d}\alpha_i\,\mathrm{d}^2k_{i\perp}\,\frac{\alpha_{i-1}^2}{\alpha_i^2}\,\frac{1}{2\alpha_{i-1}}2\pi\,\delta\big(\alpha_i\beta_{i+1}\,2\nu + (k_{i+1\,\perp}-k_{i\perp})^2\big) \\
&\quad\times\big(k_{i\perp}\big)_{\nu_3}\big(k_{i\perp}\big)_{\mu_3}\delta^{a_2b_2}\,\frac{1}{k_i^4}\,\Gamma^{a_2a_3c_2}_{\mu_3\mu_4\varrho_3}(k_{i-1},-k_{i+1},k_{i+i}-k_{i-1}) \\
&\quad\times\Gamma^{b_2b_3c_2}_{\nu_3\nu_4\varrho_4}(-k_{i-1},k_{i+1},k_{i-1}-k_{i+i})\,d^{\varrho_3\varrho_4}(k_{i-1}-k_{i+1}) \quad .
\end{aligned}
\tag{6.107}
$$

Next we simplify the transverse momenta

$$
\begin{aligned}
\int \mathrm{d}^2k_{i\perp}\,\frac{\big(k_{i\perp}\big)_{\nu_3}\big(k_{i\perp}\big)_{\mu_3}}{k_i^4} &= \frac{1}{2}\delta_{\nu_3\mu_3}\int \mathrm{d}^2k_{i\perp}\,\frac{\big(k_{i\perp}^2\big)}{k_i^4} \\
&= -\frac{1}{2}\delta_{\nu_3\mu_3}\pi\int \mathrm{d}^2k_i\,\frac{1}{k_i^2} \quad .
\end{aligned}
\tag{6.108}
$$

Note that now ν_3 and μ_3 can only be 1 or 2. Thus our general ansatz in (6.86) has to be modified accordingly to get a consistent expression which can be iterated.

$$\tilde{I} = \int \mathrm{d}\alpha_i \, \frac{\alpha_{i-1}}{\alpha_i^2} \int \mathrm{d}^2 k_i \, \frac{1}{k_i^2} \frac{\alpha_s(k_i^2)}{4\pi} \, 12 \, \Gamma^{a_2 a_3 c_2}_{\mu_3 \mu_4 \varrho_3} \, \Gamma^{b_2 b_3 c_2}_{\nu_3 \nu_4 \varrho_4} \, d^{\varrho_3 \varrho_4} \, 2\pi$$
$$\times \, \delta\big(\alpha_i \beta_{i+1} \, 2\nu + (k_{i-1\perp} - k_{i+1\perp})^2\big) \delta^{a_2 b_2} \delta_{\mu_3 \nu_3} \quad . \tag{6.109}$$

In the last step we have assumed as usual that radiative corrections turn α_s into $\alpha_s(k_i^2)$. Obviously $\tilde{I}$ has the same form as I modified by $g_{\nu_3 \mu_3} \rightarrow \delta_{\mu_3 \nu_3}$ and for a slightly different δ-function. Let us define

$$\xi = \ln \ln \left(\frac{k^2}{\Lambda^2} \right) \quad , \quad y_i = \frac{24}{b} \ln \left(\frac{1}{\alpha_i} \right) \quad , \quad b = 11 - \frac{2}{3} N_F \tag{6.110}$$

to write this expression in a simpler form.

$$\alpha_s(k^2) = \frac{4\pi}{b \ln(k^2/\Lambda^2)}$$
$$\Rightarrow \quad \xi = b \int \frac{\mathrm{d}k^2}{k^2} \frac{\alpha_s(k^2)}{4\pi} = \int \frac{\mathrm{d}k^2}{k^2 \ln(k^2/\Lambda^2)} = \ln \ln \left(\frac{k^2}{\Lambda^2} \right) \quad . \tag{6.111}$$

The parameter α_i obviously corresponds to the momentum fraction x carried by the gluon on the ladder side after the emission of i gluons. It can be at most 1 and at least $\alpha_{i+1} = x$.

$$y_i = \frac{24}{b} \int\limits_{\alpha_{i+1}}^{1} \frac{\mathrm{d}\alpha_i}{\alpha_i} = -\frac{24}{b} \ln(\alpha_{i+1}) = \frac{24}{b} \ln \left(\frac{1}{\alpha_{i+1}} \right) \quad . \tag{6.112}$$

If one puts all the expressions together, one will get a factor $\alpha_i/(\alpha_{i+1})^2$ from the next exchanged gluon instead of $\alpha_{i-1}/(\alpha_{i+1})^2$. This factor comes partially from the δ-function. To correct for this we multiply with α_i/α_{i-1} and then we can substitute α_{i-1} for α_i in the delta function. Thus we finally get

$$\tilde{I}^{(n)} = \tilde{I}^{(n-1)} \frac{1}{2} y \, \xi = \tilde{I}^{(n-1)} \frac{1}{2} \int\limits_0^{y_i} \mathrm{d}y' \int\limits_0^{\xi_i} \mathrm{d}\xi' \quad . \tag{6.113}$$

This is the crucial equation as it can be solved to obtain the explicit expression for a graph with n exchanged gluons, which we denote by $\tilde{I}^{(n)}$. From $\alpha_{i-1} \gg \alpha_i$ it follows that

$$y_{i-1} < y_i \quad . \tag{6.114}$$

Similarly

$$|k_i^2| \approx |k_{i\perp}^2| \geq |k_{i-1}^2| = |k_{i\perp}^2| \quad \Rightarrow \quad \xi_i > \xi_{i-1} \quad . \tag{6.115}$$

The last inequality is due to the fact that consecutive gluon emission increases $|\boldsymbol{k}_{i\perp}|$. With these bounds $\tilde{I}^{(n)}$ can be written as

$$\tilde{I}^{(n)} = \frac{1}{2^n} \int\limits_{\xi_0}^{\xi} d\xi_1 \int\limits_{0}^{y} dy_1 \int\limits_{\xi_1}^{\xi} d\xi_2 \int\limits_{y_1}^{y} dy_2 \ldots \int\limits_{\xi_{n-1}}^{\xi} d\xi_n \int\limits_{y_{n-1}}^{y} dy_n \quad , \tag{6.116}$$

where

$$\xi_0 = \ln \ln \left(\frac{k_\perp^2(\text{initial})}{\Lambda^2} \right) \tag{6.117}$$

describes the transverse momentum of a quark inside a nucleon and $\alpha_n = x$, such that $y = 24/b \ \ln(1/x)$. Obviously $\tilde{I}^{(n)}$ can be integrated analytically,

$$\tilde{I}^{(n)} = \frac{1}{2^n} \frac{y^n}{n!} \frac{(\xi - \xi_0)^n}{n!} \quad , \tag{6.118}$$

which also allows the sum over n to be performed:

$$\sum_n \tilde{I}^{(n)} = I_0 \left(\sqrt{2(\xi - \xi_0) y} \right) =: F(y, \xi) \quad . \tag{6.119}$$

Thus with the approximations made we get a very simple result, which allows us to proceed much further with the analyses of this complicated problem. One should, however, keep in mind that our derivation is only valid if p and q are very large and if there are many exchanged lines. Consequently $F(y, \xi)$ is physically reasonable only if it is determined by the $\tilde{I}^{(n)}$ terms with $n \gg 1$, which is the case only if

$$\frac{1}{2} y \, |\xi - \xi_0| = \frac{12}{11 - \frac{2}{3} N_f} \ln \left(\frac{1}{x} \right) \ln \left(\frac{\ln \left(k^2 / \Lambda^2 \right)}{\ln \left(k_\perp^2(\text{initial}) / \Lambda^2 \right)} \right) \gg 1 \quad . \tag{6.120}$$

Let us illustrate this requirement with a numerical example. Lets assume $k_\perp^2(\text{init}) > \Lambda^2 \approx 0.04 \, \text{GeV}^2, k^2 \approx 10 \, \text{GeV}^2$ implying

$$\frac{12}{11 - \frac{2}{3} N_f} \ln \left(\frac{\ln \left(k^2 / \Lambda^2 \right)}{\ln \left(k_\perp^2(\text{initial}) / \Lambda^2 \right)} \right) \approx 3 \quad , \quad \xi \approx 2 \quad . \tag{6.121}$$

Thus for $x = 0.01$ we get $y|\xi - \xi_0|/2 \approx 14$, from $x = 0.001$ we get 21 and for $x = 0.1$ one has $y|\xi - \xi_0|/2 \approx 7$. The approximations we made required this parameter to be really large compared to 1. From the numerical examples it is obvious that there is an experimentally accessible range in x and Q^2 in which this approximation should be valid. As $y|\xi - \xi_0|/2$ also gives the number of gluon links for the most important ladder-graphs one sees that the number of emitted gluons is very large, e.g. 21 for $x = 10^{-3}$. Gluon bremsstrahlung from gluons is thus the most important process in small-x physics. As we shall see later, the modified evolution equation, called Gribov–Levin–Ryskin equation (GLR) leads only for very small x values, like $x \sim 10^{-4}$, to noticable phenomenological corrections compared with the usual GLAP evolution. Thus the GLR equation is only important for domains in which the approximations made so far should be reliable. From our numerical example it should also be clear that the double logarithm ξ cannot be very large, such that y must be large. Generally for the DLA $y \gg \xi$ must hold.

The function $F(y,\xi)$ has a simple analytic property

$$\begin{aligned}\frac{\partial^2 F}{\partial y\,\partial \xi} &= \frac{\partial}{\partial y}\left[I_1\left(\sqrt{2y(\xi-\xi_0)}\right)\sqrt{\frac{y}{2(\xi-\xi_0)}}\right]\\ &= \frac{1}{2\sqrt{2y(\xi-\xi_0)}}\,I_1\left(\sqrt{2y(\xi-\xi_0)}\right)+\sqrt{\frac{y}{2(\xi-\xi_0)}}\\ &\quad\times\left[-\frac{1}{\sqrt{2y(\xi-\xi_0)}}\,I_1\left(\sqrt{2y(\xi-\xi_0)}\right)+I_0\left(\sqrt{2y(\xi-\xi_0)}\right)\right]\sqrt{\frac{(\xi-\xi_0)}{2y}}\\ &= \frac{1}{2}I_0\left(\sqrt{2y(\xi-\xi_0)}\right) \quad ,\end{aligned} \tag{6.122}$$

which allows us to write the recursive structure of the ladder in the form of a Bethe–Salpeter equation.

$$F(y,\xi)=1+\frac{1}{2}\int\limits_0^y \mathrm{d}y' \int\limits_{\xi_0}^{\xi} \mathrm{d}\xi'\, F(y',\xi') \quad . \tag{6.123}$$

This equation is illustrated graphically in Fig. 6.18.

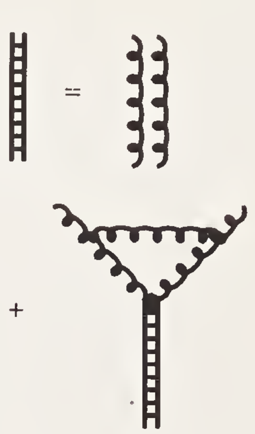


Fig. 6.18. The Bethe–Salpeter equation for the summed ladders in double-leading-logarithm approximation (DLA)

For $\sqrt{2y|\xi-\xi_0|} \gg 1$ we can use the asymptotics of the Bessel function to obtain another interesting result,

$$I_0\left(\sqrt{2y(\xi-\xi_0)}\right) \rightarrow \frac{\exp\left(\sqrt{2y(\xi-\xi_0)}\right)}{\sqrt{2\pi}\,(2y(\xi-\xi_0))^{1/4}} \quad , \tag{6.124}$$

where the exponential is a very large number. From the structure of the Bethe–Salpeter equation, on the other hand, we have

$$F(y',\xi'-\xi_0)\,F(y-y',\xi-\xi') \sim F(y,\xi-\xi_0) \tag{6.125}$$

implying to exponential precision (i.e. neglecting small factors of the order $O(1)$, namely $\sqrt{2\pi}$, $(2y(\xi-\xi_0))^{1/4}, \ldots$, which correspond to negligible terms when added to the argument of the exponential function)

$$\begin{gathered}\mathrm{e}^{\sqrt{2y'(\xi'-\xi_0)}}\,\mathrm{e}^{\sqrt{2(y-y')(\xi-\xi')}} \sim \mathrm{e}^{\sqrt{2y(\xi-\xi_0)}} \quad ,\\ \sqrt{2y'(\xi'-\xi_0)}+\sqrt{2(y-y')(\xi-\xi')} \approx \sqrt{2y(\xi-\xi_0)}\end{gathered} \tag{6.126}$$

for an arbitrary position (y',ξ') along the ladder. This equation can only be fulfilled if

$$\begin{gathered}\frac{y'}{\xi'-\xi_0} \approx \frac{y-y'}{\xi-\xi'} \approx \frac{y}{\xi-\xi_0} = c \quad ,\\ \sqrt{2c}(\xi'-\xi_0)+\sqrt{2c}(\xi-\xi') \approx \sqrt{2c}(\xi-\xi_0) \quad .\end{gathered} \tag{6.127}$$

We have thus obtained the interesting result that both y and ξ, and consequently x and $k_\perp^2$, are strictly ordered along the ladder. The more gluons are emitted, the softer the leading gluon gets and each consecutive gluon is emitted with a larger transverse momentum.

We have seen that our approximations seem to become better and better with increasing $y(\xi-\xi_0)$. It is natural to ask whether there is a bound to this, and indeed there is. We have treated QCD perturbatively adding up ladder graphs. In doing so we neglected all genuine non-perturbative processes, like complicated multi-particle processes. If these become important the DLA becomes unreliable. Typically such a breakdown of perturbation theory is signaled by an unacceptable behaviour of the S matrix in some domain and the same happens here. The crucial point is that the total cross sections cannot exceed the bounds given by general scattering theory. Using the optical theorem we shall now derive an inequality which must be satisfied by the F-function just discussed. Let us define A to be the forward amplitude for elastic photon scattering off a proton. Then the total photon-proton cross section is linked by the optical theorem to its imaginary part

$$\sigma_{\gamma \mathrm{p}} = \frac{1}{s}\,\mathrm{Im}\{A(\gamma\mathrm{p} \to \gamma'\mathrm{p})\} \quad . \tag{6.128}$$

On the other hand, the elastic photon-proton cross section in the forward direction is

$$\left.\frac{d\sigma}{dp_\perp^2}\right|_{\text{forward}} = \frac{|A^2|}{16\pi s^2} > \frac{(\mathrm{Im}\{A\})^2}{16\pi s^2} = \frac{1}{16\pi}\sigma_{\gamma\mathrm{p}}^2 \sim \sigma_{\gamma\mathrm{p}}\alpha_{\text{e.m.}}\frac{1}{q^2}F(y,\xi-\xi_0) \quad . \tag{6.129}$$

We have used the fact that

$$\sigma_{\gamma\mathrm{p}} \sim \frac{\alpha_{\text{e.m.}}}{q^2}F(y,\xi-\xi_0) \quad , \tag{6.130}$$

with q^2 coming from the quark propagator in Fig. 6.14. In (6.129) and (6.130) we neglected all factors of order $O(1)$. As the photon-proton elastic scattering cross section must be smaller, and in fact is much smaller, than the total photon-proton cross section we conclude from (6.129).

$$\sigma_{\gamma\mathrm{p}} \gg \sigma_{\gamma\mathrm{p}}\,\frac{\alpha_{\text{e.m.}}}{q^2}\,F(y,\xi-\xi_0)\,\langle p_\perp^2\rangle \sim \sigma_{\gamma\mathrm{p}}\,\alpha_{\text{e.m.}}\,\mathrm{e}^{-\exp(\xi)+\exp(\xi_0)}\,F(y,\xi-\xi_0) \tag{6.131}$$

or

$$\begin{aligned} &1 \gg \alpha_{\text{e.m.}}\,\mathrm{e}^{-\exp(\xi)+\exp(\xi_0)}\,\mathrm{e}^{\sqrt{2y(\xi-\xi_0)}} \\ \Rightarrow\quad &\mathrm{e}^{\xi} > \sqrt{2y(\xi-\xi_0)} + \mathrm{e}^{\xi_0} \quad . \end{aligned} \tag{6.132}$$

As ξ_0 is small, i.e. at most of order one, we get the additional constraint

$$y < \frac{\mathrm{e}^{2\xi}}{2\xi} \quad . \tag{6.133}$$

This constraint is very serious as it implies that we cannot just go to arbitrarily small x to make the DLA valid. To see how relevant this constraint is let us return to our numerical example:

$$\begin{aligned} &k^2 = 10\,\mathrm{GeV}^2 \quad \Rightarrow \quad \xi \approx 1.7 \\ \Rightarrow\quad &\frac{24}{11-\frac{2}{3}N_F}\ln\left(\frac{1}{x}\right) < \frac{\mathrm{e}^{3.4}}{3.4} \quad \Rightarrow \quad x > 3\times 10^{-2} \quad . \end{aligned} \tag{6.134}$$

For $k^2 = 4\,\mathrm{GeV}^2$ we get $x > 6 \times 10^{-2}$. This is a typical result, i.e. the unitarity bound restricts the validity of the simple DLA very seriously making it virtually useless. Thus one is faced with the demanding task of taking higher order ladder graphs into account. This was first attempted by Gribov, Levin and Ryskin, leading to an equation that now bears their name. This equation is quite often used for phenomenological studies, but recent work suggest that it could be invalid. Because of these controversies we shall sketch only its derivation as a typical example of how one is trying to proceed.

The basic idea is still to stick to ladder graphs, but to allow them to branch as illustrated in Fig. 6.19. The crucial new element is obviously the coupling between three gluon ladders. As a gluon ladder can be identified with the "pomeron", this coupling is also called the "three-pomeron vertex" and is subject to intense investigation. We shall not analyse this vertex in great detail. We shall just derive its form assuming that the topmost ladder is cut and shall then jump directly to the resulting iterative equation for $F(y, \xi)$. The notation we use is given in Fig. 6.20. The lower horizontal gluon line leads then to the factor

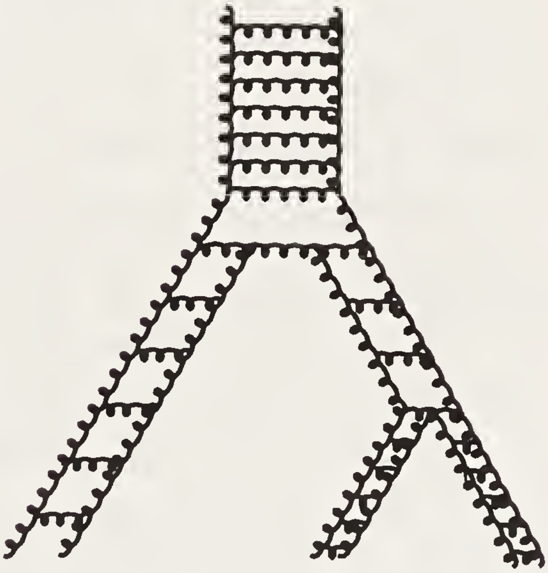


Fig. 6.19. An example of higher order ladder graphs

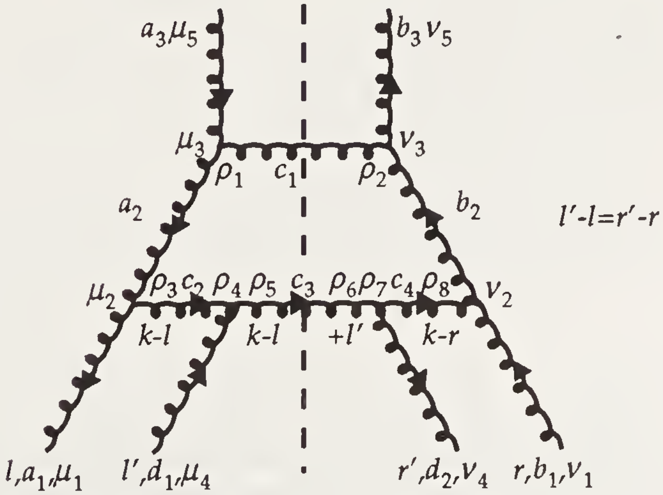


Fig. 6.20. One graph contributing to the three-ladder vertex

$$\begin{aligned}
&\int \frac{\mathrm{d}^4k}{(2\pi)^3}\, 16\pi^2\alpha_s^2(k^2)\,\delta_+\left((k-\ell+\ell')^2\right) \frac{1}{(k-\ell)^2}\,\frac{1}{(k-r)^2}\,\frac{1}{k^4} \\
&\times \Big\{ \Gamma^{\mu_1\mu_2\varrho_3}_{a_1a_2c_2}(-\ell,k,\ell-k)\,\Gamma^{\mu_4\varrho_4\varrho_5}_{d_1c_2c_3}(\ell',k-\ell,\ell-\ell'-k) \\
&\times \Gamma^{\nu_4\varrho_6\varrho_7}_{d_2c_3c_4}(-r',k-\ell+\ell',r-k)\,\Gamma^{\nu_1\varrho_8\nu_2}_{b_1c_4b_2}(r,k-r,-k) \\
&\times d_{\varrho_3\varrho_4}(k-\ell)\,d_{\varrho_5\varrho_6}(k-\ell+\ell')\,d_{\varrho_7\varrho_8}(k-r)\,d_{\mu_2\mu_3}(k)\,d_{\nu_2\nu_3}(k) \\
&\times g_{\mu_1\mu_4}\,\delta_{a_1d_1}\,g_{\nu_1\nu_4}\,\delta_{b_1d_2}\Big\} \quad . \qquad (6.135)
\end{aligned}$$

Inserting the definition of Γ the expression in braces gives

$$\begin{aligned}
\{\dots\} &= g^4 f_{b_1c_3c_4} f_{b_1c_4b_2} f_{a_1a_2c_2} f_{a_1c_2c_3} \\
&\times \left[-(k+\ell)^{\varrho_3}\, g^{\mu_1\mu_2} + (2k-\ell)^{\mu_1}\, g^{\mu_2\varrho_3} + (2\ell-k)^{\mu_2}\, g^{\varrho_3\mu_1}\right] \\
&\times \left[(\ell'-k+\ell)^{\varrho_5}\, g^{\mu_4\varrho_4} + (2k-2\ell+\ell')^{\mu_4}\, g^{\varrho_4\varrho_5} + (\ell-k-2\ell')^{\varrho_4}\, g^{\varrho_5\mu_4}\right] \\
&\times \left[(-2r'-k+r)^{\varrho_7}\, g^{\nu_4\varrho_4} + (2k-2r+r')^{\nu_4}\, g^{\varrho_6\varrho_7} + (r-k+r')^{\varrho_6}\, g^{\nu_4\varrho_7}\right] \\
&\times \left[(2r-k)^{\nu_2}\, g^{\nu_1\varrho_8} + (2k-r)^{\nu_1}\, g^{\varrho_8\nu_2} + (-k-r)^{\varrho_8}\, g^{\nu_1\nu_2}\right] \\
&\times d_{\varrho_3\varrho_4}(k-\ell)\,d_{\varrho_5\varrho_6}(k-\ell+\ell')\,d_{\varrho_7\varrho_8}(k-r)\,d_{\mu_2\mu_3}(k)\,d_{\nu_2\nu_3}(k) \\
&\times g_{\mu_1\mu_4}\,g_{\nu_1\nu_4} \quad . \qquad (6.136)
\end{aligned}$$

Again we use the strong ordering of the α's to identify the leading terms. As

$$(\ell)_{\mu_2} d^{\mu_2\mu_3}(k) = \frac{\alpha_\ell}{\alpha_k}(k_\perp)^{\mu_3} \quad \text{and} \quad \frac{\alpha_\ell}{\alpha_k} \gg 1 \quad , \qquad (6.137)$$

the terms $(2\ell-k)^{\mu_2} g^{\varrho_3\mu_1}$ and $(2r-k)^{\nu_2} g^{\nu_1\varrho_8}$ contribute in leading order in the first and last bracket. With $k^{\mu_2} d_{\mu_2\mu_3}(k) = 0$ etc. we thus have

$$\begin{aligned}
\{\dots\} &= g^4(-3\delta_{c_3b_2})(-3\delta_{c_3a_2})2\ell^{\mu_2}\, d_{\mu_2\mu_3}(k)\, 2r^{\nu_2}\, d_{\nu_2\nu_3}(k) \\
&\times \left[(\ell'-k+\ell)^{\varrho_5}\, g^{\mu_4\varrho_4} + (2k-2\ell+\ell')^{\mu_4}\, g^{\varrho_4\varrho_5} + (\ell-k-2\ell')^{\varrho_4}\, g^{\varrho_5\mu_4}\right] \\
&\times \left[(-2r'-k+r)^{\varrho_7}\, g^{\nu_4\varrho_6} + (2k-2r+r')^{\nu_4}\, g^{\varrho_6\varrho_7} + (r-k+r')^{\varrho_6}\, g^{\nu_4\varrho_7}\right] \\
&\times d_{\mu_4\varrho_4}(k-\ell)\,d_{\varrho_5\varrho_6}(k-\ell+\ell')\,d_{\varrho_7\nu_4}(k-r) \quad . \qquad (6.138)
\end{aligned}$$

Now for the three momenta $k-\ell$, $k-\ell+\ell'$ and $k-r$ there is no strict α-ordering. Using the orthogonality of the d-matrices we have

$$\begin{aligned}
\{\dots\} = 9g^4\,\delta_{a_2b_2}\, 4\,\frac{\alpha_\ell\alpha_r}{\alpha_k^2}\,(k_\perp)_{\mu_3}(k_\perp)_{\nu_3}& \\
\times \left[2\ell'^{\varrho_5}\, g^{\mu_4\varrho_4} + \ell'^{\mu_4}\, g^{\varrho_4\varrho_5} - 2\ell'^{\varrho_4}\, g^{\varrho_5\mu_4}\right]& \\
\times \left[-2r'^{\varrho_7}\, g^{\nu_4\varrho_6} + r'^{\nu_4}\, g^{\varrho_6\varrho_7} + 2r'^{\varrho_6}\, g^{\nu_4\varrho_7}\right]& \\
\times d_{\mu_4\varrho_4}(k-\ell)\,d_{\varrho_5\varrho_6}(k-\ell+\ell')\,d_{\varrho_7\nu_4}(k-r) \quad .& \qquad (6.139)
\end{aligned}$$

The two square brackets and three d factors give some scalar. According to the ladder approximation the largest scalar is $k^2 \approx -k_\perp^2$, thus neglecting numerical factors we can short cut a lengthy calculation by concluding

$$\{\ldots\} \sim \delta_{a_2b_2} \frac{\alpha_\ell \alpha_r}{\alpha_k^2} \left(k_\perp\right)_{\mu_3} \left(k_\perp\right)_{\nu_3} k_\perp^2 \quad . \tag{6.140}$$

The integral over β_k can be done easily.

$$\begin{aligned} \int \mathrm{d}\beta_k \, \delta_+\left(k^2 + \beta_k \, \alpha_{\ell'-\ell} \, s + (\ell' - \ell)^2\right) &\approx \int \mathrm{d}\beta_k \, \delta_+\left(-k_\perp^2 + \beta_k \, \alpha_{\ell'-\ell} \, s\right) \\ &= \frac{1}{\alpha_{\ell'-\ell} \, s} \quad , \end{aligned} \tag{6.141}$$

$$\beta_k = \frac{k_\perp^2}{\alpha_{\ell'-\ell} \, s} \quad . \tag{6.142}$$

And the two propagators give

$$\begin{aligned} \frac{1}{(k-\ell)^2} &= \frac{1}{k^2 - 2\,k \cdot \ell} = \frac{1}{-k_\perp^2 - \beta_k \alpha_\ell s} \\ &= \frac{-1}{\beta_k \alpha_{\ell'-\ell} s + \beta_k \alpha_\ell s} = -\frac{1}{\beta_k s} \, \frac{1}{\alpha_{\ell'-\ell} + \alpha_\ell} \\ &= \frac{\alpha_{\ell'-\ell}}{\alpha_{\ell'-\ell} + \alpha_\ell} \, \frac{-1}{k_\perp^2} = \frac{\alpha_{\ell'-\ell}}{\alpha_{\ell'-\ell} + \alpha_\ell} \, \frac{1}{k^2} \quad , \end{aligned} \tag{6.143}$$

$$\frac{1}{(k-r)^2} = \frac{\alpha_{r'-r}}{\alpha_{r'-r} + \alpha_r} \, \frac{1}{k^2} \quad , \qquad \alpha_{r'-r} = \alpha_{\ell'-\ell} \quad . \tag{6.144}$$

Alltogether this gives

$$\int \mathrm{d}^2 k_\perp \, \mathrm{d}\alpha_k \, \alpha_s^2(k_\perp^2) \, \frac{\alpha_\ell \alpha_r}{\alpha_k^2} \, \frac{\alpha_{\ell'-\ell}}{(\alpha_{\ell'-\ell} + \alpha_\ell)(\alpha_{\ell'-\ell} + \alpha_r)} \, \frac{1}{k_\perp^6} \left(k_\perp\right)_{\mu_3} \left(k_\perp\right)_{\nu_3} \delta_{a_2b_2} \quad , \tag{6.145}$$

and again only the even part in $\left(k_\perp\right)_{\mu_3} \left(k_\perp\right)_{\nu_3}$ survives:

$$\int \mathrm{d}^2 k_\perp \, \mathrm{d}\alpha_k \, \alpha_s^2(k_\perp^2) \, \frac{\alpha_\ell \, \alpha_r}{\alpha_k^2} \, \frac{\alpha_{\ell'-\ell}}{(\alpha_{\ell'-\ell} + \alpha_\ell)(\alpha_{\ell'-\ell} + \alpha_r)} \, \frac{1}{k_\perp^4} \, g_{\mu_3\nu_3} \, \delta_{a_2b_2} \quad . \tag{6.146}$$

As we discussed after (6.112), the rung just above the ladder-branching provides a factor α_k. On the other hand the factors containing no α_k are absorbed into the lower rungs and the ℓ, ℓ' integrations. The remaining k-dependent term then simply reads

$$\int \mathrm{d}^2 k_\perp \, \mathrm{d}\alpha_k \, \alpha_s^2(k_\perp^2) \, \frac{1}{k_\perp^4} \, \frac{1}{\alpha_k} \, g_{\mu_3\nu_3} \, \delta_{a_2b_2} \sim \int \mathrm{d}y \, \mathrm{d}\xi \, \frac{\alpha_s(k_\perp^2)}{k_\perp^2} \, g_{\mu_3\nu_3} \, \delta_{a_2b_2} \quad , \tag{6.147}$$

and the tripple-ladder vertex is simply $\sim \alpha_s(k_\perp^2)/k_\perp^2$. Substituting $\alpha_s(k_\perp^2)/k_\perp^2 \sim \exp\left(-\xi - \exp(\xi)\right)$ one can than write down a self-consistent equation illustrated heuristically by Fig. 6.21. From the form of the vertex derived, the following form of this equation seems plausible:

$$\frac{\partial^2 F(y,\xi)}{\partial y \, \partial \xi} = \frac{1}{2} F(y,\xi) - \exp\left(-\mathrm{e}^\xi - \xi\right) F^2(y,\xi) \quad . \tag{6.148}$$

This equation is actually the result of a much more careful derivation, which can be found in GLR, Phys. Rep. **100**, 1 (1983). The complete argumentation would be far

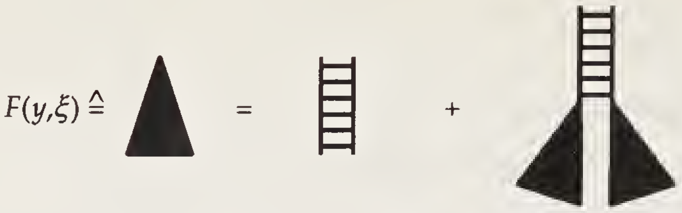


Fig. 6.21. The self-consistent equation for ladder-branching (Fan diagrams)

too lengthy and, furthermore, the validity of this equation is presently very much under discussion. Here we just wanted to present the fundamental ideas central to all discussions of low-x physics. Hopefully this whole field will be much better understood in a few years from now.

Leaving all the theoretical problems connected with (6.148) aside, many people have analysed its phenomenological consequences, substituting typically $F \Rightarrow x\,G(x)$ (the gluons are by for dominant at the small x-values considered). It turned out that, if correct, the non-linear term should lead to observable effects at the lowest x-values attainable at HERA with sufficiently large Q^2 to allow for QCD perturbation theory.

One should also notice that (6.148) has a rather simple intuitive meaning. The GLAP equations are linear in the distribution functions. This is justified as long as they are sufficiently small to make $\alpha_s\, G(x) \ll 1$, i.e. for reasonably large x. If the distribution function get larger, non-linear effects have to become large at some point. While the usual GLAP evolution increases all distribution functions with increasing Q^2, at some point e.g. gluon-gluon fusion should start to be important, leading to a saturation at extremely small x.

7. Nonperturbative QCD

Our discussion began with the baryon spectra shown in Fig. 1.1. It is believed that exact solutions of the QCD equations reproduce these spectra, just as atomic spectra are explained by QED. Since such exact calculations are not yet available, one cannot be absolutely sure that QCD is really the correct theory of strong interactions.

Many different nonperturbative approaches to the QCD equation have been proposed, frequently employing assumptions and approximations whose justification is not quite clear. We shall therefore discuss only the technique of lattice calculations, which is accepted to be the most exact, and the technique of QCD sumrules (Sect. 7.3) which is conceptually somewhat problematic but extremely successfull. Later some phenomenological ideas are considered that can help achieve a better understanding of the features of QCD.

7.1 Lattice Gauge Theories

Lattice gauge theories try to solve the QCD equations in an approximate way by drastically reducing the number of field degrees of freedom. Continuous space-time is replaced by discrete points. Usually one is restricted to quite small lattices in order to achieve practicable computing times. Even a lattice with 16 points in each direction, i.e., a total of 16^4 points, is considered large, and this is a major problem of these calculations. One expects that for a sufficiently large number of points N the obtained results should no longer depend on N, i.e., they converge to a limiting value. It can be argued whether the lattices used at the moment have already reached this limiting value.

A further problem is the violation of Lorentz invariance. The regular lattices usually employed have a discrete symmetry, which does not correspond to that of physical space. One cannot exclude the possibility that this leads to mistakes. However, it has been shown, for example, in two-dimensional calculations with cylindrical symmetry, that the physical symmetry is conserved for sufficiently large N. A possibility of avoiding artificial features of the lattice is to use irregular Monte Carlo lattices. Averaging many results of such lattices then cancels spurious effects. But at present this approach still demands too much computing time.

In addition special features of QCD result in difficulties. An increasing number of lattice points leads to a diminishing distance between them. Correspondingly one also has to modify the running coupling constant, which vanishes in the limit $N \to \infty$ (asymptotic freedom). But such a modification can cause phase transitions resulting in sudden changes at a specific value of N. These phase transitions, however, are artificial effects due to the discrete treatment of the theory and have no physical meaning. Because of their possible existence it is very difficult to decide at what value of N the continuum limit has really been reached.

Finally the lattice formulation of QCD is not unique and a complete treatment of fermions faces tremendous difficulties.

Although a number of unsettled questions remain, the results obtained with lattice gauge calculations have so far substantially improved our knowledge of QCD. An example of its successes can be seen in Table 7.1, which shows ratios of hadron masses obtained by the IBM lattice groups.[1] The agreement is typically better than 5%. In addition the QCD scaling parameter in the $\overline{MS}$ regularization scheme is compared with the value derived from various analyses of the experimental data. The very good agreement obvious from Table 7.1 probably implies that dynamical gluon degrees of freedom, which can be treated exactly, are much more important than the dynamical quark degrees of freedom, which were approximated rather crudely in this calculation.

Table 7.1. The ratio of hadron masses calculated on the lattice

Ratio	Lattice calculation	Experimental value
m_{K^*}/m_ρ	1.167 ± 0.016	1.164
m_ϕ/m_ρ	1.333 ± 0.032	1.327
m_{N}/m_ρ	1.219 ± 0.105	1.222
$(m_\Xi + m_\Sigma - m_{\mathrm{N}})/m_\rho$	1.930 ± 0.073	2.047
m_Δ/m_ρ	1.595 ± 0.111	1.604
m_{Σ^*}/m_ρ	1.821 ± 0.075	1.803
m_{Ξ^*}/m_ρ	2.063 ± 0.067	1.996
m_Ω/m_ρ	2.298 ± 0.098	2.177
$\Lambda_{\mathrm{QCD}}(\overline{MS})/m_\rho$	0.319 ± 0.012	$0.30 - 0.33$

To understand how gauge theories can be formulated on a lattice, it is helpful to rewrite the QCD equations. In Example 4.1 we sketched the geometric interpretation of QCD and discussed the change of a quark spinor owing to an infinitesimal *parallel transport*:

$$\begin{aligned}\Psi(x+\mathrm{d}x) &= (1+\mathrm{d}x^\mu \cdot \hat{D}_\mu)\Psi(x) \\ &= \Psi(x) + \mathrm{d}x^\mu \partial_\mu \Psi(x) + \mathrm{d}x^\mu \left[-\mathrm{i}g\hat{\tilde{A}}_\mu(x)\right]\Psi(x) \\ &= (1+\mathrm{d}x^\mu\partial_\mu)\left[1-\mathrm{d}x^\mu \mathrm{i}g\hat{\tilde{A}}_\mu(x)\right]\Psi(x) \end{aligned} \tag{7.1}$$

with the abbreviation introduced earlier

$$\hat{\tilde{A}}_\mu = \sum_{a=1}^{8} A_\mu^a \frac{\tilde{\lambda}^a}{2} \quad .$$

The first bracket contains the basic dependence on the position and the second bracket describes the effect of the parallel transport on $\Psi(x)$. This expression can be written as

---

[1] See F. Butler et al.: Phys. Rev. Lett. **70**, 2849 (1993).

$$\hat{U}(x + \mathrm{d}x, x) = \mathrm{e}^{-\mathrm{i}g\hat{\hat{A}}_\mu(x)\mathrm{d}x^\mu} \quad . \tag{7.2}$$

The combined effect of many infinitesimal parallel transports leads to the integral

$$\hat{U}(y, x) = P\left\{\mathrm{e}^{-\mathrm{i}g\int_C \hat{\hat{A}}_\mu(z)\mathrm{d}z^\mu}\right\} \quad , \tag{7.3}$$

where C denotes the path from x to y. Since the operators $\hat{\hat{A}}_\mu(z)$ do not commute with each other, a so-called path-ordering P has been introduced. P is defined in such a way that $\hat{\hat{A}}_\mu(z_1)$ stands to the right of $\hat{\hat{A}}_\mu(z_2)$ if the path ℓ reaches z_1 earlier than z_2. Under a gauge transformation

$$\Psi(x) \to \mathrm{e}^{\mathrm{i}g\hat{\hat{\theta}}(x)}\Psi(x) \tag{7.4}$$

$\hat{U}(y, x)$ transforms like

$$\hat{U}(y, x) \to \mathrm{e}^{\mathrm{i}g\hat{\hat{\theta}}(y)}\hat{U}(y, x)\mathrm{e}^{-\mathrm{i}g\hat{\hat{\theta}}(x)} \quad . \tag{7.5}$$

It should be noted that QCD can also be formulated as a theory of the path-dependent phase factors (7.3). But we shall not discuss this further.

From (7.5) the gauge invariance of $\mathrm{tr}\left[\hat{U}_C(x, x)\right]$ follows directly. Furthermore Definition (7.3) shows that the specific choice of x is not significant. Therefore $\mathrm{tr}\left[\hat{U}_C(x, x)\right] \equiv \mathrm{tr}\left[\hat{U}_C\right]$ is an interesting invariant known as *Wilson's loop operator*. Because of (7.3) the gauge fields can be completely replaced by the translation operators $\hat{U}(x, y)$. We restrict our discussion to regular rectangular lattices. The transition to other lattice structures, however, is by no means trivial.

Let us denote the lattice points by

$$x_i = a(k_i, l_i, m_i, n_i) \quad k_i, l_i, m_i, n_i \in Z \quad . \tag{7.6}$$

The quark fields $\Psi_C(x_i)$ are defined at these points. The translation operators can then be built from those fields, which connect two adjoining lattice points:

$$\hat{U}(x_i, x_j) \quad , \quad |x_i - x_j| = a \quad . \tag{7.7}$$

For a lattice with N^4 points the quark fields are therefore described by N^4 spinors $\Psi_C(x_i)$ and the gauge fields by $(8N^4)/2 = 4N^4$ matrix-valued *link* variables:

$$\hat{U}(x_i, x_j) = \hat{U}^{-1}(x_j, x_i) \quad . \tag{7.8}$$

Each point has 8 links to its neighboring lattice points and the factor 1/2 is due to the double-counting of the links. In Fig. 7.1 this is depicted for a two-dimensional lattice. Usually the edges of the lattice are identified with each other, i.e., periodic boundary conditions are employed.

Discrete gauge field variables correspond to discrete variables $\hat{U}(x_i, x_j)$. If we denote the links of a lattice by ℓ_i, the $\hat{U}(\ell_i)$ then depend only on the integrals

$$\int_{\ell_i} \hat{\hat{A}}_\mu(z)\mathrm{d}z^\mu =: a\hat{A}(\ell_i) \quad . \tag{7.9}$$

Here $\hat{A}(\ell_i)$ parametrizes the component of $\hat{\hat{A}}_\mu(x)$ that points in the corresponding direction ℓ_i. Therefore in the continuum limit ($a \to 0$)

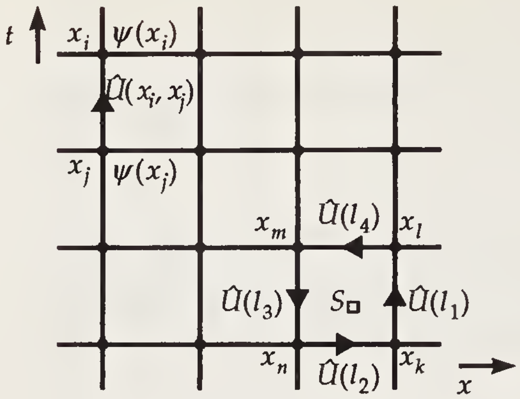


Fig. 7.1. A two-dimensional lattice. The quark fields are defined on the lattice points. The translation operators $\hat{U}(x_m, x_l)$, which describe the gauge fields, correspond to the lines below two lattice points (the so-called *links*). In order to define the action one considers the product of the $\hat{U}$ operators on an elementary square (or *plaquette*)

$$\begin{aligned}
&\hat{A}(\ell_1) \hat{=} \hat{\hat{A}}_0(x_k) \quad , \\
&\hat{A}(\ell_2) \hat{=} \hat{\hat{A}}_1(x_k) \quad , \\
&\hat{A}(\ell_3) \hat{=} -\hat{\hat{A}}_0(x_n) \quad , \\
&\hat{A}(\ell_4) \hat{=} -\hat{\hat{A}}_1(x_l) \quad ,
\end{aligned} \tag{7.10}$$

where a denotes the lattice constant, i.e., the length of the links. The links can be associated with either end point, but this has to be done consistently. We have associated ℓ_1 and ℓ_3 with the lower points x_n and x_k:

$$\hat{U}(\ell_i) = \mathrm{e}^{-\mathrm{i}g\hat{A}(\ell_i)a} \quad . \tag{7.11}$$

Every path C can be constructed from links, i.e., we only consider such paths. Hence

$$\hat{U}_C(y,x) = P \prod_{\ell_i \in C} \hat{U}(\ell_i) \quad . \tag{7.12}$$

The Wilson operator is particularly interesting. For an elementary square, a so-called *plaquette*, (7.11) gives

$$\begin{aligned}
W_\Box &= \mathrm{tr}\left[\hat{U}(\ell_1)\hat{U}(\ell_2)\hat{U}(\ell_3)\hat{U}(\ell_4)\right] \\
&= \mathrm{tr}\left[\mathrm{e}^{-\mathrm{i}ga\hat{A}(\ell_1)}\mathrm{e}^{-\mathrm{i}ga\hat{A}(\ell_2)}\mathrm{e}^{-\mathrm{i}ga\hat{A}(\ell_3)}\mathrm{e}^{-\mathrm{i}ga\hat{A}(\ell_4)}\right] \\
&= \mathrm{tr}\Bigg\{ \exp\Bigg[-\mathrm{i}ga\left(\hat{A}(\ell_1) + \hat{A}(\ell_2) + \hat{A}(\ell_3) + \hat{A}(\ell_4)\right) \\
&\quad -\frac{g^2a^2}{2}\left(\left[\hat{A}(\ell_1), \hat{A}(\ell_2)\right]_- + \left[\hat{A}(\ell_1), \hat{A}(\ell_3)\right]_- + \left[\hat{A}(\ell_1), \hat{A}(\ell_4)\right]_- \right. \\
&\quad \left. + \left[\hat{A}(\ell_2), \hat{A}(\ell_3)\right]_- + \left[\hat{A}(\ell_2), \hat{A}(\ell_4)\right]_- + \left[\hat{A}(\ell_3), \hat{A}(\ell_4)\right]_- \right) \\
&\quad + O(g^3a^3)\Bigg]\Bigg\} \quad ,
\end{aligned} \tag{7.13}$$

where we have used the notation of Fig. 7.1. From (7.8) and (7.9) it follows that $\hat{A}(\ell)$ changes its sign if the starting and the ending points are exchanged:

$$\hat{A}(\ell_3) = -\hat{A}(-\ell_3) \quad . \tag{7.14}$$

The sum $\hat{A}(\ell_1) + \hat{A}(\ell_3)$ is therefore a measure for the x_1 dependence of $\hat{\hat{A}}$:

$$\begin{aligned} \hat{A}(\ell_1) + \hat{A}(\ell_3) &= a \left.\frac{\partial \hat{\hat{A}}_0}{\partial x^1}\right|_{x=x_k} + O(a^2) \quad , \\ \hat{A}(\ell_2) + \hat{A}(\ell_4) &= a \left.\frac{\partial \hat{\hat{A}}_1}{\partial x^0}\right|_{x=x_k} + O(a^2) \quad . \end{aligned} \tag{7.15}$$

Owing (7.10) and the fact that

$$\begin{aligned} A_1(x_n) &= A_1(1 - a\partial_1 + \dots)A_1(x_k) \\ &= A_1(x_k) + O(a) \quad , \end{aligned}$$

we identify in addition the following:

$$\begin{aligned} \left[\hat{A}(\ell_1), \hat{A}(\ell_2)\right]_- &= \left[\hat{\hat{A}}_0, \hat{\hat{A}}_1\right]_- (x_k) \quad , \\ \left[\hat{A}(\ell_1), \hat{A}(\ell_3)\right]_- &= \left[\hat{\hat{A}}_0, -\hat{\hat{A}}_0\right]_- (x_k) + O(a) = O(a) \quad , \\ \left[\hat{A}(\ell_1), \hat{A}(\ell_4)\right]_- &= \left[\hat{\hat{A}}_0, -\hat{\hat{A}}_1\right]_- (x_k) + O(a) = -\left[\hat{\hat{A}}_0, \hat{\hat{A}}_1\right]_- (x_k) + O(a) \quad , \\ \left[\hat{A}(\ell_2), \hat{A}(\ell_3)\right]_- &= \left[\hat{\hat{A}}_1, -\hat{\hat{A}}_0\right]_- (x_k) + O(a) = \left[\hat{\hat{A}}_0, \hat{\hat{A}}_1\right]_- (x_k) + O(a) \quad , \\ \left[\hat{A}(\ell_2), \hat{A}(\ell_4)\right]_- &= \left[\hat{\hat{A}}_1, -\hat{\hat{A}}_1\right]_- (x_k) + O(a) = O(a) \quad , \\ \left[\hat{A}(\ell_3), \hat{A}(\ell_4)\right]_- &= \left[\hat{\hat{A}}_0, \hat{\hat{A}}_1\right]_- (x_k) + O(a) \quad , \end{aligned} \tag{7.16}$$

The Wilson operator can then be expressed as

$$\begin{aligned} W_\Box &= \mathrm{tr}\left\{\exp\left[-iga^2\left(\frac{\partial \hat{\hat{A}}_0}{\partial x^1} - \frac{\partial \hat{\hat{A}}_1}{\partial x^0}\right) - g^2a^2\left[\hat{\hat{A}}_0, \hat{\hat{A}}_1\right]_- + O(a^3)\right]\right\} \\ &= \mathrm{tr}\left\{\exp\left[-iga^2\left(\partial_1\hat{\hat{A}}_0 - \partial_0\hat{\hat{A}}_1 - \mathrm{i}g\left[\hat{\hat{A}}_0, \hat{\hat{A}}_1\right]_-\right) + O(a^3)\right]\right\} \\ &= \mathrm{tr}\left\{\exp\left[-iga^2\hat{\hat{F}}_{10} + O(a^3)\right]\right\} \quad . \end{aligned} \tag{7.17}$$

In an expansion of the exponential function the linear term proportional to $\hat{\hat{F}}_{10} = F^a_{10} \cdot 1/2 \cdot \hat{\lambda}^a$ vanishes owing to the evaluation of the trace.

$$\begin{aligned} W_\Box &= 1 \cdot \mathrm{tr}(1) - iga^2F^a_{10}\frac{1}{2}\mathrm{tr}\left(\hat{\lambda}^a\right) - \frac{1}{2}g^2a^4F^a_{10}F^b_{10}\frac{1}{4}\mathrm{tr}\left(\hat{\lambda}^a\hat{\lambda}^b\right) + O(a^5) \\ &= 3 - \frac{1}{4}g^2a^4F^a_{10}F^a_{10} + O(a^5) \quad , \end{aligned} \tag{7.18}$$

which implies that

$$\frac{2}{g^2}(3 - W_\Box) = \frac{1}{2}a^4 F^a_{10}F^a_{10} + O(a^5) \quad . \tag{7.19}$$

If this is summed over all plaquettes and the sums are replaced by integrals, we obtain the well-known expression for the action

$$\sum_\Box \frac{2}{g^2}(3 - W_\Box) = \frac{1}{4}\int \mathrm{d}^4x F^a_{\mu\nu}F^{a\mu\nu} + O(a^5) \quad . \tag{7.20}$$

The additional factor 1/2 is due to the fact that $F^a_{\mu\nu}F^{a\mu\nu}$ contributes $F^a_{10}F^{a10}$ and $F^a_{01}F^{a01} = F^a_{10}F^{a10}$. In the limit $a \to 0$, i.e., for arbitrarily dense lattices, the so-called *Wilson action*

$$S_\mathrm{W} = \sum_\Box \frac{2}{g^2}(3 - W_\Box) = \sum_\Box \frac{2}{g^2}\left\{3 - \mathrm{Re}\left[\mathrm{tr}(U_\Box)\right]\right\} \tag{7.21}$$

therefore assumes the correct continuum value. In (7.21) we have denoted the product between the U operators of an elementary plaquette by $U_\Box$. Since $Tr(U_\Box)$ is complex for a finite lattice constant, we have in addition taken the real part of the trace, making the Wilson action a real positive number. The properties of $U_\Box$ imply that S_W is positive: with $U(\ell_i)$, $U_\Box$ also is unitary and can therefore be diagonalized, yielding eigenvalues $\exp[\mathrm{i}\alpha]$. Hence

$$3 - \mathrm{Re}\left[\mathrm{tr}(U_\Box)\right] = 3 - \mathrm{Re}\left[\exp(\mathrm{i}\alpha_1) + \exp(\mathrm{i}\alpha_2) + \exp(\mathrm{i}\alpha_3)\right] \geq 0 \quad . \tag{7.22}$$

At finite lattice constants a the choice (7.21) for the action is not unique. One can think of different expressions that have the same limiting value but that can lead to different numerical results for small lattices, which can be treated with present day computers. To eliminate these uncertainties it is essential to employ different methods for formulating QCD on a lattice. By comparing the results obtained with different approaches one gets a feeling for the remaining uncertainties. Furthermore different methods are required to describe different aspects of QCD correctly and it is very important to find out for which problems a given method fails.

If a specific form of the action has been assumed, for example S_W from (7.21), the partition function is an integral over all values of the translation operators $\hat{U}(\ell_i)$:

$$\mathcal{Z} = \int D[U(\ell_i)] \exp\{-S_\mathrm{W}[U(\ell_i)]\} \quad . \tag{7.23}$$

All thermodynamical quantities of interest can be derived from the partition function.

Up to now we have considered only the gauge fields. To achieve a complete description of QCD the quarks also have to be treated dynamically. However, if one tries to apply the lattice formulation discussed above to fermions, a new problem is encountered, the so-called *fermion doubling problem*. The fermion fields are defined on the lattice points x_i and the usual Dirac Lagrangian is discretized according to, for example,

$$
\begin{aligned}
\frac{1}{a^4} I &= \sum_{x_i} \mathcal{L}_{\text{lattice}}(x_i) \\
&= \sum_{x_i} \left[\sum_{\mu} \frac{-1}{2a\mathrm{i}} \left[\bar{\Psi}(x_i)\gamma_\mu \Psi(x_i + a_\mu) \right.\right. \\
&\quad \left.\left. - \bar{\Psi}(x_i + a_\mu)\gamma_\mu \Psi(x_i) \right] - m\bar{\Psi}(x_i)\Psi(x_i) \right] \qquad (7.24)
\end{aligned}
$$

with the lattice constant a. Here a_μ denotes a vector of length a in the μ direction. In the continuum limit (7.24) corresponds to the symmetric Lagrangian

$$
\mathcal{L} = \bar{\Psi} \left[\frac{1}{2}\mathrm{i} \left(\gamma_\mu \overrightarrow{\partial}^{\mu} - \gamma_\mu \overleftarrow{\partial}^{\mu} \right) - m \right] \Psi \quad .
$$

Now we introduce the translation operators into (7.24) (note that we do not sum over the index μ)

$$
\Psi(x_i + a_\mu) = \exp\left(a_\mu \partial^\mu\right) \Psi(x_i) \quad . \qquad (7.25)
$$

In position space the Dirac operator then becomes

$$
\frac{1}{2a\mathrm{i}} \gamma_\mu \left[\exp\left(a_\mu \overrightarrow{\partial}^{\mu} \right) - \exp\left(a_\mu \overleftarrow{\partial}^{\mu} \right) \right] - m \qquad (7.26)
$$

and the corresponding propagator in momentum space

$$
S(p) = \left[-\frac{1}{2ai} \gamma_\mu \left(\mathrm{e}^{-\mathrm{i}a_\mu p^\mu} - \mathrm{e}^{\mathrm{i}a_\mu p^\mu} \right) - m \right]^{-1} = \left(\frac{1}{a} \gamma_\mu \sin a_\mu p^\mu - m \right)^{-1} \quad . \qquad (7.27)
$$

Now we consider the two limiting cases $p_\mu \to 0$ and $a_\mu p^\mu \to \pi$. The former yields

$$
\lim_{p_\mu \to 0} S(p) = \left[\gamma_\mu p^\mu - m \right]^{-1} \quad , \qquad (7.28)
$$

i.e., the usual continuum expression. In the latter case we have

$$
\lim_{a_\mu p^\mu \to \pi} S(p) = \left[\frac{1}{a} \gamma_\mu (\pi - a_\mu p^\mu) - m \right]^{-1} \quad . \qquad (7.29)
$$

We define $\tilde{p}_\mu$ by $\tilde{p}_\mu = a_\mu \frac{\pi}{a^2} - p_\mu$ and obtain

$$
\lim_{a_\mu p^\mu \to \pi} S(p) = \left[\gamma_\mu \tilde{p}^\mu - m \right]^{-1} \quad . \qquad (7.30)
$$

Hence there are two free solutions for every given lattice constant a and for every quark mass. In the limit $a \to 0$ one of them becomes the continuum solution. The energy of the second solution becomes infinitely large in this limit, so that these solutions decouple from all processes and become unobservable. This therefore has to be interpreted as a lattice artefact. Because of the periodicity of (7.27), which restricts the momentum eigenvalues to $-\pi a_\mu / a^2 < p_\mu < +\pi a_\mu / a^2$, only these two solutions exist (see Fig. 7.2).

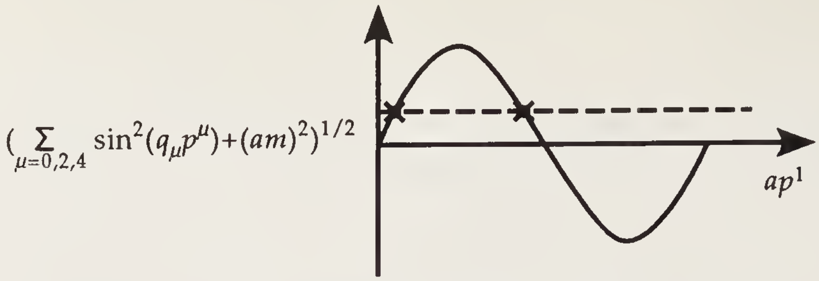


Fig. 7.2. Graphical solution of the eigenvalue problem for the discrete Dirac operator and the problem of fermion doubling. For each momentum component p^μ two different solutions exist for the eigenvalue problem. There are therefor a total of $2^4 = 16$ solutions

Since the fermion doubling discussed above occurs for every momentum component, fifteen nonphysical states are obtained in addition to every physical state, provided the most obvious ansatz (7.24) for the lattice Lagrangian is employed (see Exercise 7.1).

Here the question arises whether this problem can be avoided by a different choice for the lattice Lagrangian than (7.24). One finds, however, that those Lagrangians which can actually avoid fermion doubling look very unnatural. We can only sketch the reason for this.

All obvious fermionic lattice Lagrangians yield in the limit $m \to 0$ a chiral invariant theory, i.e., a theory that is invariant under the transformation

$$\Psi(x_i) \to e^{i\theta(x_i)\gamma_5}\Psi(x_i) \quad . \tag{7.31}$$

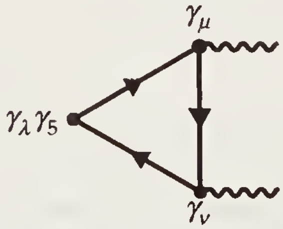
Correspondingly both the vector current

$$\bar{\Psi}(x_i)\gamma_\mu\Psi(x_i) \tag{7.32}$$

and the axial vector current

$$\bar{\Psi}(x_i)\gamma_\mu\gamma_5\Psi(x_i) \tag{7.33}$$

would be conserved if there were no fermion doubling. But this is not true for the common continuum QCD. Divergent triangle graphs of the kind



violate the conservation of the axial vector current. This feature is known as the *anomaly of the axial vector current*. The anomaly is strongly related to the renormalization properties of quantum field theories. Since the regularization of lattice gauge theories is automatically provided by the lattice constant, every lattice gauge theory differs on this point from its corresponding continuum theory. Chiral invariant lattice gauge theories, i.e., those whose Lagrangians are invariant for $m = 0$ under transformation (7.27), have no anomalies. But on the other hand the limit $a \to 0$ yields the continuum theory with the anomaly. This seeming contradiction can be resolved by noting that a lattice formulation based on (7.24) contains additional nonphysical fermions, which cancel the contribution of the physical fermions to the anomaly at finite values of a. In the limit $a \to 0$ these unphysical degrees of freedom disappear and one is left with only one fermion-species and an anomaly. This will be further illustrated in Exercise 7.2.

Clearly one has to change the lattice Lagrangian (7.24) in order to avoid fermion doubling. There are basically two possible ways of achieving this. The first is to make sure that in momentum space the propagator assumes the common unique form

$$S(p) \to (p - m)^{-1} \quad . \tag{7.34}$$

If this is transformed back to position space, one obtains a nonlocal lattice Lagrangian. While (7.24) only contains products of fermion fields at adjacent lattice points, terms of the form $\Psi(x_i)\gamma_\mu\Psi(x_i + na_\mu)$ with arbitrary n now occur. In Monte Carlo calculations, however, where a huge number of field configurations has to be evaluated, this approach leads to impracticable large computing times.

The second possibility is therefore choosen. Instead of (7.24), the following lattice Lagrangian, for example, is assumed (r is constant):

$$\begin{aligned}\mathcal{L}_{\text{lattice}}(x_i) = {} & \sum_\mu -\frac{1}{2a\mathrm{i}}\left[\bar{\Psi}(x_i)\gamma_\mu\Psi(x_i + a_\mu) - \bar{\Psi}(x_i + a_\mu)\gamma_\mu\Psi(x_i)\right] - m\bar{\Psi}(x_i)\Psi(x_i) \\ & + \frac{r}{2a}\sum_\mu\left[\bar{\Psi}(x_i)\Psi(x_i + a_\mu) + \bar{\Psi}(x_i + a_\mu)\Psi(x_i)\right] - \frac{4r}{a}\bar{\Psi}(x_i)\Psi(x_i) \quad .\end{aligned} \tag{7.35}$$

EXERCISE

7.1 Fermion Doubling

Problem. Show that the discretized Lagrangian (7.24) has a symmetry group with the transformation law

$$\Psi(x_i) \to \hat{T}_l\Psi(x_i) \quad , \quad \bar{\Psi}(x_i) \to \bar{\Psi}(x_i)\hat{T}_l^{-1} \quad , l = 1, 2, \dots, 16 \tag{1}$$

and the operators

$$\begin{aligned}& \hat{T}_1 = \hat{1} \quad , \hat{T}_2 = \gamma_0\gamma_5(-)^{\frac{x_0}{a}} \quad , \quad \hat{T}_3 = \gamma_1\gamma_5(-)^{\frac{x_1}{a}} \quad , \\ & \hat{T}_4 = \gamma_2\gamma_5(-)^{\frac{x_2}{a}} \quad , \quad \hat{T}_5 = \gamma_3\gamma_5(-)^{\frac{x_3}{a}} \quad , \quad \hat{T}_6 = \hat{T}_2 \cdot \hat{T}_3 \quad , \\ & \hat{T}_7 = \hat{T}_2 \cdot \hat{T}_4 \quad , \quad \hat{T}_8 = \hat{T}_2 \cdot \hat{T}_5 \quad , \quad \hat{T}_9 = \hat{T}_3 \cdot \hat{T}_4 \quad , \\ & \hat{T}_{10} = \hat{T}_3 \cdot \hat{T}_5 \quad , \quad \hat{T}_{11} = \hat{T}_4 \cdot \hat{T}_5 \quad , \quad \hat{T}_{12} = \hat{T}_2 \cdot \hat{T}_3 \cdot \hat{T}_4 \quad , \\ & \hat{T}_{13} = \hat{T}_2 \cdot \hat{T}_3 \cdot \hat{T}_5 \quad , \quad \hat{T}_{14} = \hat{T}_2 \cdot \hat{T}_4 \cdot \hat{T}_5 \quad , \quad \hat{T}_{15} = \hat{T}_3 \cdot \hat{T}_4 \cdot \hat{T}_5 \quad , \\ & \hat{T}_{16} = \hat{T}_2 \cdot \hat{T}_3 \cdot \hat{T}_4 \cdot \hat{T}_5\end{aligned} \tag{2}$$

Exercise 7.1.

and that one obtains the *nonphysical* fermion states by applying these operators to the physical states.

Solution. In order to prove that (1) and (2) in fact define a symmetry group of (7.24), we investigate the effects of $\hat{T}_2$ to $\hat{T}_5$. Only the kinetic term is important.

$$\begin{aligned}&\sum_\mu -\frac{1}{2ai}\left[\bar{\Psi}(x)\gamma_\mu\Psi(x+a_\mu)-\bar{\Psi}(x+a_\mu)\gamma_\mu\Psi(x)\right]\\ &\to\sum_\mu -\frac{1}{2ai}\Big[\bar{\Psi}(x)\gamma_5\gamma^\nu(-)^{\frac{x_\nu}{a}}\gamma_\mu\gamma_\nu\gamma_5(-)^{\frac{x_\nu}{a}+\delta_{\mu\nu}}\Psi(x+a_\mu)\\ &\quad-\bar{\Psi}(x+a_\mu)\gamma_5\gamma^\nu(-)^{\frac{x_\nu}{a}+\delta_{\mu\nu}}\gamma_\mu\gamma_\nu\gamma_5(-)^{\frac{x_\nu}{a}}\Psi(x)\Big]\quad .\end{aligned}\tag{3}$$

Because $x_\nu = N_\nu \cdot a$, $N_\nu \in Z$, we have $(-)^{\frac{x_\nu}{a}} = (-)^{-\frac{x_\nu}{a}} = \pm 1$, and therefore the kinetic term becomes just

$$\begin{aligned}&\sum_\mu -\frac{1}{2ai}\left[\bar{\Psi}(x)(-)^{\delta_{\mu\nu}+1}\gamma^\nu\gamma_\mu\gamma_\nu\Psi(x+a_\mu)\right.\\ &\quad\left.-\bar{\Psi}(x+a_\mu)(-)^{\delta_{\mu\nu}+1}\gamma^\nu\gamma_\mu\gamma_\nu\Psi(x)\right]\quad .\end{aligned}\tag{4}$$

In the last step we have permuted γ_5 with the other γ matrices and combined the minus sign with $\delta_{\mu\nu}$. For $\mu \neq \nu$ we have (no summation over ν)

$$(-)^{\delta_{\mu\nu}+1}\gamma^\nu\gamma_\mu\gamma_\nu = -\gamma^\nu\gamma_\mu\gamma_\nu = \gamma_\mu \quad . \tag{5}$$

For $\mu = \nu$

$$(-)^{\delta_{\mu\nu}+1}\gamma^\nu\gamma_\mu\gamma_\nu = (-)^2\gamma_\mu = \gamma_\mu \quad . \tag{6}$$

Therefore (4) reduces to

$$\sum_\mu -\frac{1}{2ai}\left[\bar{\Psi}(x)\gamma_\mu\Psi(x+a_\mu)-\bar{\Psi}(x+a_\mu)\gamma_\mu\Psi(x)\right] \tag{7}$$

and $\hat{T}_2$ to $\hat{T}_5$ are the symmetry operators of the lattice action function in (7.24). Since all $\hat{T}_l$ can be constructed from these operators, our calculations can be applied directly to all $\hat{T}$'s.

Now we show that $\hat{T}_l\Psi(x)$, $l = 2, \dots, 16$ are the non-physical fermion states. The phase factor $(-)^{\frac{x_\mu}{a}}$ can also be written as $\exp(-i\pi\frac{x_\mu}{a})$. Then the Fourier transformation of, e.g., $\hat{T}_2\Psi(x)$ yields

$$\begin{aligned}\sum_x e^{ip^\mu x_\mu}\hat{T}_2\Psi(x) &= \gamma_0\gamma_5\sum_x e^{i(p^\mu-\frac{\pi}{a}\delta_{0\mu})x_\mu}\Psi(x)\\ &= \gamma_0\gamma_5\Psi\left(p^0-\frac{\pi}{a},p^1,p^2,p^3\right)\quad ,\end{aligned}\tag{8}$$

i.e., a fermion state with a momentum component p^2 diminished by $\frac{\pi}{a}$ (modulo $\frac{2\pi}{a}$), which is one of the non-physical states. In an analogous way the application of all operators $\hat{T}_l$ yields the 16 momentum states

$$\left(p^0-j\frac{\pi}{a},p^1-k\frac{\pi}{a},p^2-l\frac{\pi}{a},p^3-m\frac{\pi}{a}\right)\quad \text{with}\quad j,k,l,m=0,1\quad . \tag{9}$$

EXERCISE

7.2 Anomalies in Lattice Gauge Theories

Problem. Show that for $m = 0$ the contributions of the 16 fermion fields $\Psi_l(x) = \hat{T}_l\Psi(x)$ (with $\hat{T}_l$ from Problem 7.1) to the axial vector anomaly cancel each other. Make use of the fact that the axial vector A_μ, which is treated as a background field, couples to the fermions according to

$$\mathcal{L}(x) = \sum_\mu \frac{-1}{2ai}\Big[\overline{\Psi}(x)\gamma_\mu e^{ia\gamma_5 A_\mu(x)}\Psi(x+a_\mu) \\ - \overline{\Psi}(x+a_\mu)\gamma_\mu e^{-ia\gamma_5 A_\mu(x)}\Psi(x)\Big] \quad . \tag{1}$$

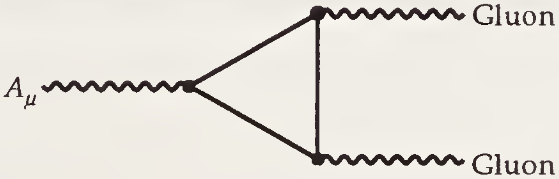
Equation (1) is obtained by performing a chiral transformation (see (7.31)) and writing the gauge angles as

$$\theta(x+a_\mu) - \theta(x) := aA_\mu(x) \quad . \tag{2}$$

In the same manner the gauge angles of other symmetries are also replaced by vector gauge fields. Correspondingly in the continuum limit one would write

$$\frac{\partial\theta(x)}{\partial x^\mu} = A_\mu(x) \quad . \tag{3}$$

Solution. To evaluate the anomaly graph we also need the quark gluon coupling.



However, it is sufficient to know the quark A_μ vertex in order to prove that its contribution vanishes. The coupling to the gluons and the fermion propagators are the same for all kinds of fermions. By expanding (1) into powers of A_μ we obtain for the quark A_μ vertex

$$-\frac{1}{2ai}iaA_\mu(x)\Big[\overline{\Psi}(x)\gamma_\mu\gamma_5\Psi(x+a_\mu) + \overline{\Psi}(x+a_\mu)\gamma_\mu\gamma_5\Psi(x)\Big] \quad . \tag{4}$$

Repeating steps (7.23–26) yields the momentum-space expression

$$-A_\mu(k)\overline{\Psi}(q)\gamma_\mu\gamma_5\cos\left[\frac{1}{2}(p+q)\cdot a_\mu\right]\Psi(p)\delta^4(q-k-p)e^{-\frac{1}{2}k\cdot a_\mu} \quad . \tag{5}$$

Now we determine the contributions of $\Psi_l(x) = \hat{T}_l\Psi(x)$, choosing as an example $l = 2$:

Exercise 7.2.

$$\begin{aligned}\Psi(p) &\to \gamma_0\gamma_5\Psi\left(p^0-\frac{\pi}{a},p^1,p^2,p^3\right) \quad ,\\ \overline{\Psi}(q) &\to \overline{\Psi}\left(q^0-\frac{\pi}{a},q^1,q^2,q^3\right)\gamma_5\gamma_0 \quad ,\end{aligned} \tag{6}$$

$$\begin{aligned}&\overline{\Psi}(q)\gamma_\mu\gamma_5\cos\left[\frac{1}{2}(p+q)\cdot a_\mu\right]\Psi(p)\\ &\Rightarrow \overline{\Psi}\left(q-\frac{a_0}{a^2}\pi\right)\gamma_5\gamma^0\gamma_\mu\gamma_5\cos\left[\frac{1}{2}(p+q-\frac{a_0}{a^2}\pi)a_\mu\right]\gamma_0\gamma_5\Psi\left(p-\frac{a_0}{a^2}\pi\right)\\ &=\overline{\Psi}\left(q-\frac{a_0}{a^2}\pi\right)(-)^{\delta_{\mu 0}+1}\gamma_\mu\gamma_5\cos\left[\frac{1}{2}(p+q)a_\mu\right](-)^{\delta_{\mu 0}}\Psi\left(p-\frac{a_0}{a^2}\pi\right) \quad .\end{aligned} \tag{7}$$

Since $\Psi\left(p-\frac{a_0}{a^2}\pi\right)$ has the same propagator as $\Psi(p)$, it can be replaced by $\Psi(p)$. The vertex contribution of $\hat{T}_2\Psi(x)$ then becomes:

$$-A_\mu(k)\overline{\Psi}(q)(-)\gamma_\mu\gamma_5\cos\left[\frac{1}{2}(p+q)\cdot a_\mu\right]\Psi(p) \quad . \tag{8}$$

Correspondingly the sum over all anomaly graphs, i.e. the sum over all 16 fermions, that can build the fermion loop yields a contribution proportional to

$$1+(-)\cdot 4+(+)\cdot 6+(-)\cdot 4+(+)\cdot 1=0 \quad . \tag{9}$$

The first term is due to Ψ_2 to Ψ_5, the second to Ψ_6 to Ψ_{11}, the third to Ψ_{12} to Ψ_{15}, and the last to Ψ_{16}.

Hence the lattice fermion theory defined by (7.23) yields no axial vector anomaly and for this reason alone does not provide the correct continuum theory.

In position space the additional term gives the following contribution to the Dirac operator:

$$\frac{r}{2a}\left(\mathrm{e}^{a_\mu\overrightarrow{\partial}^\mu}+\mathrm{e}^{a_\mu\overleftarrow{\partial}^\mu}\right)-\frac{4r}{a} \quad , \tag{7.36}$$

which, after Fourier transformation, becomes

$$\frac{r}{2a}\cdot 2\cos a_\mu p^\mu-\frac{4r}{a}=-\frac{r}{a}\sum_\mu\left(1-\cos a_\mu p^\mu\right) \quad . \tag{7.37}$$

The modified Dirac operator is now

$$S(p)=\left\{\sum_\mu\frac{1}{a}\gamma_\mu\sin(a_\mu p^\mu)-\left[m+\frac{r}{a}\sum_\mu(1-\cos a_\mu p^\mu)\right]\right\}^{-1} \quad . \tag{7.38}$$

The additional term is now no longer the same at $a_\mu p^\mu$ and at $\pi-a_\mu p^\mu$. Moreover, it diverges for all unphysical states in the limit $a\to 0$ because $a_\mu p^\mu\to\pi$. Only for the physical state does the additional term vanish for $a\to 0$. Obviously the unphysical states acquire an arbitrarily large mass in the continuum limit and thus play no role. In this way, we do not get rid of the unphysical states but they are guaranteed to vanish in the continuum limit. The unspoken hope in this approach is that reaching this limit is not made any harder by the additional term in (7.35), i.e., that larger grids are not required to reach the limit. It must be emphasized that there are many choices for the additional term besides the one discussed above.

Finally let us consider Exercise 7.2 again. There we illustrated the general statement that a chirally invariant lattice theory cannot have a γ^5 anomaly and therefore must contain unphysical fermions. The modified Lagrange density (7.35) now obviously no loger leads to a chirally invariant theory. For $a, m \to 0$, L in (7.35) becomes

$$\mathcal{L} = \bar{\Psi}(x)\not{\partial}\Psi(x) - \frac{3r}{a}\bar{\Psi}(x)\Psi(x) \quad . \tag{7.39}$$

The last term here is not invariant under the chiral transformation

$$\begin{aligned}\bar{\Psi}(x)\Psi(x) &= [\Psi(x)]^{+}\gamma_0\Psi(x) \\ &\to \left[\mathrm{e}^{\mathrm{i}\gamma_5\theta(x)}\Psi(x)\right]^{+}\gamma_0\mathrm{e}^{\mathrm{i}\gamma_5\theta(x)}\Psi(x) = \bar{\Psi}(x)\mathrm{e}^{2\mathrm{i}\gamma_5\theta(x)}\Psi(x) \quad .\end{aligned} \tag{7.40}$$

Accordingly the unphysical fermions do not contribute to the triangle anomaly (in the limit $a \to 0$) and we obtain the usual continuum limit.

When discussing fermion lattice theories, we left out the gauge fields for simplicity. It is easy to see from (7.3–5) how they have to be included in (7.35). They describe the change of the phase between the point x_i and the point $x_i + a_\mu$:

$$\begin{aligned}\mathcal{L}_{\text{lattice gauge theory}}(x_i) = &\sum_\mu \frac{\mathrm{i}}{2a}\,[\bar{\Psi}(x_i)\gamma_\mu U(x_i, x_i + a_\mu)\Psi(x_i + a_\mu) \\ &- \bar{\Psi}(x_i + a_\mu)\gamma_\mu U^{-1}(x_i, x_i + a_\mu)\Psi(x_i)] \\ &- \left(m + \frac{4r}{a}\right)\bar{\Psi}(x_i)\Psi(x_i) \\ &+ \frac{r}{2a}\sum_\mu [\bar{\Psi}(x_i)U(x_i, x_i + a_\mu)\Psi(x_i + a_\mu) \\ &+ \bar{\Psi}(x_i + a_\mu)U^{-1}(x_i, x_i + a_\mu)\Psi(x_i)] \quad .\end{aligned} \tag{7.41}$$

The fermions introduced according to (7.41) are called Wilson fermions. However, this formulation has a major disadvantage. The problem lattice QCD is particularly concerned with is the QCD phase transition from the confined hadronic phase to the so-called quark gluon plasma. This phase transition is discussed in detail in Chap. 8. A phase transition can actually be observed in two different observables (see Fig. 7.13), namely the free energy of a particle and the expectation value of the product $\bar{\Psi}\Psi$. This expectation value vanishes for a massless fermion field. Since it drops to zero at the same (or nearly the same) temperature where the free energy abruptly diminishes, we conclude that the mechanism that generates a finite quark mass stops working at the same time when the color interaction is weakened and deconfinement takes place. Since these two effects are initially totally unrelated, one speaks of two different phase transitions, namely the deconfinement phase transition and the chiral phase transition. The relation between these two processes and the question of which is more fundamental is not yet understood, although there are many different theories. In general one does not assume that this coincidence is purely accidental. All of this creates a problem for ansatz (7.41), where it is just the mass term that is modified, and it is thus unclear whether this leads to an unphysical modification of the chiral phase transition. Owing to the suspected interrelationship between both phase transitions, even the deconfinement phase transition might be modified in an unphysical way. For these reasons, calculations of the QCD phase transitions with Wilson fermions must be regarded with a certain amount of doubt.

The alternative route usually taken, although also not fully satisfactory, is to accept fermion doubling and to identify the excessive fermions with additional fermion degrees of freedom such as spin, isospin, and flavor degrees of freedom.

The kinetic fermion term in (7.41) shows that the field Ψ at the point $(0,0,0,0)$ only couples to the field $\gamma_0\Psi$ at the point $(1,0,0,0)$, to the field $\gamma_1\Psi$ at $(0,1,0,0)$, the field $\gamma_0\gamma_1\Psi$ at $(1,1,0,0)$, and so on. In general all fields which couple can be written as

$$\gamma_0^{m_0}\gamma_1^{m_1}\gamma_2^{m_2}\gamma_3^{m_3}\Psi(m_0,m_1,m_2,m_3) \quad . \tag{7.42}$$

Since Ψ has four components, there are four independent collections of fields. Starting from $(0,0,0,0)$, we can collect all fields coupling to the first component of Ψ. Starting then with the second component, we obtain a linearly independent field configuration at each lattice site.

Now what happens if the fields are defined relative to an arbitrary point $n = (n_0,n_1,n_2,n_3)$ instead of $(0,0,0,0)$. Then the term

$$\left(\gamma_0^{m_0}\gamma_1^{m_1}\gamma_2^{m_2}\gamma_3^{m_3}\Psi\right)^\dagger \gamma_0\gamma_2 U_{n,2}\gamma_0^{m_0}\gamma_1^{m_1}\gamma_2^{m_2+1}\gamma_3^{m_3}\Psi \tag{7.43}$$

changes into

$$\left(\gamma_0^{m_0+n_0}\gamma_1^{m_1+n_1}\gamma_2^{m_2+n_2}\gamma_3^{m_3+n_3}\Psi\right)^\dagger \gamma_0\gamma_2 U_{n,2}\gamma_0^{m_0+n_0}\gamma_1^{m_1+n_1}\gamma_2^{m_2+n_2+1}\gamma_3^{m_3+n_3}\Psi \quad . \tag{7.44}$$

The additional commutation (in this case of γ_2) leads to an additional sign $(-1)^{n_0+n_1}$. Thus in general an additional sign must be introduced in order to have a consistent definition:

$$(-1)^{n_0+\dots+n_{\mu-1}}\bar{\Psi}_m U_{n,\mu}\Psi_{m+\mu} \quad .$$

The γ_μ matrix has been absorbed into the definition of Ψ_m and is no loger present. The fermion contribution to the action is now

$$S = S_G + \sum_{n,m} \bar{\Psi}_n (Q^F)_{n,m}\Psi_m \tag{7.45}$$

with

$$Q^F_{n,m} = ma\delta_{n,m} + \frac{1}{2}\sum_{\mu=0}^{3}(-1)^{n_0+\dots+n_{\mu-1}}\left[U_{n,\mu}\delta_{n,m-\mu} - U^\dagger_{n,\mu}\delta_{n,m+\mu}\right] \quad . \tag{7.46}$$

The fermions thus defined are called staggered fermions or Kogut–Susskind fermions. Expression (7.41) can be greatly simplified by a simple trick. Remembering that for Grassmann variables

$$\int D[\bar{\Psi}_i,\Psi_j]\mathrm{e}^{\bar{\Psi}_i Q^F_{ij}\Psi_j} = \det Q^F \tag{7.47}$$

holds, one can use the same trick as when introducing ghost fields[2]. This replacement has the big advantage that the fermion fields disappear and only the link variables U are needed.

The sum over states of a lattice SU(3) theory with fermions is therefore

$$\mathcal{Z} = \int D[U_{n,\mu}] \cdot \det Q^F \mathrm{e}^{-S_G(U)} \quad , \tag{7.48}$$

with

$$S_G(U) = \beta \sum_{n,\mu\nu} \left(1 - \frac{1}{3}\,\mathrm{Re\,tr}\left\{U_{n,\mu}U_{n+\mu,\nu}U^+_{n+\nu,\mu}U^+_{n,\nu}\right\}\right) \quad . \tag{7.49}$$

A disadvantage of this procedure is that the matrices Q^F are very large and their calculation thus very time consuming. Another disadvantage is of course that there are still too many fermion degrees of freedom. Since each fermion flavor corresponds to four independent sets of fields in the above definition, the 16 fermionic degrees of freedom describe four quark flavors, while only two light, up and down, and one medium heavy (strange) quarks are physically present, rendering all calculations with this action problematic. There are two ways to confront this problem. The first denies it exists by assuming that the number of quark flavors does not play an important role. In the second, following the Columbia collaboration[3], one replaces the determinant by the following expression:

$$\det Q^F \longrightarrow \exp\left[\frac{2}{4}\ln\det Q^F(m_\mathrm{u},d) + \frac{1}{4}\ln\det Q^F(m_\mathrm{s})\right] \quad . \tag{7.50}$$

This should reflect the fact that the contribution of one flavor to the action should be one fourth of the contribution of four flavors. Although this sounds reasonable it should be borne in mind that the determinant trick cannot yield a factor like this. The result of the above-mentioned calculations was that the type of the phase transition depends critically on the individual quark masses. For a light strange quark, $m_s = m_u = m_d$, one obtains, for example a first-order deconfinement phase transition, while realistic masses for the strange quark yield a second-order transition (see Fig. 7.3). As the nature of the phase transition is essential for its phenomenological properties, one has to realize that an ideal procedure of describing fermions on the lattice has not yet been found, and all calculations with fermions have to be interpreted with care. This conclusion is reinforced by the numerical problems of these calculations. To save time and thus to be able to use larger lattices one often neglects the determinant det Q^F all together, i.e., sets it equal to 1. This is called the "quenched fermion approximation". It is motivated by many indications that the gluon dynamic is essential to our understanding QCD, while the role of dynamic quarks, for example, virtual quark antiquark pairs, is very limited. If this is true, then it is reasonable to concentrate on reducing finite-size corrections for

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[2] See W. Greiner and B. Müller: Theoretical Physics, Vol. 5, *Gauge Theory of Weak Interactions* (Springer, Berlin, Heidelberg, New York 1993).

[3] F. R. Brown: Phys. Rev. Lett. **65**, 2491 (1990).

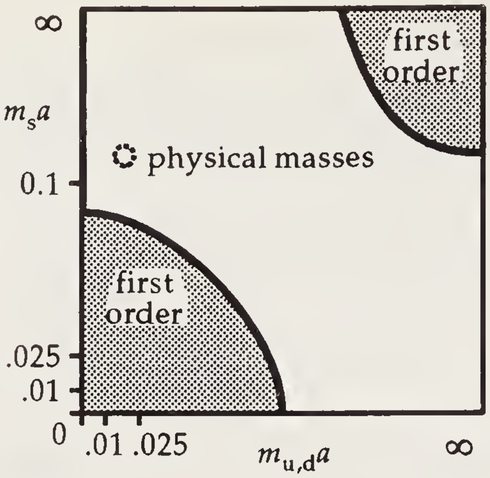

Fig. 7.3. The phase transition diagram for QCD indicating the nature of the phase transition as a function of the quark masses. If the strange quark mass is much larger than the up and down quark masses, the phase transition is of higher order. This is true in particular for the physical values (dashed circle). If all three quark masses are small or very large (the last case is equivalent to a pure gauge theory), one observes a first-order phase transition. The lines separating the regions of first-order and higher-order phase transitions are only meant as an illustration of what the complete diagram might look like

the gluon sector (by using larger lattices) and to neglect the fermion dynamics completely. The results presented in Table 7.1 were obtained, for example with quenched Wilson quarks.

The problems of quenched calculations can also be illustrated at the level of hadrons. We know, for example, that a nucleon is surrounded by a pion cloud. There are even models for the nucleon in which this pion cloud plays a major role in our understanding of the properties of the nucleon (e.g., in so-called Skyrme-type models). Such a pion cloud, however, corresponds to the presence of virtual quark antiquark pairs, as illustrated in Fig. 7.4.

To obtain the action of a lattice gauge theory with fermions, we just add (7.41) to the Wilson action:

$$S = \sum_{\Box} \frac{2}{g^2}(3 - W_{\Box}) + a^4 \mathcal{L}_{\text{lattice gauge theory}} \quad . \tag{7.51}$$

Usually one introduces

$$\beta = \frac{6}{g^2} \quad . \tag{7.52}$$

and the sum over states is written as

$$\mathcal{Z} = \int D[U(\ell_i), \Psi(x_i), \bar{\Psi}(x_i)] \mathrm{e}^{-S}, \quad S = \beta \sum_{\Box} \left(1 - \frac{1}{3} W_{\Box}\right) + \dots \quad . \tag{7.53}$$

Here β cannot be interpreted as the inverse temperature

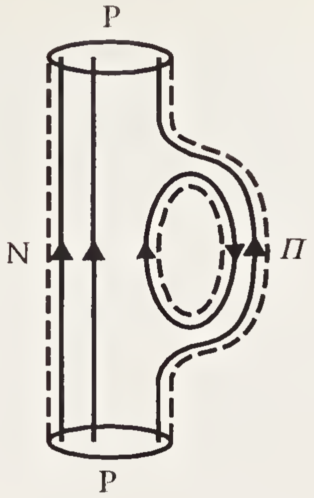

Fig. 7.4. Virtual quark-pair production as a QCD analog to the pion cloud around hadrons. Such terms are neglected in quenched calculations, which could lead to severe errors for a number of phenomenological properties

$$\beta = \frac{6}{g^2} \neq \frac{1}{kT} \tag{7.54}$$

as it is in thermodynamics. In fact the nomenclature is extremely misleading, since β is actually used with three different meanings, namely $\beta = 6/g^2$, $\beta = 1/T$, and β the β function of the renormalization group equation. The actual relation between β and the temperature is discussed in Sect. 7.2.

To calculate (7.53), we must rewrite the integral over all gauge transformations $U(\ell_i)$ into a parametric integral. For the simplest groups such as U(1) this can be done immediately. From

$$U(\ell_i) = \mathrm{e}^{\mathrm{i}\Theta(\ell_i)} \tag{7.55}$$

it follows that

$$\int D[U(\ell_i)] \ldots = \frac{1}{2\pi} \prod_i \int_0^{2\pi} \mathrm{d}\Theta(\ell_i) \ldots \quad . \tag{7.56}$$

For more complex symmetry groups we use instead a matrix representation of the symmetry group and apply the following formula from differential geometry:

$$\begin{gathered} \int D[U(\ell_m)] \ldots = \mathrm{const} \prod_{\ell_m} \int \mathrm{d}\alpha_n(\ell_m) |\mathrm{det} M|^{1/2} \quad , \\ M_{i,j} = \mathrm{tr} \left\{ U^{-1}(\alpha) \left[\frac{\partial}{\partial \alpha_i} U(\alpha) \right] U^{-1}(\alpha) \frac{\partial}{\partial \alpha_j} U(\alpha) \right\} \quad . \end{gathered} \tag{7.57}$$

There are two principal ways of calculating the integral (7.53): either the exponential function is expanded in a series, or the integral is calculated using a Monte Carlo method. The perturbative calculation is obviously only possible for small exponents. In pure SU(3) gauge theory, this means that $\beta = 6/g^2$ is small and thus g is large. It is therefore termed the strong-coupling limit. In this case, (7.53) is reduced to an integral over a finite polynomial of gauge matrices $U(\ell_i)$ that can be calculated by group-theoretical means. The strong-coupling approximation yields some interesting results and can further be understood as a test of Monte Carlo calculations (see Exercise 7.3).

The principle of Monte Carlo integration is comparatively simple. Each integral can be calculated as the average value of the integrand measured at random points in the integration domain. Let us consider for illustration a simple one-dimensional integral

$$\int_0^a \mathrm{d}x\; x = \frac{1}{2}a^2 \quad . \tag{7.58}$$

Let y_i be random numbers between zero and one. Then we can calculate the average,

$$\frac{a}{N}\sum_{i=1}^{N} x_i \quad , \quad x_i = a\; y_i \quad , \tag{7.59}$$

and its value approaches the integral (7.58) for large N:

$$\frac{a}{N}\sum_{i=1}^{N} x_i = a^2\,\frac{1}{N}\sum_{i=1}^{N} y_i = a^2\,\frac{1}{2} \quad . \tag{7.60}$$

In a similar way integrals of the type

$$\int D[U(\ell_i), \Psi(x_i), \bar{\Psi}(x_i)]\mathrm{e}^{-S} f\left(U(\ell_j), \Psi(x_j), \bar{\Psi}(x_j)\right) \tag{7.61}$$

can be calculated by using random values for $U(\ell_i), \Psi(x_i), \bar{\Psi}(x_i)$ and averaging. This procedure is not really effective since $\exp(-S)$ is a very small number for nearly all random values and there is thus only a negligible contribution to (7.61) from these values. It is much more sensible to choose a distribution of the random values such that large $\exp(-S)$ are favored, namely in such a way that the probability of finding a combination of certain random values is proportional to $\exp(-S)$ calculated with these values. To obtain such a distribution one proceeds as follows.

1. Starting from an arbitrary initial configuration $C = [U(\ell_i), \Psi(x_i), \bar{\Psi}(x_i)]$ one makes a small change leading to a configuration C'. The probability $p_0(C \rightarrow C')$ for this change obeys

   $$p_0(C \rightarrow C') = p_0(C' \rightarrow C) \quad . \tag{7.62}$$

   One possibility is to make a random modification of U on a certain link of Ψ, $\bar{\Psi}$ on the neighboring points.
2. One calculates the difference

   $$\Delta S = S(C') - S(C) \quad . \tag{7.63}$$

   If ΔS is negative, C is replaced by C'. If ΔS is positive, one chooses a random number $0 \leq y \leq 1$ and uses C' if $y < \exp(-\Delta S)$. Otherwise one retains the old configuration C.

The resulting transition probabilities in this approach are

$$\begin{aligned} p(C \to C') &= p_0(C \to C') \begin{cases} 1 & \text{for } \Delta S < 0 \\ \mathrm{e}^{-\Delta S} & \text{for } \Delta S > 0 \end{cases} , \\ p(C' \to C) &= p_0(C' \to C) \begin{cases} \mathrm{e}^{\Delta S} & \text{for } \Delta S < 0 \\ 1 & \text{for } \Delta S > 0 \end{cases} , \end{aligned} \tag{7.64}$$

and therefore

$$\frac{p(C \to C')}{p(C' \to C)} = \mathrm{e}^{-\Delta S} \quad . \tag{7.65}$$

With

$$\sum_{C'} p(C \to C') = 1 \tag{7.66}$$

(7.65) implies that

$$\sum_{C} \mathrm{e}^{-S(C)} p(C \to C') = \sum_{C} \mathrm{e}^{-S(C')} p(C' \to C) = \mathrm{e}^{-S(C')} \quad . \tag{7.67}$$

If the configurations C are Boltzmann distributed, so are the configurations C'. The Boltzmann distribution is thus a fixpoint of our algorithm, and indeed any arbitrary distribution $P(C)$ of configurations approaches this fixpoint under this algorithm. Each step maps $P(C)$ onto a new distribution $P(C')$:

$$P(C') = \sum_{C} P(C) p(C \to C') \quad . \tag{7.68}$$

The distance E of $P(C)$ from the Boltzmann distribution is

$$E = \sum_{C} \left| P(C) - N \mathrm{e}^{-S(C)} \right| \tag{7.69}$$

(N being a normalization constant, (7.72)). Under transformation (7.68), E changes according to

$$\begin{aligned} E \to E' &= \sum_{C'} \left| P(C') - N \mathrm{e}^{-S(C')} \right| \\ &= \sum_{C'} \left| \sum_{C} P(C) p(C \to C') - N \mathrm{e}^{-S(C')} \right| \\ &= \sum_{C'} \left| \sum_{C} p(C \to C') \left[P(C) - N \mathrm{e}^{-S(C)} \right] \right| \\ &\leq \sum_{C',C} p(C \to C') \left| P(C) - N \mathrm{e}^{-S(C)} \right| \\ &= \sum_{C} \left| P(C) - N \mathrm{e}^{-S(C)} \right| \\ &= E \end{aligned} \tag{7.70}$$

owing to the relation

$$\sum_{C} \left(P(C) - N\mathrm{e}^{-S(C)}\right) = 0 \quad , \tag{7.71}$$

which expresses the fact that $P(C)$ and $N\mathrm{e}^{-S(C)}$ are both normalized to one, $E_{C'}$ is strictly smaller than E_C as long as $P(C)$ is different from $N\exp(-S(C))$.

This so-called "Metropolis algorithm"[4] therefore yields, after sufficient equilibration time a Boltzmann-distributed random configuration. If C_i are such configurations, the integral (7.61) is simply

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} f(C_i) \quad . \tag{7.72}$$

In actual calculations one chooses $p_0(C \to C')$ in such a way that the averaging is done in a certain but not too large area around that configuration C for which $S(C)$ is minimal. The usual criterion is to choose $p_0(C \to C')$ such that about every second modification $C \to C'$ is accepted. For the initial equilibration, $p_0(C \to C')$ must allow large changes, but when approaching the Boltzmann distribution the variations $C \to C'$ become smaller.

As a first example of Monte Carlo calculations we shall consider the "average plaquette"

$$\begin{aligned} P &= \left\langle 1 - \frac{1}{3}\mathrm{Re}[\mathrm{tr}(U_\Box)] \right\rangle \\ &:= \frac{\int D[U(\ell_i)] \exp(-S) \sum_\Box \left\{1 - \frac{1}{3}\mathrm{Re}[\mathrm{tr}(U_\Box)]\right\}}{6N \int D[U(\ell_i)] \exp(-S)} \end{aligned} \tag{7.73}$$

in a pure $SU(3)$ gauge theory (i.e., without fermions). $U_\Box$ represents the product of the four parallel-transport matrices along the links of a plaquette (see (7.13)). We can rewrite P as

$$P = \frac{1}{6N} \frac{\partial}{\partial \beta} \ln \mathcal{Z} \quad . \tag{7.74}$$

and we realize that this looks like the internal energy of thermodynamics with kT substituted by β. N here is the number of lattice sites.

The result of a Monte Carlo calculation is shown in Fig. 7.5. Since $\beta = 6/g^2$ (see (7.52)), $\beta \to \infty$ corresponds to the free limit in which the internal energy must obviously go to zero. For $g \to \infty$ and $\beta \to 0$ it reaches a maximum value, which has been normalized to unity.

How can we check that this result is correct? First, one can check the consistency of the procedure by comparing calculations at different lattice sizes (here two lattices with 4^4 and 6^4 sites, have been used). More important is that P can be calculated analytically in the two limits of β small and β large. Asymptotically the function $P_{\mathrm{MC}}(\beta)$ should approach the asymptotic functions $P_{\beta\,\mathrm{small}}(\beta)$ and $P_{\beta\,\mathrm{large}}(\beta)$. The first is calculated by expanding the exponential in (7.74), the second

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[4] See N. Metropolis, A. W. Rosenbluth, A. H. Teller, E. Teller: J. Chem. Phys. **21**, 1087 (1953).

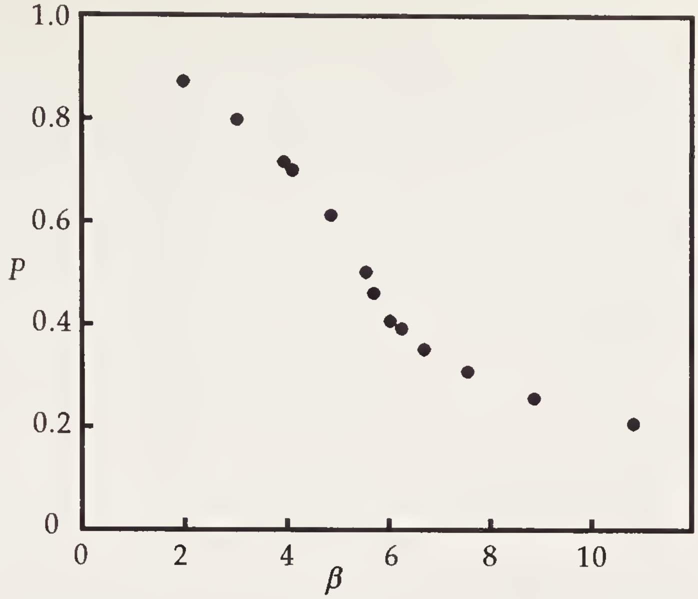

Fig. 7.5. The result of a Monte Carlo calculation for the average plaquette function $P(\beta)$. (From M.Creutz: *Quarks, Gluons and Lattices* (Cambridge University Press, Cambridge 1983), p. 79.)

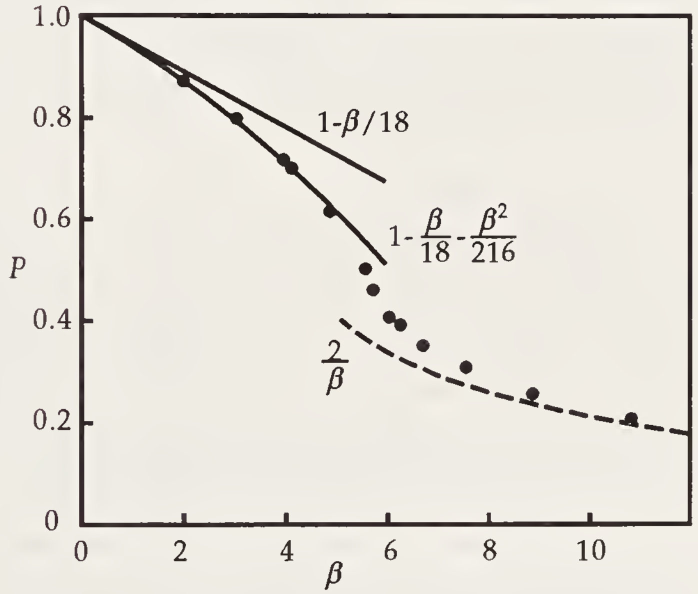

Fig. 7.6. Comparison of the Monte Carlo calculation from Fig. 7.5 with a perturbative calculation is in first and second orders and the result of a saddle-point calculation (*dashed line*). (From M. Creutz: *Quarks, Gluons and Lattices* (Cambridge University Press, Cambridge 1983).)

from a saddle-point approximation (see Examples 7.3 and 7.4). The result is shown in Fig. 7.6. The Monte Carlo calculation gives an excellent interpolation between the asymptotic domains.

While calculations of pure gauge theories can be performed quite reliably on today's computers, the available computing power is not yet quite sufficient for calculations with dynamic fermions, i.e., calculations are only done on comparably small lattices where it is known that errors due to the finite size of the lattice are still large. This fact is illustrated in Fig. 7.7, where the temperature of the QCD phase transition is plotted for calculations with different lattice sizes. The lattice size is $N_\tau \times N_\sigma^3$. Here N_σ is typically one and a half to two times the size of N_τ. For simplicity, only N_τ is given in Fig. 7.7.

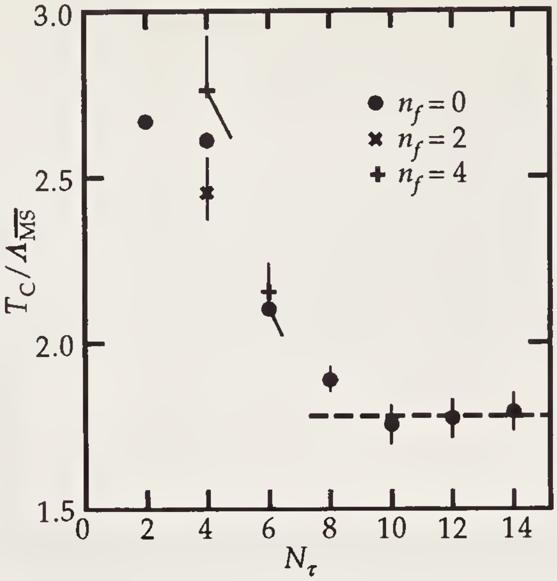

Fig. 7.7. Results for the QCD phase-transition temperature T_C on lattices of different sizes. The size of the lattices is characterized by the number of lattice sites in the Euclidian time direction. The critical temperature is measured in terms of the scale parameter. The dots are the results of a pure-gauge calculation, the crosses are from a calculation with dynamical fermions. Note the size of the error bars, which give a measure of how much more time is required to do calculations with fermions

One can see that the error bars are small for the pure gauge theory, that one can perform calculations up to very large lattices, and that the continuum limit is reached, i.e., that for large lattices the resulting phase-transition temperature does not depend on the lattice size. This limit can hardly be reached with existing computers. Estimates show that really reliable calculations with dynamical fermions require computers about 100 times more powerful than today's. The construction of such computers (typically with 1 teraflop) is currently being attempted by several groups and should be successful in the near future.

The reasons that lattice gauge calculations are computationally so intensive are manifold. First, the matrix Q^F is just very large; for example, its dimension is 165.888 for a 12×24^3 lattice. However, determinant calculation is numerically quite a fast operation, and most components of Q^F are zero, so that it is not obvious why this leads to such long computation times. The reason for this is more subtle. To be able to make a statistically reliable prediction, one should generate of the order of 10.000 configurations. The simplest way to generate these is to start the Metropolis algorithm with correspondingly many different initial configurations and to wait until the configurations are convergent. The number of iterations to do this is critical, since a 12×24^3 lattice can have 665.552 link variables and it will usually take an extremely long time to reach optimal configurations in this immense space of states. In fact, the effective generation of configurations is probably the problem into which most effort is being put at present.

Accordingly the collection of configurations generated in computer centers by lattice-gauge experts is most valuable. If a sufficiently large and reliable set of configurations is obtained, all relevant observables can be calculated in a comparatively short time. The problem is then to find sensible observables and in particular to account for deviations from the continuum theory caused by the lattice.

To obtain converged configurations at all in a sensible time, one must choose the changes in each Monte Carlo step such that the acceptance probability is about 50 percent. This is often very difficult to obtain in actual calculations, and the speed of the calculation depends critically on the search algorithm used. Extremely sophisticated algorithms have been developed to solve these problems. One of these, the hybrid Monte Carlo algorithm, will be sketched now.

Its basic idea is to introduce an auxiliary field coupled to the link variables and governed by dynamics of its own. Once the whole system is equilibrated, the equilibration is conserved under 'time' evolution and this hopefully will also hold for the subspace of link variables. The Hamiltonian

$$H = \sum_{n,\mu} \frac{1}{2}(\pi_{n\mu}^2) + S_{\text{eff}} \tag{7.75}$$

is defined with a new field π. The equations of motion are

$$U_{n\mu}(\tau_{k+1}) = \exp\left[\Delta\tau\pi_{n\mu}\left(\tau_k + \frac{\Delta\tau}{2}\right)\right] U_{n,\mu}(\tau_k) \quad , \tag{7.76}$$

$$\pi_{n\mu}\left(\tau_k + \frac{\Delta\tau}{2}\right) = \pi_{n\mu}\left(\tau_k - \frac{\Delta\tau}{2}\right) - \Delta\tau\frac{\partial S_{\text{eff}}(\tau_k)}{\partial U_{n\mu}} \quad . \tag{7.77}$$

The equations of motion guarantee that H is τ independent:

$$\begin{aligned}\frac{\Delta H}{\Delta\tau} &= \sum_{n,\mu}\left\{\pi_{n\mu}(\tau)\frac{1}{\Delta\tau}\left[\pi_{n\mu}\left(\tau + \frac{\Delta\tau}{2}\right) - \pi_{n\mu}\left(\tau - \frac{\Delta\tau}{2}\right)\right]\right. \\ &\quad \left. + \left(\frac{\Delta U_{n\mu}}{\Delta\tau}\right)\frac{\partial S_{\text{eff}}(\tau)}{\partial U_{n\mu}}\right\} \\ &= \sum_{n\mu}\left(-\pi_{n\mu}(\tau)\frac{\partial S_{\text{eff}}}{\partial U_{n\mu}} + \pi_{n\mu}\frac{\partial S_{\text{eff}}}{\partial U_{n\mu}}\right) = 0 \quad .\end{aligned} \tag{7.78}$$

Once an optimum configuration is found, τ is set to τ_1 and the system evolves to a 'time' τ_2. The probability that this new configuration will be accepted is

$$P = \min\{1, \exp[-H(\tau_1) + H(\tau_2)]\} \quad . \tag{7.79}$$

If one were to solve the exact equations of motion, H would be conserved. The discretization is the entire source of change in H. It must be taken as an empirical fact that this procedure yields particularly good results.

The big disadvantage, among all the advantages, of the hybrid Monte Carlo algorithm is that the derivative of the determinant term, $\ln \operatorname{tr}(Q^F)$, when written as part of S_{eff}, is proportional to $\operatorname{tr}[(Q^F)^{-1}]$, which leads to time-consuming inversions of the huge fermion matrices when solving the equations of motion.

Finally there is another, very fundamental problem. Near the phase transition, which is the domain one is most interested in, the configurations are quite unstable. Totally different configurations, namely those belonging to different phases, have nearly the same action and drastically worsen the convergence of the Metropolis algorithm. This is called "critical slowing down".

For all these reasons QCD calculations with dynamical fermions in the domain of the phase transition on reasonably large lattices and with sufficient statistics remain problematic today. In the following we shall thus restrict ourselves to the results of pure gauge theories.

EXAMPLE

7.3 The Strong Coupling Approximation for the Average Plaquette Function

We expand the exponential function in (7.73) up to first order:

$$
\begin{aligned}
P = \frac{1}{6N} & \left\{ \int D[U(\ell_i)] \left[1 - \beta \sum_{\Box} \left(1 - \frac{1}{3} \mathrm{Re}\big[\mathrm{tr}(U_{\Box})\big] \right) \right] \right. \\
& \left. \times \sum_{\Box'} \left(1 - \frac{1}{3} \mathrm{Re}\big[\mathrm{tr}(U_{\Box'})\big] \right) \right\} \\
& \Big/ \left\{ \int D[U(\ell_i)] \left[1 - \beta \sum_{\Box} \left(1 - \frac{1}{3} \mathrm{Re}\big[\mathrm{tr}(U_{\Box})\big] \right) \right] \right\} \quad . \qquad (1)
\end{aligned}
$$

For a lattice the functional integral consists of the product of integrals over the link variables:

$$\int D[U(\ell_i)] \ldots = \prod_i \int \mathrm{d}[U(\ell_i)] \ldots \quad . \qquad (2)$$

Now

$$\int \mathrm{d}[U(\ell_i)] 1 = 1 \quad , \qquad (3)$$

$$\int \mathrm{d}[U(\ell_i)] U_{mn}(\ell_i) = 0 \quad , \qquad (4)$$

$$\int \mathrm{d}[U(\ell_i)] U_{mn}(\ell_j) U^*_{pq}(\ell_k) = \frac{1}{3} \delta_{mq} \delta_{np} \delta_{\ell_j \ell_k} \quad . \qquad (5)$$

The factor 1/3 in (5) can be understood in the following way. From $\ell_j = \ell_k$, $n = p$ and a summation over n it follows that

$$\int \mathrm{d}[U(\ell_i)] \left(U(\ell_j) U^*(\ell_j) \right)_{mq} = \int \mathrm{d}[U(\ell_i)] (\mathbb{1})_{mq} = \delta_{mq} \quad . \qquad (6)$$

On the other hand this is equal to

$$\frac{1}{3} \sum_n \delta_{mq} \delta_{nn} \delta_{\ell_j \ell_j} = \frac{1}{3} \cdot 3 \delta_{mq} = \delta_{mq} \quad . \qquad (7)$$

Equations (3) to (5) then yield

$$\int D[U(\ell_i)] 1 = 1 \quad , \qquad (8)$$

$$
\begin{aligned}
& \int D[U(\ell_i)] \mathrm{Re}\big[\mathrm{tr}(U_{\Box})\big] \\
& = \int D[U(\ell_i)] \frac{1}{2} \big[U_{jk}(1) U_{kl}(2) U_{lm}(3) U_{mj}(4) \\
& \quad + U^*_{jm}(4) U^*_{ml}(3) U^*_{lk}(2) U^*_{kj}(1) \big] = 0 \qquad (9)
\end{aligned}
$$

and

$$\begin{aligned}
&\int D[U(\ell_i)]\mathrm{Re}\big[\mathrm{tr}(U_\Box)\big]\mathrm{Re}\big[\mathrm{tr}(U_{\Box'})\big]\\
&\quad= \delta_{\Box\Box'}\int \mathrm{d}[U(1)]\mathrm{d}[U(2)]\mathrm{d}[U(3)]\mathrm{d}[U(4)]\frac{1}{4}\,\big(U_{jk}(1)U_{kl}(2)U_{lm}(3)U_{mj}(4)\\
&\qquad +U^*_{jm}(4)U^*_{ml}(3)U^*_{lk}(2)U^*_{kj}(1)\big)\,\big(U_{j'k'}(1)U_{k'l'}(2)U_{l'm'}(3)U_{m'j'}(4)\\
&\qquad +U^*_{j'm'}(4)U^*_{m'l'}(3)U^*_{l'k'}(2)U^*_{k'j'}(1)\big)\\
&\quad= \delta_{\Box\Box'}\sum_{\substack{jklm\\ j'k'l'm'}}\frac{1}{3^4}\cdot\frac{2}{4}\delta_{jj'}\delta_{kk'}\delta_{ll'}\delta_{mm'}\\
&\quad= \frac{1}{2}\delta_{\Box\Box'}\sum_{j,k,l,m=1}^{3}\frac{1}{3^4} = \frac{1}{2}\delta_{\Box\Box'} \quad . \qquad (10)
\end{aligned}$$

By means of these relations we obtain

$$\begin{aligned}
P &= \frac{1}{6N}\,\frac{\sum\limits_{\Box'}1-\beta\sum\limits_{\Box\Box'}1-\frac{\beta}{9}\frac{1}{2}\sum\limits_{\Box\Box'}\delta_{\Box\Box'}}{1-\beta\sum\limits_{\Box}1}\\
&= \frac{1}{6N}N_\Box\frac{1-\beta N_\Box-\frac{\beta}{18}}{1-\beta N_\Box}\\
&= \frac{N_\Box}{6N}\left(1-\frac{\beta}{18}+O(\beta^2)\right) \quad . \qquad (11)
\end{aligned}$$

The number of elementary plaquettes in four dimensions is six times as big as the number of lattice points N. This can easily be understood if all plaquettes pointing in the positive coordinate direction are assigned to a given lattice point. For the point $0_\mu = (0,0,0,0)$ these are

$$\begin{aligned}
&\{(0,0,0,0)(1,0,0,0)(1,1,0,0)(0,1,0,0)\} \quad ,\\
&\{(0,0,0,0)(1,0,0,0)(1,0,1,0)(0,0,1,0)\} \quad ,\\
&\{(0,0,0,0)(1,0,0,0)(1,0,0,1)(0,0,0,1)\} \quad ,\\
&\{(0,0,0,0)(0,1,0,0)(0,1,1,0)(0,0,1,0)\} \quad ,\\
&\{(0,0,0,0)(0,1,0,0)(0,1,0,1)(0,0,0,1)\} \quad ,\\
&\{(0,0,0,0)(0,0,1,0)(0,0,1,1)(0,0,0,1)\} \quad .
\end{aligned} \qquad (12)$$

For arbitrary dimensions d it clearly holds that $N_\Box = d(d-1)N/2$. We therefore obtain in the limit $\beta \to 0$ for the averaged plaquette function

$$P = 1 - \frac{\beta}{18} + O(\beta^2) \quad . \qquad (13)$$

Because $\beta = 6/g^2$ the approximation considered here is referred to as the *strong-coupling approximation.*

EXAMPLE

7.4 The Method of Steepest Descent for the Average Plaquette Function

For very large values of β the partition function

$$\mathcal{Z} = \int D[U] \mathrm{e}^{-\beta \sum\limits_{\Box} \left(1 - \frac{1}{3} \mathrm{Re}\left\{\mathrm{tr} U_{\Box}\right\}\right)} \tag{1}$$

is dominated by gauge field configurations, whose $U_{\Box}$ is approximately equal to unity:

$$U_{\Box} \approx 1 \Rightarrow \left(1 - \frac{1}{3} \mathrm{Re}\left\{\mathrm{tr} U_{\Box}\right\}\right) \approx 0 \quad . \tag{2}$$

Using

$$U_{\Box} = \mathrm{e}^{\mathrm{i}\lambda^a \varepsilon^a_{\Box}} = 1 + \mathrm{i}\lambda^a \varepsilon^a_{\Box} - \frac{1}{2} \left(\lambda^a \varepsilon^a_{\Box}\right)^2 + O\left(\varepsilon^3_{\Box}\right) \tag{3}$$

yields

$$\begin{aligned} \mathcal{Z} &\approx \int D[U] \mathrm{e}^{-\beta \sum\limits_{\Box} \left[1 - \frac{1}{3}\left(3 - \varepsilon^a_{\Box} \varepsilon^a_{\Box}\right)\right]} \\ &= \int D[U] \mathrm{e}^{-\frac{\beta}{3} \sum\limits_{\Box} \varepsilon^a_{\Box} \varepsilon^a_{\Box} + O\left(\varepsilon^4_{\Box} \beta\right)} \quad . \end{aligned} \tag{4}$$

The exact evaluation of this integral leads to a number of difficulties. But for the averaged plaquette function only the β dependence is important, and this can be easily determined. To this end we make the transformation

$$\varepsilon^a_{\Box} = \beta^{-\frac{1}{2}} \tilde{\varepsilon}^a_{\Box} \quad , \tag{5}$$

$$\begin{aligned} \mathcal{Z} &= \left(\beta^{-\frac{1}{2}}\right)^{N_{\mathrm{FG}}} \int D[\tilde{U}] \mathrm{e}^{-\frac{1}{3} \sum\limits_{\Box} \tilde{\varepsilon}^a_{\Box} \tilde{\varepsilon}^a_{\Box} + O\left(\frac{1}{\beta}\right)} \\ &= \beta^{-\frac{N_{\mathrm{FG}}}{2}} \cdot \mathrm{const} \quad . \end{aligned} \tag{6}$$

Here N_{FG} denotes the total number of degrees of freedom that are integrated over in $\int D[U]$. Since there are 8 gluon fields and $4N$ links, there are $4 \cdot 8 \cdot N$ link variables. However, not all of them are independent. The necessary gauge fixing eliminates a quarter of these degrees of freedom, just as in the continuum case. Therefore we obtain

$$N_{\mathrm{FG}} = 3 \cdot 8 \cdot N \quad , \tag{7}$$

$$\mathcal{Z} \approx \beta^{-12N} \cdot \mathrm{const} \quad . \tag{8}$$

The averaged plaquette function then assumes the form

$$P = \frac{1}{6N} \left| \frac{\partial}{\partial \beta} \ln \mathcal{Z} \right| = \frac{1}{6N} \frac{12N}{\beta} = \frac{2}{\beta} + O\left(\beta^{-2}\right) \quad . \tag{9}$$

In the last step we have modified the definition of P by taking the absolute value of $\partial \ln \mathcal{Z}/\partial\beta$. This is necessary because the derivative of $\mathcal{Z}$ can become negative, while P is positive definite.

Example 7.4.

When calculating an arbitrary physical quantity on the lattice, one is in the end interested in the continuum limit. If M is the mass of the lightest glueball (a glueball is a meson with a dominant gluonic component), for example, one wants to know the limit

$$\lim_{a\to 0} M(g(a),a) = M_{\text{phys.}} \quad (\text{in units of } \Lambda) \quad . \tag{7.80}$$

For this continuum limit to exist,

$$\frac{\mathrm{d}M}{\mathrm{d}a} = \frac{\partial M}{\partial g}\frac{\partial g}{\partial a} + \frac{\partial M}{\partial a} = 0 \quad , \quad \frac{\partial M}{\partial g}\frac{\partial g}{\partial a} = -\frac{\partial M}{\partial a} \tag{7.81}$$

must hold. As the discretization by the lattice is just another renormalization procedure, the function $\partial g/\partial a$ is known, since the coupling constant on the lattice must have the form

$$\frac{g_0^2(a)}{4\pi} = \frac{4\pi}{\left(11-\frac{2}{3}N_f\right)\ln\frac{Q_a^2}{\Lambda_a^2}} = \frac{4\pi}{\left(11-\frac{2}{3}N_f\right)\ln\frac{1}{a^2\Lambda_a^2}} \quad , \tag{7.82}$$

where $Q_a \sim 1/a$ is a typical cut-off momentum. Furthermore

$$\frac{g^2(-q^2)}{4\pi} = \frac{4\pi}{\left(11-\frac{2}{3}N_f\right)\ln\frac{-q^2}{\Lambda^2}} \tag{7.83}$$

(see Equation (7) in Example 4.3). $g_0(a)$ is the bare coupling constant on a lattice of spacing a and $g(-q^2)$ is the resulting physical coupling constant at momentum transfer q^2. $g(a)$ is therefore a function of a single parameter. It turns out numerically that Λ_a must be chosen very small, $\Lambda_a \sim 3$ MeV, to obtain the correct limiting behavior. How can this be reconciled with the value $\Lambda_{\text{QCD}} \approx 100-300$ MeV usually obtained from QCD fits? The answer is that Λ depends on the renormalization scheme. Let us consider one-loop contributions to the gluon propagator in a pure gauge theory (Fig. 7.8).

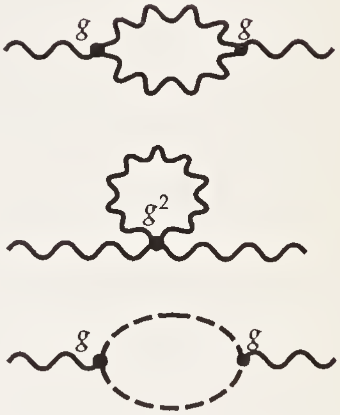

Fig. 7.8. One-loop corrections to the gluon propagator in pure gauge theory. The dashed line represents a ghost field

Renormalization absorbs these graphs into a redefinition of the coupling constant. The contribution of the loops is different in different renormalization procedures, and thus the bare coupling constants must be chosen differently. If g_0 and g_0' belong to two different renormalization schemes, it holds in general that

$$\begin{aligned} g_0 &= g_0'\left[1 + c{g'_0}^2 + O({g'_0}^4)\right] \quad , \\ \frac{1}{{g_0}^2} &= \frac{1}{{g'_0}^2}(1-2c{g'_0}^2) = \frac{1}{{g'_0}^2} - 2c \quad , \end{aligned} \tag{7.84}$$

where the loop graphs contribute to the second order in g. By explicit calculation[5] for a pure SU(3) gauge theory one obtains

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[5] See R. Dashen and D.J. Gross: Phys. Rev. **D23**, 2340 (1981).

$$\frac{1}{g_0{}^2(a)} - \frac{1}{g'_0{}^2(m)} \approx \frac{11}{8\pi^2}[3.444 - \ln(ma)] \quad , \tag{7.85}$$

where $g_0(a)$ is the bare coupling constant of a lattice gauge theory with the Wilson action and $g_0(m)$ is the corresponding coupling constant in a Pauli Villars regularized continuum theory. In the Pauli Villars method, auxiliary fields of mass m are introduced to cancel divergences of the original theory. The mass m is then sent to infinity. Both $g_0(a)$ and Λ_a depend, of course, on the action used, and different relations would result for any action other than the Wilson action.

$$\begin{aligned} g_0^2(a) &= -\frac{16\pi^2}{11\ln a^2\Lambda_a^2} \quad , \\ g^2(m) &= \frac{16\pi^2}{11\ln \frac{m^2}{\Lambda_m^2}} \quad . \end{aligned} \tag{7.86}$$

Using (7.85) it follows that

$$\begin{aligned} &[g_0(a)]^{-2} - [g_0(m)]^{-2} = \frac{11}{8\pi^2}[-\ln a - \ln \Lambda_a - \ln m + \ln \Lambda_m] \\ &= \frac{11}{8\pi^2}\left[-\ln(am) + \ln\frac{\Lambda_m}{\Lambda_a}\right] = \frac{11}{8\pi^2}[3.444 - \ln(am)] \end{aligned} \tag{7.87}$$

$$\Rightarrow \frac{\Lambda_m}{\Lambda_a} = 31.31 \quad . \tag{7.88}$$

Therefore the Λ parameter of the lattice theory is indeed much smaller than the Λ parameter of a Pauli Villars regularized continuum theory.

In general, one finds quite different values for different regularization procedures. Most commonly used is the so-called modified subtraction scheme, in short $\overline{MS}$. The relations between the different Λ parameters are[6]

$$\begin{aligned} &\frac{\Lambda_m}{\Lambda_{\overline{MS}}} = e^{1/12} \approx 1.087 \quad , \\ &\frac{\Lambda_a}{\Lambda_{\overline{MS}}} = 0.0347 \quad , \\ &\frac{\Lambda_{MS}}{\Lambda_{\overline{MS}}} = 0.3765; \quad \frac{\Lambda_{MOM}(\text{Landau gauge})}{\Lambda_{\overline{MS}}} = 3.336 \quad , \\ &\frac{\Lambda_{MOM}(\text{Feynman gauge})}{\Lambda_{\overline{MS}}} = 2.895 \quad . \end{aligned} \tag{7.89}$$

In (7.89) we give the relations between various common renormalization schemes without discussing these schemes in detail.

Next we consider another interesting quantity, the "string constant" K. Color confinement should lead to the creation of a color string between two heavy quarks.

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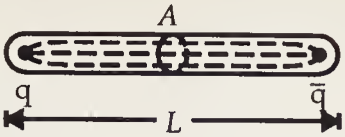
[6] See R. Dashen and D. Gross: Phys.Rev. **D23**, 2340 (1981).



For large seperation of the quarks the total energy is approximately

$$E = A \cdot L \cdot B + 2m_q = V(L) + 2m_q \tag{7.90}$$

with the bag constant B and the two-loop potential $V(L)$. If we further assume that the cross section A of the string does not depend on L, then $V(L)$ is proportial to L and the constant of proportionality is called the “string constant”:

$$V(L) = K \cdot L \quad , \quad K = A \cdot B \quad . \tag{7.91}$$

Indeed, lattice theory leads to such a linear potential, as we shall see shortly, and thus justifies the heuristic picture of a color string. From (7.91) we obtain the estimate

$$K \approx \pi \cdot 1\ \mathrm{fm}^2 \cdot (150\mathrm{MeV})^4 \approx (200\mathrm{MeV})^2 \quad . \tag{7.92}$$

How can a two-quark potential be defined on the lattice? To this end, we consider an idealized rectangular quark loop (Fig. 7.9).

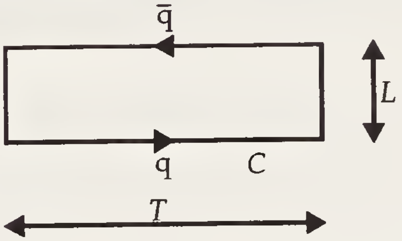


Fig. 7.9. The Idealized quark loop $\mathcal{C}$. T is the extent in the time direction, L the spatial extent

Wilson showed that the $q\overline{q}$ potential can be obtained as

$$V(L) = -\lim_{T\to\infty} \frac{1}{T} \ln W(\mathcal{C}), \tag{7.93}$$

$$W(\mathcal{C}) = \frac{1}{3Z} \int D[U] e^{-\beta} \mathrm{tr} U_{\mathcal{C}} \quad . \tag{7.94}$$

$U_{\mathcal{C}}$ is the product of all link variables along the path $\mathcal{C}$. $W(\mathcal{C})$ is called the “Wilson loop”. To motivate ansatz (7.93), we shall show in Example 7.5 that it gives the Coulomb potential in the case of QED. $W(\mathcal{C})$ is most easily calculated for small values of β. The high-temperature expansion (see Example 7.6) yields for SU(3)

$$\begin{aligned} W(\mathcal{C}) &= \left(\frac{\beta}{18}\right)^{LT/a^2} \quad , \\ V(L) &= -\lim_{T\to\infty} \frac{1}{T} \frac{LT}{a^2} \ln \frac{\beta}{18} \cdot L \quad . \end{aligned} \tag{7.95}$$

In the high-temperature limit the string constant of SU(3) is thus

$$K = \frac{1}{a^2} \ln \frac{\beta}{18} \quad . \tag{7.96}$$

For numerical calculations (7.93) is replaced by

$$K_{\text{num}} = -\frac{1}{a^2} \ln \frac{W(L,T)W(L-a,T-a)}{W(L,T-a)W(L-a,T)} \quad . \tag{7.97}$$

For sufficiently large values of T it holds by (7.93) that

$$W(C) = W(L,T) = \mathrm{e}^{-KLT} \tag{7.98}$$

and

$$K_{\text{num}} = -\frac{1}{a^2} \ln \mathrm{e}^{-K(LT+(L-a)(T-a)-L(T-a)-(L-a)T)} = K \quad . \tag{7.99}$$

Figure 7.7 shows the result of a Monte Carlo calculation[7] on a 6^4 lattice. The number pairs are $(L/a, T/a)$. Also shown is the prediction of a weak-coupling approximation (see Example 7.7). The Monte Carlo calculation again yields an excellent interpolation of both approximations. As with the average-plaquette function (mean free energy), the transition between both regions is at $\beta \approx 6$.

EXAMPLE

7.5 The QED Wilson Loop

We consider QED continued to Euclidian space (i.e., $x_0 \to \mathrm{i}x_0$, $A_0 \to \mathrm{i}A_0$), employing the Feynman gauge $\partial_\mu A^\mu = 0$. The action is then

$$\begin{aligned} S &= \frac{1}{4} \int \mathrm{d}^4x \, \left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) \\ &= -\frac{1}{2} \int \mathrm{d}^4x \, A_\nu \Box A_\nu \quad . \end{aligned} \tag{1}$$

We attempt to evaluate the integral

$$W(C) = \frac{1}{\mathcal{Z}} \int D[A_\mu] \exp\left[\frac{1}{2} \int \mathrm{d}^4x \, A_\nu \Box A_\nu + \int \mathrm{d}^4x \, j_\mu(x) A_\mu(x)\right] \quad . \tag{2}$$

The difference between (2) and (7.94) is that the factor $1/3$ has been replaced by 1. In the case of the groups SU(N) and U(N) the functional $W(C)$ is defined with the factor $1/N$. The coupling to the charge is ensured by the additional term

$$\int \mathrm{d}^4x \, j_\mu(x) A_\mu(x) = \mathrm{i}Q \int\limits_C \mathrm{d}s_\mu \, A_\mu(s) \quad , \tag{3}$$

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[7] See M. Creutz and K.J.M. Moriarty: Phys. Rev. **D26**, 2166 (1982).

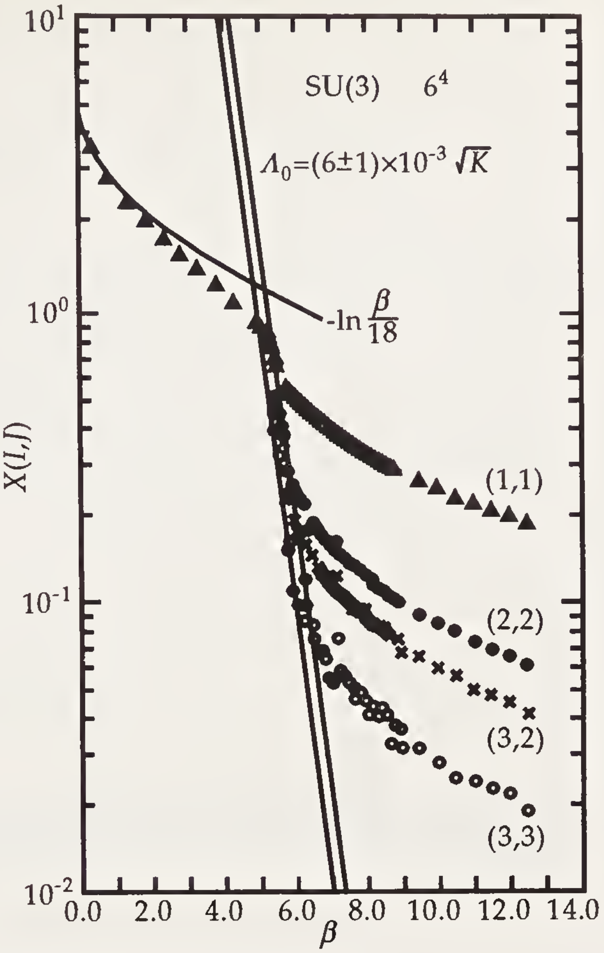


Fig. 7.10. Results of a Monte-Carlo calculation for the string constant compared to a high-temperature (single line) and a large-β approximation (double line). (From M. Creutz and K. J. M. Moriarty: Phys. Rev. **D26**, 2166 (1982).)

where s denotes the length of the path C. The functional integral is performed by making the substitution

$$A_\mu(x) \to A_\mu(x) + \mathrm{i}Q \int\limits_C \mathrm{d}s_\nu \; D_{\mu\nu}[x - x(s)] \tag{4}$$

with the Feynman propagator

$$\Box D_{\mu\nu}(x) = -\delta_{\mu\nu}\;\delta^4(x) \quad .$$

Equation (4) then yields

$$\begin{aligned}
&\frac{1}{2}\int \mathrm{d}^4x \; A_\nu \Box A_\nu + \mathrm{i}Q \int\limits_C \mathrm{d}s_\mu \; A_\mu(s) \\
&\Rightarrow \frac{1}{2}\int \mathrm{d}^4x \; A_\nu \Box A_\nu - \mathrm{i}Q \int \mathrm{d}s_\mu \; A_\mu(s) + \mathrm{i}Q \int \mathrm{d}s_\mu \; A_\mu(s) \\
&\quad - \frac{Q^2}{2} \int\limits_C \mathrm{d}s_\nu \; \mathrm{d}s'_\mu \; D_{\mu\nu}[x(s) - x(s')] \quad ,
\end{aligned} \tag{5}$$

and therefore

Example 7.5.

$$W(C)=\frac{1}{\mathcal{Z}}\int D[A_\mu]\,\exp\left[\frac{1}{2}\int \mathrm{d}^4x\; A_\nu\Box A_\nu+\mathrm{i}Q\int\limits_C \mathrm{d}s_\mu\; A_\mu(s)\right]$$

$$=\frac{1}{\mathcal{Z}}\mathcal{Z}\exp\left\{-\frac{Q^2}{2}\int\limits_C \mathrm{d}s_\nu\;\mathrm{d}s'_\mu\; D_{\mu\nu}[x(s)-x(s')]\right\}\quad . \tag{6}$$

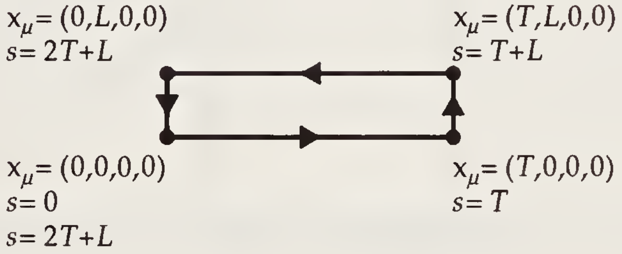
In position space the Feynman propagator is simply

$$D_{\mu\nu}[x(s)-x(s')]=\frac{1}{4\pi^2}\frac{\delta_{\mu\nu}}{[x(s)-x(s')]^2}\quad . \tag{7}$$

Now we are left with the evaluation of the integral

$$I=\frac{1}{4\pi^2}\int\limits_C \mathrm{d}s_\nu\int\limits_C \mathrm{d}s'_\mu\;\frac{1}{[x(s)-x(s')]^2}\quad . \tag{8}$$

To this end we consider the rectangular path:



s_ν and $s_{\nu'}$ then only point in the same direction if both lie on the same side of C. Correspondingly they point in opposite directions when positioned on opposite sides of C. In the former case (6) yields the action of a charge onto itself, i.e., the self-energy. The latter case has to reproduce the Coulomb potential. Therefore we obtain

$$I_{cb}=-\frac{1}{2\pi^2}\left[\int\limits_0^T \mathrm{d}s\int\limits_{T+L}^{2T+L}\mathrm{d}s'\frac{1}{L^2+(2T+L-s'-s)^2}\right.$$
$$\left.+\int\limits_T^{T+L}\mathrm{d}s\int\limits_{2T+L}^{2T+2L}\mathrm{d}s'\frac{1}{T^2+(3T+2L-s'-s)^2}\right]\quad . \tag{9}$$

For $T\to\infty$ the second contribution vanishes. In the first term we substitute $s'\to 2T+L-s'$:

$$I_{cb}=-\frac{1}{2\pi^2}\int\limits_0^T\mathrm{d}s\int\limits_0^T\mathrm{d}s'\;\frac{1}{L^2+(s-s')^2}$$
$$=-\frac{1}{2\pi^2}\int\limits_0^T\mathrm{d}s\int\limits_{-s}^{T-s}\mathrm{d}y\;\frac{1}{L^2+y^2}$$

Example 7.5.

$$= -\frac{1}{2\pi^2}\int_0^T ds\,\frac{1}{L}\left[\arctan\left(\frac{T-s}{L}\right)+\arctan\left(\frac{s}{L}\right)\right]$$

$$= -\frac{1}{\pi^2 L}\int_0^T ds\,\arctan\left(\frac{s}{L}\right)$$

$$= -\frac{T}{\pi^2 L}\arctan\left(\frac{T}{L}\right)+\frac{1}{2\pi^2}\ln\left(\frac{T^2+L^2}{L^2}\right) \quad . \tag{10}$$

The limiting case from (7.93) finally yields

$$V(L) = -\lim_{T\to\infty}\frac{1}{T}\ln\left[\exp\left\{\frac{Q^2}{2\pi^2}\left[\frac{T}{L}\arctan\left(\frac{T}{L}\right)-\frac{1}{2\pi^2}\ln\left(\frac{T^2+L^2}{L^2}\right)\right]\right\}\right]$$

$$= -\lim_{T\to\infty}\frac{1}{T}\frac{Q^2}{2\pi^2}\frac{T}{L}\frac{\pi}{2}$$

$$= -\frac{Q^2}{4\pi}\frac{1}{L} \quad , \tag{11}$$

which in fact is the Coulomb potential.

EXAMPLE

7.6 The Strong Coupling Approximation for the Wilson Loop

We attempt to evaluate the integral of (7.94) to the lowest order of β:

$$W(C) = \frac{1}{3\mathcal{Z}}\int D[U]\mathrm{tr}[U(C)]\exp\left[-\beta\sum_{\square}\left\{1-\frac{1}{3}\mathrm{Re}\left[\mathrm{tr}(U_{\square})\right]\right\}\right] \quad . \tag{1}$$

Owing to equation (3) of Example 7.3 the integral over $D[U]$ is equal to zero unless every link variable $U(\ell)$ is compensated by the corresponding $U^{-1}(\ell)$. Graphically this means that each line in Fig. 7.11 must be accompanied by a line with the opposite orientation. Since the exponent in (1) only yields complete plaquettes, the lowest-order contribution to β corresponds to the graph in which the inside of the Wilson loop is completely filled with plaquettes (see Fig. 7.12).

Figure 7.13 depicts a contribution to the next order in β, which is neglected here. Just as either a quark and an antiquark or three quarks can be coupled to a color singlet, both the integral

$$\int D[U]\;U_{ij}(\ell)U^{\dagger}_{jk}(\ell) \tag{2}$$

and the integral

$$\int D[U]\;U_{ij}(\ell)U_{jk}(\ell)U_{kn}(\ell) \tag{3}$$

are nonzero.

Fig. 7.11. A Wilson loop with $L = 3a$ and $T = 4a$

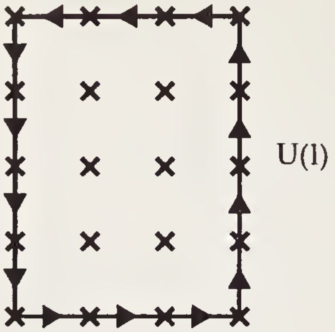


Fig. 7.12. The lowest order contribution to the Wilson loop

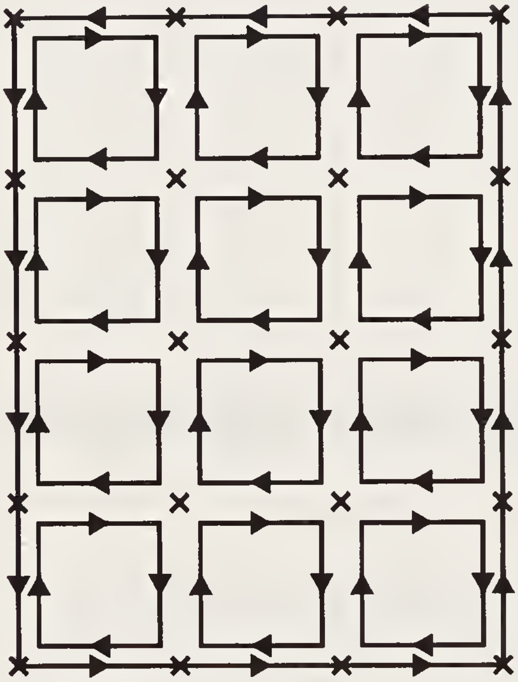
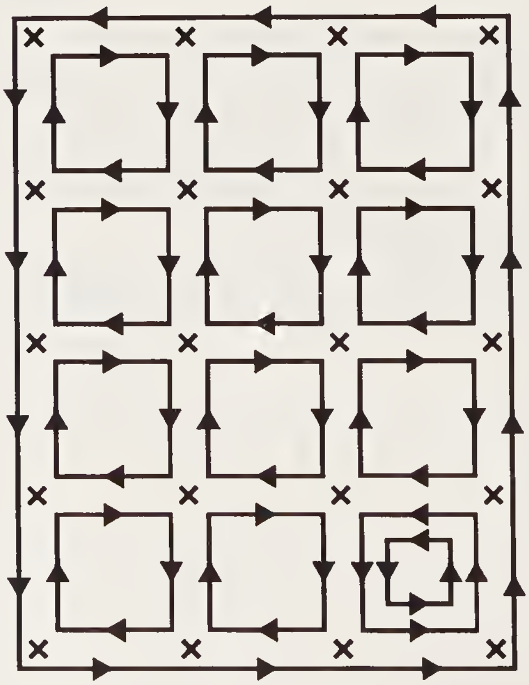


Fig. 7.13. A contribution to the Wilson loop of the next order



Employing relation (3) of Example 7.3,

Example 7.6.

$$\int D[U]\; U_{ij}(\ell)U^*_{mn}(\ell) = \frac{1}{3}\delta_{in}\delta_{jm} \quad , \tag{4}$$

we can easily deduce the value of $W(C)$ from Fig. 7.12. The first nonvanishing contribution is due to the $(L \cdot T/a^2)$th term of the power series. Because $I = L/a$ and $J = T/a$,

$$W(C) \sim \frac{1}{3Z}\int D[U]\left\{\beta\sum_{\square}\frac{1}{3}\mathrm{Re}\left[\mathrm{tr}(U_\square)\right]\right\}^{I\cdot J}\frac{1}{(I\cdot J)!}\mathrm{tr}[U(C)] \quad . \tag{5}$$

From (4) we obtain a factor $1/3$ for every link. The total number of these links is $2I \cdot J + I + J$. In addition we get a factor $(I \cdot J)!$ for the number of permutations of the plaquettes shown in Fig. 7.12. Finally

$$\left\{\beta\sum_{\square}\frac{1}{3}\mathrm{Re}\left[\mathrm{tr}(U_\square)\right]\right\}^{I\cdot J} = \left\{\frac{\beta}{6}\sum_{\square}\left[\mathrm{tr}(U_\square) + \mathrm{tr}(U^\dagger_\square)\right]\right\}^{I\cdot J} \tag{6}$$

yields a further factor $(\beta/6)^{I\cdot J}$. Therefore we have

$$\begin{aligned} W(C) &= \frac{1}{3Z}\left(\frac{\beta}{6}\right)^{I\cdot J}\left(\frac{1}{3}\right)^{2I\cdot J+I+J} \\ &\quad\times \sum_{i_1,\ldots,i_{4I\cdot J+2(I+J)}=1}^{3} \underbrace{\delta_{i_1 i_n}\ldots\delta_{i_m i_{4I\cdot J+2(I+J)}}}_{[3IJ+I+J-1]\text{Kronecker deltas}} \quad . \end{aligned} \tag{7}$$

Here we have made use of the fact that the integration over $D[U]$ yields two Kronecker deltas for every link (see (4)). Hence the total number of deltas is $2 \cdot [2I \cdot J + I + J]$. But one delta per lattice point is contained in the others (see Fig. 7.14).

Because there are $(I + 1)(J + 1)$ lattice points, we have

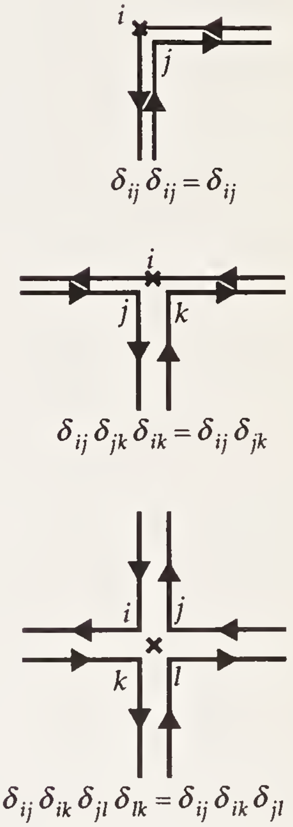
$$2[2I \cdot J + I + J] - (I + 1)(J + 1) = 3I \cdot J + I + J - 1 \tag{8}$$

Kronecker-deltas. Finally we have to count the number of summation indices. Every plaquette contributes four indices and the Wilson loop itself $2(I + J)$ indices, both numbers correspond to the number of involved lattice points. Therefore there is a total number of

$$4I \cdot J + 2(I + J) \tag{9}$$

summation indices. $3I \cdot J + I + J - 1$ of these sums cancel the Kronecker deltas, and the remaining sums yield a factor 3 each. Finally we make use of the fact that to lowest order $1/Z$ is equal to one and obtain

$$\begin{aligned} W(C) &= \frac{1}{3}\left(\frac{\beta}{6}\right)^{I\cdot J}\left(\frac{1}{3}\right)^{2I\cdot J+I+J} 3^{4I\cdot J+2(I+J)-(3I\cdot J+I+J-1)} \\ &= \left(\frac{\beta}{18}\right)^{I\cdot J} \\ &= \left(\frac{\beta}{18}\right)^{LT/a^2}\left[1 + O(\beta)\right] \quad . \end{aligned} \tag{10}$$



$\delta_{ij}\delta_{ij} = \delta_{ij}$

$\delta_{ij}\delta_{jk}\delta_{ik} = \delta_{ij}\delta_{jk}$

$\delta_{ij}\delta_{ik}\delta_{jl}\delta_{lk} = \delta_{ij}\delta_{ik}\delta_{jl}$

Fig. 7.14. Graphical deduction of (7)

EXAMPLE

7.7 The Weak-Coupling Approximation for the Wilson Loop

We are interested in the quantity Ka^2 in the limit of weak-coupling constants. In this limiting case (7.66)

$$\frac{g_0^2(a)}{4\pi} = \frac{4\pi}{\left(11 - \frac{2}{3}N_f\right) \ln\left[\frac{1}{a^2 \Lambda_a^2}\right]} \tag{1}$$

holds. We consider a pure gauge theory, i.e., N_f is equal to zero. Then one can solve (1) for a:

$$\begin{aligned} a^2 &= \frac{1}{\Lambda_a^2} \exp\left(-\frac{16\pi^2}{11 g_0^2}\right) \\ &= \frac{1}{\Lambda_a^2} \exp\left(-\frac{8\pi^2}{33}\beta\right) \\ &= \frac{1}{\Lambda_a^2} 10^{-1.04\cdot\beta} \quad . \end{aligned} \tag{2}$$

Since the limiting case $\beta \to \infty$ corresponds to the continuum limit, K should become constant for large values of β (in units of Λ_{QCD}). It can be seen from Fig. 7.7 that Monte Carlo calculations converge approximately toward

$$K\ a^2 \approx 2 \cdot 10^5 \cdot 10^{-1.04\cdot\beta} \quad \text{for} \quad \beta = 7 \quad . \tag{3}$$

This corresponds to

$$K = 2 \cdot 10^5 \Lambda_a^2 \quad . \tag{4}$$

The second order yields[8]

$$a^2 = \frac{1}{\Lambda_a^2} \left(\frac{33}{8\pi^2} \frac{1}{\beta}\right)^{\frac{102}{121}} 10^{-1.04\beta} \quad . \tag{5}$$

This gives an even better approximation, shown by the double line in Fig. 7.7. In this case we obtain

$$K \approx 3 \times 10^4 \Lambda_a^2 \quad . \tag{6}$$

Employing (7.89) we can estimate the value of Λ_a. The number $\Lambda_{\overline{MS}} \approx 100\mathrm{MeV}$ then yields

$$K \approx 3 \times 10^4 \times [0.0347 \times 100\mathrm{MeV}]^2 = [580\mathrm{MeV}]^2 \quad . \tag{7}$$

This is approximately the same order of magnitude obtained by our heuristic estimate in (7.92).

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[8] See A. Hasenfratz and P. Hasenfratz: Phys. Lett. **B93**, 165 (1980).

The result of a Monte Carlo calculation yields

$$K \approx (580\mathrm{MeV})^2 \quad , \tag{7.100}$$

as discussed in Example 5.7. This must be compared to the heuristic estimate of (7.92):

$$K \approx (200\mathrm{MeV})^2 \tag{7.101}$$

and to the result from a phenomenological string model

$$K \approx (400\mathrm{MeV})^2 \quad . \tag{7.102}$$

The latter describes higher resonances (see Fig. 1.1.) as rotational states of strings. The length and mass of the string then depend on K. Conversely K can be obtained from a fit of the masses, which leads to (7.96). In comparing (7.100) with the other estimates, one must bear in mind that we have only considered a pure gauge theory. With fermions, (7.100) should change. This granted, the order-of-magnitude agreement of the different results for K is a pleasing success.

Another very interesting result of lattice calculations is the discovery of a phase transition between two domains. For low temperatures, single quarks have an infinite energy, and quarks thus appear confined. At higher temperatures, a phase transition to another phase occurs where the quark condensate vanishes and a free quark has a finite energy that goes to zero as $T \to \infty$. This new, hypothetical phase is nonconfining; quarks and gluons behave like free particles with only a weak interaction that can be calculated in perturbation theory.

To get this result, we first must find a suitable observable on the lattice. For the quark condensate this is trivially done by calculating

$$\langle \bar{\Psi}\Psi \rangle = \frac{1}{\mathcal{Z}} \int D\left[U(\ell), \Psi(x), \bar{\Psi}(x)\right] \mathrm{e}^{-S} \bar{\Psi}(x)\Psi(x) \quad , \tag{7.103}$$

where S is choosen to avoid fermion doubling. In analogy to the Wilson loop, which yields the interaction energy of a $q\bar{q}$ pair, we consider the expectation value of link operators along the time axis to calculate the interaction energy of a single quark (see Fig. 7.15).

To obtain a gauge-invariant expression, we demand a periodicity condition in the time direction or, equivalently, identity points at the edge of the lattice. The expectation value

$$\langle \mathcal{L} \rangle := \frac{1}{\mathcal{Z}} \int D\left[U(\ell), \Psi(x), \bar{\Psi}(x)\right] \mathrm{e}^{-S} \mathrm{tr}\left[U(L)\right] \tag{7.104}$$

then yields at large times $\exp(-\beta F)$ with free energy F, as does the Wilson loop of (7.93). The result of Monte Carlo calculations in general looks as sketched in Fig. 7.16.

Calculations differ mainly in the abruptness of the phase transition, i.e., in deciding the question whether it is of first or of higher order. This question is essential to deciding what experimentally observable quantities are related to this phase transition. Unluckily it is still controversial (see Fig. 7.16). As a further illustration, Fig. 7.17 shows the result of a calculation with four light fermions, showing a first-order phase transition. (Nonetheless, the majority of published papers seem to favor a higher order.)

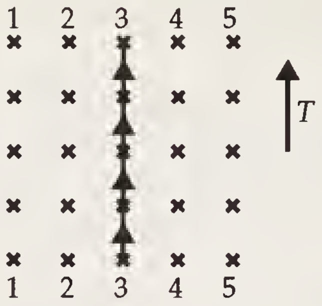


Fig. 7.15. The definition of the Wilson line. The lattice sites $1, 2, \dots, 5$ are identified with each other

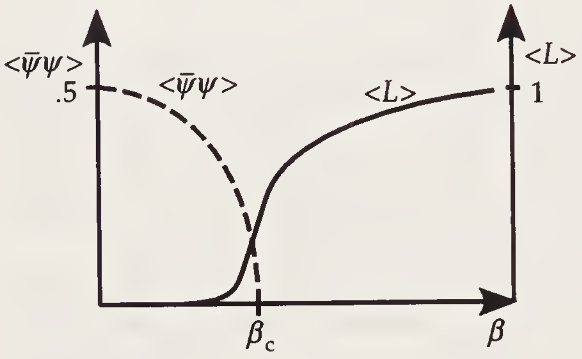


Fig. 7.16. Schematic results of a Monte Carlo calculation for the confinement–deconfinement and the chiral-phase transition

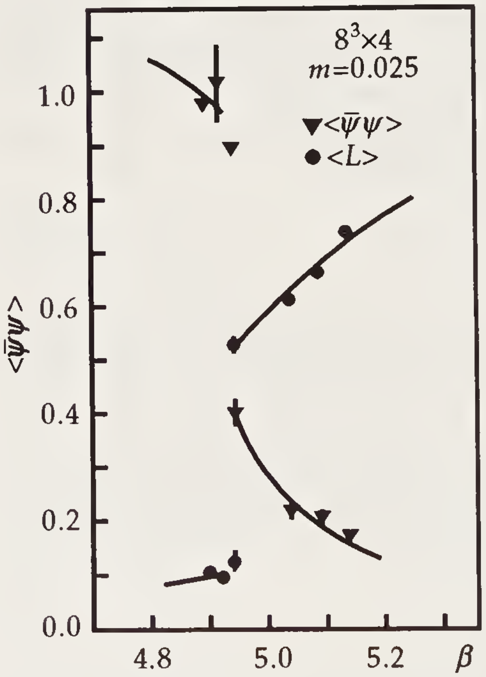
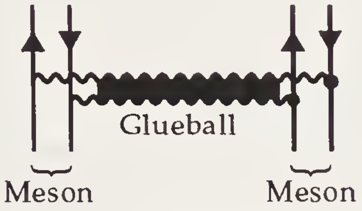


Fig. 7.17. The result of a Monte Carlo calculation for the Wilson line L and the quark condensate $\langle\bar{\Psi}\Psi\rangle$ (From F. Karsch et al.: Phys. Lett. **B118**, 353 (1981).)

From these calculations a critical value β_c is obtained. Converting this to a critical temperature is somewhat troublesome and thus often not stated. We shall discuss this in detail in the next section. The result will be a critical temperature in the range $T_c = (100\text{–}300)$ MeV.

Physicists are currently looking for signals of this phase transition in collisions of nuclei with extremely high energy. The hypothetical nonconfining state has been termed the "quark–gluon plasma". Its predicted properties are discussed in Sect. 8.2.

As a last example of the application of lattice QCD we shall now discuss the calculation of glueball masses. The basic idea here is to consider two objects which couple to gluons and can thus exchange glueballs. This is then manifested by a Yukawa potential.

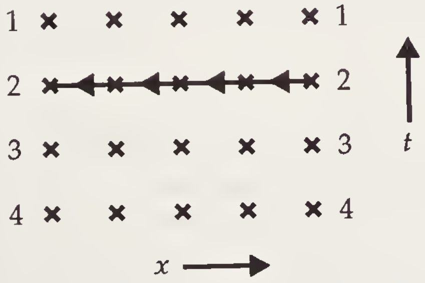


Describing mesons by a field Φ we expect a correlation function that falls off as

$$\langle \Phi(x)\Phi(x+X)\rangle \approx \mathrm{e}^{-mX} \quad . \tag{7.105}$$

The major problem in lattice QCD is now to find sensible lattice representations for mesons Φ. The simplest object, a Wilson loop, leads to problems owing to the missing translational invariance. A translationally invariant Wilson loop, called a Polyakov loop, is therefore constructed by demanding periodic boundary conditions, for example for the x direction, and considering Wilson lines in the x direction instead of the t direction: $L_x(y,z,t)$. The sum over all values of z and y is the Polyakov loop

$$P(t) = \sum_{y,z} L_x(y,z,t) \quad . \tag{7.106}$$



We then look at the correlation function

$$c(t) = \langle 0|P(0)P(t)|0\rangle \tag{7.107}$$

and determine the relation

$$\frac{c(t+a)}{c(t)} \approx \frac{\mathrm{e}^{-m(t+a)}}{\mathrm{e}^{-mt}} = \mathrm{e}^{-ma} \quad , \tag{7.108}$$

$$am = -\lim_{t\to\infty} \ln \frac{c(t+a)}{c(t)} \quad . \tag{7.109}$$

The physical mass is identified with

$$M = \lim_{\substack{a \text{ little} \\ \beta \text{ large}}} \frac{m}{L} = - \lim_{\substack{a \text{ little} \\ \beta \text{ large} \\ t \to \infty}} \frac{1}{L} \ln \frac{c(1+a)}{c(t)} \quad , \tag{7.110}$$

where L is the extent of the lattice in the x direction (in units of a). As a typical result masses between 0.5 and 2 GeV are obtained for the lightest glueball. One challenging problem of QCD is now to observe these states experimentally. Great effort has been made in this direction, but so far without success. It therefore seems possible that the decay width is so large and the mixing to mesons so strong that no glueballs in the sense of well-defined resonances exist.

To conclude, it must again be noted that all the results of lattice QCD must still be considered with caution, owing to the great conceptual and numerical problems. In the end, the justification of lattice QCD will come from its success in describing and predicting experimental data. These successes are so far very encouraging.

7.2 Lattice Theories at Finite Temperature

Let us first review how temperature is treated in continuum field theory. It will then be relatively easy to adapt the resulting expression for the partition function to the case of a discrete lattice.

The time-evolution operator of quantum mechanics is $\exp(-\mathrm{i}Ht)$, while the thermodynamic partition function contains the weights $\exp(-E/T)$. It is therefore natural to assume that the thermodynamic formulation can be obtained from a field theory by means of the analytic continuation to imaginary times ($t \to -\mathrm{i}\tau$). This is in fact an exact statement. It is easy to see that the usual Schrödinger equation

$$\left(\mathrm{i}\frac{\partial}{\partial t} + \frac{1}{2m}\triangle \right) \phi = 0 \quad , \tag{7.111}$$

for example, turns into a diffusion equation when $-\mathrm{i}\tau$ is substituted for t:

$$\left(\frac{\partial}{\partial \tau} - \frac{1}{2m}\triangle \right) \phi = 0 \quad , \tag{7.112}$$

describing, for example, Brownian motion. Also the resulting solutions are just analytic continuations of one another: oscillations in time for quantum mechanics and decaying exponentials for thermodynamics. Owing to this exact formal equivalence it is relatively easy to derive the form of the partition function. In quantum mechanics the transition matrix element between two states ϕ_i and ϕ_f can be written as

$$\langle \phi_\mathrm{f} | \mathrm{e}^{-\mathrm{i}\hat{H}t_\mathrm{f}} | \phi_\mathrm{i} \rangle = \int \mathcal{D}[\pi] \int_{\phi(\boldsymbol{x},0)=\phi_\mathrm{i}(\boldsymbol{x})}^{\phi(\boldsymbol{x},t_\mathrm{f})=\phi_\mathrm{f}(\boldsymbol{x})} \mathcal{D}[\phi] \mathrm{e}^{\mathrm{i}\int_0^{t_\mathrm{f}} \mathrm{d}t \int \mathrm{d}^3x \left\{ \pi(\boldsymbol{x},t) \frac{\partial \phi(\boldsymbol{x},t)}{\partial t} - \mathcal{H}[\pi(\boldsymbol{x},t), \phi(\boldsymbol{x},t)] \right\}} \quad . \tag{7.113}$$

This is the path-integral formulation for the propagator, which is completely equivalent to the canonical formulation of quantum mechanics.[9]

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[9] See, for example, R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York 1965).

As the thermodynamic partition function is simply

$$\mathcal{Z}(T, V, \ldots) = N \int \mathcal{D}[\phi_\alpha] \langle \phi_\alpha | \mathrm{e}^{-\hat{H}/T} | \phi_\alpha \rangle \quad , \tag{7.114}$$

we can reexpress it by the analytic continuation of the right-hand side of (7.113)

$$\begin{aligned} \mathcal{Z}(T, V, \cdots) &= N' \int \mathcal{D}[\pi] \int \mathcal{D}[\phi_\alpha] \int_{\phi(\boldsymbol{x},0)=\phi_\alpha}^{\phi(\boldsymbol{x},-\mathrm{i}\tau)=\phi_\alpha} \mathcal{D}[\phi]\, \mathrm{e}^{\mathrm{i} \int_0^{-\mathrm{i}\tau} \mathrm{d}t \int \mathrm{d}^3x \left[\pi \frac{\partial \phi}{\partial t} - \mathcal{H}(\pi,\phi)\right]} \\ &= N' \int \mathcal{D}[\pi] \int_{\text{periodic}} \mathcal{D}[\phi]\, \mathrm{e}^{\mathrm{i} \int_0^{-\mathrm{i}\tau} \mathrm{d}t \int \mathrm{d}^3x \left(\sum_\alpha \pi_\alpha \frac{\partial \phi_\alpha}{\partial t} - \mathcal{H}'\right)} \quad . \end{aligned} \tag{7.115}$$

For bosons the boundary conditions are periodic and thus the energy can only take discrete values

$$\omega = \frac{2\pi n}{T} \quad , \quad n = 0, 1, 2, \cdots \quad , \tag{7.116}$$

while for fermions we would have got

$$\omega = \frac{(2n+1)\pi}{T} \quad , \quad n = 0, 1, 2, \cdots \quad . \tag{7.117}$$

The frequencies (7.116) and (7.117) are called Matsubara frequencies. For gauge theories the partition function then simply has the same form as (7.115):

$$\mathcal{Z} = N \int \mathcal{D}\left[A_\mu^a\right] \mathrm{e}^{-\frac{1}{4} \int_0^\tau \mathrm{d}t \int_V \mathrm{d}^3x F_{\mu\nu} F_{\mu\nu}} \quad , \tag{7.118}$$

where we have substituted t for $-\mathrm{i}t$. By analogy with (7.114), we must conclude that τ is now playing the role of the inverse temperature $\tau \mathrel{\hat{=}} 1/T$. Unluckily this is again usually called β, leading to substantial confusion. Further on we shall discuss the connection between $\beta = 6/g^2$ and $\beta = 1/T$: it is given by β, being the QCD beta function

$$\begin{aligned} \beta_{\mathrm{QCD}} &= -\frac{1}{16\pi^2}\left(11 - \frac{2}{3} N_f\right) g_0^3 - \frac{1}{\left(16\pi^2\right)^2}\left(102 - \frac{38}{3} N_f\right) g_0^5 + \cdots \\ &= -a \frac{\partial g_0}{\partial a} \quad , \end{aligned} \tag{7.119}$$

which describes the running of the coupling constant with the scale-size, which for lattice QCD is given by the lattice constant a. Finally the temperature T should not be confused with the length in time direction T appearing in the Wilson loop.

The finite temperature partition function can be discretized in analogy to the usual action, but with a slight modification. One has to fulfill the thermodynamic equations for pressure and energy density

$$\begin{aligned} p &= T \left. \frac{\partial \ln \mathcal{Z}}{\partial V} \right|_T \quad , \\ \varepsilon &= \frac{T^2}{V} \left. \frac{\partial \ln \mathcal{Z}}{\partial T} \right|_V \quad . \end{aligned} \tag{7.120}$$

This requires that the temperature, or rather $\tau = N_c a_c = 1/T$, and the volume can be varied independently, which is only possible if the lattice spacing in the Euclidean time direction a_τ and in the space directions a_σ can be varied independently (assuming that N_τ and N_σ are constants as usual):

$$V = (N_\sigma a_\sigma)^3 \quad , \quad T = \frac{1}{N_\tau a_\tau} \quad . \tag{7.121}$$

With $\xi = a_\sigma / a_\tau$ the discretized action thus reads for a pure SU(N) gauge theory

$$\begin{aligned} S[U] &= \beta_\sigma \sum_{P_\sigma} \left[1 - \frac{1}{N} \mathrm{Re}\left[\mathrm{tr}(U_{ij} U_{jk} U_{kl} U_{li}) \right] \right] \\ &\quad + \beta_\tau \sum_{P_\tau} \left[1 - \frac{1}{N} \mathrm{Re}\left[\mathrm{tr}(U_{ij} U_{jk} U_{kl} U_{li}) \right] \right] \\ &\quad \beta_\sigma = \frac{2N}{g_\sigma^2 \xi} \quad , \quad \beta_\tau = \frac{2N\xi}{g_\tau^2} \quad . \end{aligned} \tag{7.122}$$

For $\xi = 1$ and the continuum limit $a \to 0$ one must have $\beta = \beta_\sigma = \beta_\tau$. The P_σ are plaquettes with one space and one time link. With the average plaquettes $\bar{P}_{\sigma,\tau}$

$$\bar{P}_{\sigma,\tau} = \frac{1}{3N_\sigma^3 N_\tau \mathcal{Z}} \int \prod_{\{ij\}} dU_{ij} \mathrm{e}^{-S[U]} \sum_{P_{\sigma,\tau}} \left[1 - \frac{1}{N} \mathrm{Re}\left[\mathrm{tr}(U_{ij} U_{jk} U_{kl} U_{li}) \right] \right] \quad , \tag{7.123}$$

pressure and density for example read

$$\begin{aligned} p\, a_\sigma^4 &= -\xi^2 \left[\left. \frac{\partial \beta_\sigma}{\partial \xi} \right|_{a_\sigma} \bar{P}_\sigma + \left. \frac{\partial \beta_\tau}{\partial \xi} \right|_{a_\sigma} \bar{P}_\tau \right. \\ &\quad \left. + \frac{a_\sigma}{\xi} \left\{ \left. \frac{\partial \beta_\sigma}{\partial a_\sigma} \right|_\xi \bar{P}_\sigma + \left. \frac{\partial \beta_\tau}{\partial a_\sigma} \right|_\xi \bar{P}_\tau \right\} \right] \quad , \\ \varepsilon\, a_\sigma^4 &= -3\xi^2 \left[\left. \frac{\partial \beta_\sigma}{\partial \xi} \right|_{a_\sigma} \bar{P}_\sigma + \left. \frac{\partial \beta_\tau}{\partial \xi} \right|_{a_\sigma} \bar{P}_\tau \right] \quad , \end{aligned} \tag{7.124}$$

and the partial derivatives are given by

$$\begin{aligned} \left. \frac{\partial \beta_\sigma}{\partial \xi} \right|_{a_\sigma} &= -\frac{2N g_\sigma^{-2}}{\xi^2} + \frac{2N}{\xi} \left. \frac{\partial g_\sigma^{-2}}{\partial \xi} \right|_{a_\sigma} \quad , \\ \left. \frac{\partial \beta_\tau}{\partial \xi} \right|_{a_\sigma} &= 2N g_\tau^{-2} + 2N\xi \left. \frac{\partial g_\tau^{-2}}{\partial \xi} \right|_{a_\sigma} \quad , \\ \left. \frac{\partial \beta_\sigma}{\partial a_\sigma} \right|_\xi &= \frac{2N}{\xi} \left. \frac{\partial g_\sigma^{-2}}{\partial a_\sigma} \right|_\xi \quad , \\ \left. \frac{\partial \beta_\tau}{\partial a_\sigma} \right|_\xi &= 2N\xi \left. \frac{\partial g_\tau^{-2}}{\partial a_\sigma} \right|_\xi \quad . \end{aligned} \tag{7.125}$$

The parameter ξ was introduced to allow for the required partial derivatives but it has no physical significance. Physics cannot depend on the relative discretisation length for time and space directions. This requirement can be used to determine the ξ dependence of g_τ and g_σ. If one defines the quantities $c_{\sigma,\tau}$ by

$$g_{\sigma,\tau}^{-2}(a_\sigma,\xi) = g_{\sigma,\tau}^{-2}(a_\sigma)\left(1 + c_{\sigma,\tau}(\xi)\, g^2(a_\sigma) + O\left(g^4\right)\right) \quad , \tag{7.126}$$

one finds[10]

$$\begin{aligned} p\, a^4 &= 2N\left\{g^{-2}(\bar{P}_\sigma - \bar{P}_\tau) - (c'_\sigma\, \bar{P}_\sigma + c'_\tau\, \bar{P}_\tau) - a\frac{\mathrm{d}g^{-2}}{\mathrm{d}a}(\bar{P}_\sigma + \bar{P}_\tau)\right\} , \\ \varepsilon\, a^4 &= 6N\left\{g^{-2}(\bar{P}_\sigma - \bar{P}_\tau) - (c'_\sigma\, \bar{P}_\sigma + c'_\tau\, \bar{P}_\tau)\right\} \quad , \end{aligned} \tag{7.127}$$

and numerically for SU(N)

$$\begin{aligned} c'_\sigma &= 4N\left\{0.586844\frac{N^2-1}{32N^2} + 0.000499\right\} \quad , \\ c'_\tau &= 4N\left\{-0.586844\frac{N^2-1}{32N^2} + 0.005306\right\} \quad . \end{aligned} \tag{7.128}$$

To obtain these results one has to calculate the effective action (7.122) on lattices with different values of ξ. The results have to be the same, as ξ is an unphysical parameter and this leads to (7.128).

To evaluate (7.127) and obtain the pressure and energy density which characterize the physical properties of the quark-gluon plasma phase transition, further corrections are necessary. The expressions (7.117) still contain the vacuum energy which has to be subtracted to get a physical result. In principle the pure vacuum contributions can be identified by simply choosing $N_\tau V$ very large which corresponds to small $T = 1/N_\tau a$. In practice, however, one faces severe numerical problems. A typical simulation uses $N_\sigma^3 \times N_\tau = 16^3 \times 4$ lattice sites and determines the vacuum contribution from a run with $16^3 \times 16$ sites (or sometimes even with a smaller N_τ). This might raise some doubts with respect to the validity of the results, but one should keep in mind that the thermal energy density is proportional to T^4 such that a change in T by a factor 4 (or even 2) already nearly eliminates the contributions of interest, such that the remainder is indeed a good approximation for the $T = 0$ contribution, or at least an upper bound. Let us denote the average plaquette for $N_\sigma = N_\tau$ by $\bar{P}$ (obviously there is now no difference between $\bar{P}_\sigma$ and $\bar{P}_\tau$). The vacuum-subtracted pressure and energy density are then

$$\begin{aligned} p(\text{vacuum}-\text{subtracted})a^4 &= pa^4(N_\sigma^3 \times N_\tau) - pa^4(N_\sigma^3 \times N_\sigma) \\ &= 2N\Big\{g^{-2}(\bar{P}_\sigma - \bar{P}_\tau) - \big(c'_\sigma(\bar{P}_\sigma - \bar{P}) + c'_\tau(\bar{P}_\tau - \bar{P}) \\ &\qquad - a\frac{\mathrm{d}g^{-2}}{\mathrm{d}a}(\bar{P}_\sigma + \bar{P}_\tau - 2\bar{P})\Big\} \quad , \end{aligned} \tag{7.129}$$

$$\begin{aligned} \varepsilon(\text{vacuum}-\text{subtracted})a^4 &= \varepsilon a^4(N_\sigma^3 \times N_\tau) - \varepsilon a^4(N_\sigma^3 \times N_\sigma) \\ &= 6N\big\{g^{-2}(\bar{P}_\sigma - \bar{P}_\tau) \\ &\qquad - \big(c'_\sigma(\bar{P}_\sigma - \bar{P}) + c'_\tau(\bar{P}_\tau - \bar{P})\big)\big\} \quad . \end{aligned} \tag{7.130}$$

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[10] See F. Karsch: Nucl. Phys. **B205**, 285 (1982).

The deviation from an ideal gas can be parametrized by the function

$$\Delta \equiv \frac{\varepsilon^L_{\mathrm{SU}(N)} - 3\,p^L_{\mathrm{SU}(N)}}{T^4} = 6NN_\tau^4\, a\frac{\mathrm{d}g^{-2}}{\mathrm{d}a}\,(\bar{P}_\sigma + \bar{P}_\tau - 2\bar{P}) \quad , \tag{7.131}$$

where Δ is a measure of the strength of interaction.

Monte-Carlo simulations allow us now to obtain $\bar{P}_\tau$, $\bar{P}_\sigma$ and $\bar{P}$ for various values of N_σ, N_τ and β and thus

$$p^{\mathrm{MC}}a^4\,(N_\sigma, N_\tau, \beta) \quad , \quad \varepsilon^{\mathrm{MC}}a^4\,(N_\sigma, N_\tau, \beta) \quad , \tag{7.132}$$

with the temperature $T = 1/aN_\tau$.

As can be seen in Fig. 7.18, a phase transition occurs for some value $\beta = \beta_c$. Next we have to translate this number into a physical transition temperature. The relation connecting the coupling constant g, and thus β, with the lattice spacing a is the usual equation for the running coupling constant (see Sect. 4.3):

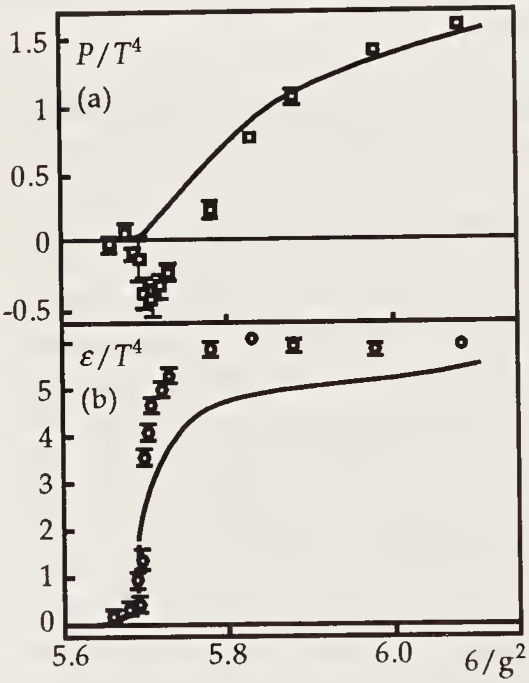

Fig. 7.18. The phase transition for a SU(3) pure gauge theory on a $16^3 \times 4$ lattice

$$\alpha_N(q^2) = \frac{4\pi}{\left(11\cdot\frac{N}{3} - \frac{2}{3}N_f\right)\ln\frac{-q^2}{\Lambda_N^2}} \quad . \tag{7.133}$$

In lattice calculations the resolution q^2 is $-q^2 = 1/a^2$ and $\Lambda = \Lambda_L$ is the lattice value, which is substantially different from the one in the $\overline{MS}$ scheme. Thus we have

$$\beta_L = \frac{2N}{g^2} = \frac{6}{4\pi}\frac{1}{\alpha} = \frac{12}{(4\pi)^2}(11\cdot\frac{N}{3} - \frac{2}{3}N_f)\ln a\Lambda_L \tag{7.134}$$

where we have introduced the subscript L to distinguish the lattice β function from the QCD beta function β_{QCD}.

Inverting (7.134) gives

$$a = \frac{1}{\Lambda_L} \mathrm{e}^{4\pi^2 \beta_L(N_\tau, N_\sigma)/(33 - N - 2N_f)} \tag{7.135}$$

and in second order

$$\begin{aligned} \beta_{\mathrm{QCD}} &= -\beta_0 g_0^3 - \beta_1 g_0^5 + O(g_0^7) \quad , \\ \beta_0 &= \frac{1}{16\pi^2}\left(11 \cdot \frac{N}{3} - \frac{2}{3} N_f\right) \quad , \\ \beta_1 &= \frac{1}{\left(16\pi^2\right)^2}\left(102 \cdot \frac{N^2}{9} - 10 \cdot \frac{N}{3} N_f - \frac{N^2 - 1}{N} N_f\right) \quad , \end{aligned} \tag{7.136}$$

$$a = \frac{1}{\Lambda_L} \cdot \left(\beta_0 g_0^2\right)^{-\frac{\beta_1}{2\beta_0^2}} \mathrm{e}^{-\frac{\beta_L(N_\tau, N_\sigma)}{12\beta_0}} = \frac{1}{\Lambda_L} \cdot R(\beta(N_\tau, N_\sigma)) \quad , \tag{7.137}$$

and thus

$$T = \frac{\Lambda_L}{N_\tau R(\beta(N_\tau, N_\sigma))} \quad . \tag{7.138}$$

To finally get physical values for T one has to determine Λ_L by fitting to some physical known value, e.g. some hadron mass or to α_s at some given momentum scale. Alternatively one can relate T to some definite temperature like T_c and plot all data as a function of

$$\frac{T}{T_c} = \frac{R(\beta_c(N_\tau, N_\sigma))}{R(\beta(N_\tau, N_\sigma))} \quad . \tag{7.139}$$

Following this procedure one obtains the open symbols in Fig. 7.18 and quickly realizes that something has gone wrong as, close to the phase transition, the pressure becomes negative. The reason is that the perturbative relation for the running coupling constant is not really applicable. The phase transition occurs for

$$\beta_L = \frac{6}{g^2} = 5.9 \pm 0.2 \qquad \Rightarrow \qquad g^2 = 1.02 \pm 0.03 \quad , \tag{7.140}$$

such that an expansion in g^2 is not valid. The point is again that the lattices are too small. For larger N_σ and N_τ and given T the lattice spacing a would be smaller, implying that g^2 would be smaller and β_L larger. For sufficiently small g^2 the approach just discussed would thus be reliable. At present one is stuck, since really large lattices can only be treated with the next (or next-to-next) computer generation. To treat the higher orders, at least approximately, purely phenomenological β_{QCD} functions were proposed, e.g. by J. Hoek[11]

$$a \frac{\mathrm{d}g}{\mathrm{d}a} = b_0 g^3 \frac{(1 - a_1 g^2)^2 + a_2^2 g^4}{\left(1 - \left[a_1 + \frac{b_1}{2b_0}\right] g^2\right)^2 + a_3 g^4} \quad , \tag{7.141}$$

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[11] See J. Hoek: Nucl. Phys. **B339**, 732 (1990).

with $a_1 = 0.853572$, $a_2 = 0.93 \cdot 10^{-5}$, $a_3 = 0.0157993$. As can be seen from Fig. 7.19, this form leads to acceptable values for p/T^4 and ε/T^4. Equations (7.137) and (7.139) now read

$$a\Lambda_L = (b_0 g^2)^{-b_1/2b_0^2} \exp\left\{-\frac{1}{2b_0 g^2} + \frac{1}{2b_0}\left[\frac{1}{a_2}\left(a_2^2 - a_3 - \left(\frac{b_1}{2b_0}\right)^2\right)\right.\right.$$
$$\left.\left. \times \left(\arctan\left(\frac{g^{-2} - a_1}{a_2}\right) - \frac{\pi}{2}\right) + \frac{b_1}{2b_0} \ln\left(1 - 2a_1 g^2 + (a_1^2 + a_2^2) g^4\right)\right]\right\} , \tag{7.142}$$

$$\frac{T}{T_c} \equiv \frac{a_c}{a} = \exp\left\{-\frac{1}{2b_0}\left(\frac{\beta_c - \beta}{2N} - \frac{b_1}{2b_0} \ln\left(\frac{(a_1 - \beta_c/2N)^2 + a_2^2}{(a_1 - \beta/2N)^2 + a_2^2}\right)\right.\right.$$
$$+ \frac{a_3 - a_2^2 + (b_1/2b_0)^2}{a_2}$$
$$\left.\left. \times \left[\arctan\left(\frac{a_1 - \beta/2N}{a_2}\right) - \arctan\left(\frac{a_1 - \beta_c/2N}{a_2}\right)\right]\right)\right\} . \tag{7.143}$$

To extract the physical significance of Fig. 7.18 it is useful to compare ρ and ξ with the results for an ideal gas of massless gluons. We shall show in problem 7.8 that a Stefan-Boltzmann gas gives (for SU(3) the colour factor is 16)

$$\varepsilon_{\mathrm{SB}}^L a^4 = -\frac{1}{N_\sigma^3} {\sum_j}' \left(\frac{1}{N_\tau} \frac{\sin^2(\pi j_0/N_\tau)}{b^2 + \sin^2(\pi j_0/N_\tau)} - \frac{1}{N_\sigma} \frac{\sin^2(\pi j_0/N_\sigma)}{b^2 + \sin^2(\pi j_0/N_\sigma)}\right) , \tag{7.144}$$

with

$$b^2 = \sum_{\mu=1}^{3} \sin^2(\pi j_\mu/N_\sigma) . \tag{7.145}$$

The sum runs over the values $j_0 = 0, \pm 1, \pm 2, \cdots \pm (N_\tau/2 - 1), +N_\tau/2$ and $j_\mu = 0, \pm 1, \pm 2 \cdots \pm (N_\sigma/2 - 1), +N_\sigma$. In the primed sum the term with $j_0 = j_1 = j_2 = j_3 = 0$ is left out. Again the contribution for $N_\tau = N_\sigma$ was subtracted to get rid of the vacuum contribution. We still have to make finite-size corrections. The real continuum Stefan-Boltzmann energy density is

$$\varepsilon_{\mathrm{SB}} = \frac{\pi^2}{30} T^4 = \frac{\pi^2}{30} \frac{1}{a^4 N_\tau^4} . \tag{7.146}$$

For a $16^3 \times 4$ lattice (7.144) gives instead

$$\frac{\varepsilon_{\mathrm{SB}}^L}{\varepsilon_{\mathrm{SB}}} = 1.4922 \tag{7.147}$$

and the quantities plotted in Fig. 7.19 are

$$\frac{\varepsilon_{\mathrm{lattice}}}{1.4922 \cdot 16 \frac{\pi^2}{30} T^4} \quad \text{and} \quad \frac{p_{\mathrm{lattice}}}{1.4922 \cdot \frac{1}{3} \cdot 16 \frac{\pi^2}{30} T^4} . \tag{7.148}$$

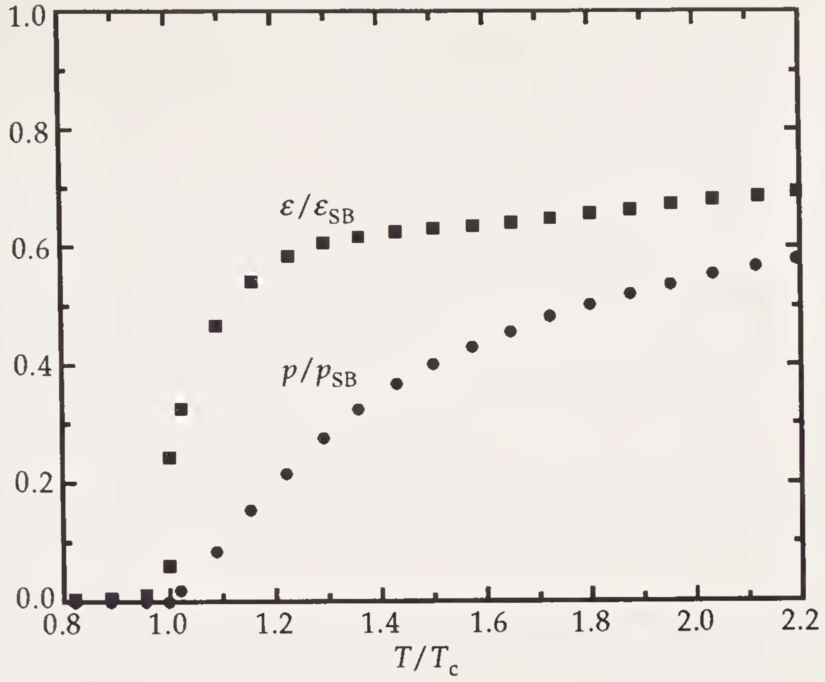

Fig. 7.19. Corrected lattice data obtained with a $16^3 \times 4$ lattice for a pure SU(3) gauge theory

We have seen that to obtain this figure several assumptions were made, and various corrections were applied, but it represents the present level of understanding of the quark-gluon phase transition and all further analyses have to be based on such data, and on those also including dynamical fermions. The latter we do not discuss here, as all the problems just explained are even more severe for fermions, and probably a final understanding has not yet been reached, whereas the gluonic part seems to be reasonably under control. The phenomenological implications of Fig. 7.19 will be discussed in Chap. 8.

EXAMPLE

7.8 The Ideal Bose Gas on the Lattice

We consider the partition function for non-interacting neutral spin-zero particles with mass m on an Euclidian $(N_\sigma^3 \times N_\tau)$ lattice. a_σ and a_τ are the lattice constants in spatial and time directions, respectively. The continuum Lagrangian reads

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2\right) \quad , \tag{1}$$

and

$$\pi \equiv \frac{\partial\mathcal{L}}{\partial(\partial_0\phi)} = \partial_0\phi \tag{2}$$

is the field conjugate to ϕ. Since the system has no conserved quantum number, the continuum partition function reads

Example 7.8.

$$\mathcal{Z}(T,V;m)=\operatorname{tr}\left\{\exp\left[-\frac{\hat{H}}{T}\right]\right\} \quad , \tag{3}$$

$$\hat{H}=\int_V d^3x\,\mathcal{H}(\pi(\boldsymbol{x},t),\phi(\boldsymbol{x},t)) \quad , \tag{4}$$

$$\begin{aligned}\mathcal{H}(\pi,\phi)&=\pi\partial_0\phi-\mathcal{L}(\phi,\partial_0\phi)\\ &\equiv\frac{1}{2}\left(\pi^2+[\nabla\phi]^2+m^2\phi^2\right) \quad .\end{aligned} \tag{5}$$

$\mathcal{Z}$ can be written as the following functional integral.

$$\mathcal{Z}(T,V;m)=N\int\mathcal{D}\pi\int_P\mathcal{D}\phi\exp\left\{\mathrm{i}\int_0^{1/\mathrm{i}T}dt\int_V d^3x(\pi\partial_0\phi-\mathcal{H}(\pi,\phi))\right\} \tag{6}$$

Using the Hamiltonian (5) the argument of the exponential function can be rewritten in the following way.

$$\mathrm{i}\int_0^{1/\mathrm{i}T}dt\int_V d^3x\left(-\frac{1}{2}(\pi-\partial_0\phi)^2+\mathcal{L}(\phi,\partial_0\phi)\right) \quad . \tag{7}$$

Therefore one can eliminate the π-field as in the case of gauge field theories. However, in a Euclidean space-time lattice it is not necessarily more practicable to expand the π-field into Fourier components, i.e. to transfrom the functional integrals into products of ordinary integrals. Since our space-time lattice consists of a finite countable set of points

$$\begin{aligned}&x_\alpha=(\alpha_0a_\tau,\boldsymbol{\alpha}a_\sigma) \quad ,\\ &\alpha_0=0,\pm1,\pm2,\dots,\pm\left(\frac{N_\tau}{2}-1\right),\frac{N_\tau}{2} \quad ,\\ &\alpha_i=0,\pm1,\pm2,\dots,\pm\left(\frac{N_\sigma}{2}-1\right),\frac{N_\sigma}{2} \quad ,\\ &\qquad i=1,2,3\end{aligned}$$

(we restrict our treatment on even values of N_σ, N_τ), there is also only a finite countable set of fields to be integrated,

$$N\mathcal{D}\pi\mathcal{D}\phi\sim a_\sigma^{3N_\sigma^3N_\tau}\left[\prod_\alpha\frac{d\pi(x_\alpha)}{2\pi}d\phi(x_\alpha)\right] \quad . \tag{8}$$

The normalization factor on the right hand side is due to $N\sim V^{F'}\sim a_\sigma^{3F'}$ and $F'\equiv\Sigma_x=N_\sigma^3N_\tau$. The discretized ($\mathrm{i}t\to t$) version of (6) taking into account (7) and the substitution $\pi-\partial_0\phi\to\pi$ reads

$$\begin{aligned}\mathcal{Z}(T,V;m)\sim a_\sigma^{3N_\sigma^3N_\tau}&\int\left[\prod_\alpha\frac{d\pi(x_\alpha)}{2\pi}d\phi(x_\alpha)\right]\\ &\times\exp\left\{-\frac{1}{2}a_\sigma^3a_\tau\sum_\alpha\left[\pi^2(x_\alpha)+\left(\frac{\phi(x_\alpha+e_0)-\phi(x_\alpha)}{a_\tau}\right)^2\right.\right.\\ &\left.\left.+\sum_{i=1}^3\left(\frac{\phi(x_\alpha+e_i)-\phi(x_\alpha)}{a_\sigma}\right)^2+m^2\phi^2(x_\alpha)\right]\right\} \quad .\end{aligned} \tag{9}$$

Example 7.8.

Here the partial differentiations have been written in discrete form as quotients of differences employing the lattice unit vectors e_μ. The integration over the π-fields yields a factor $(a_\sigma^3 a_\tau)^{-N_\sigma^3 N_\tau/2}$, and hence one has

$$\mathcal{Z}(T,V;m) \sim \left(\frac{a_\sigma^3}{a_\tau}\right)^{\frac{N_\sigma^3 N_\tau}{2}} \int \left[\prod_\alpha d\phi(x_\alpha)\right] \mathrm{e}^{-S[\phi]} \quad , \tag{10}$$

$$\begin{aligned} S[\phi] &\equiv \frac{1}{2} a_\sigma^3 a_\tau \sum_\alpha \left[\left(\frac{\phi(x_\alpha+e_0)-\phi(x_\alpha)}{a_\tau}\right)^2 \right. \\ &\quad \left. + \sum_{i=1}^{3}\left(\frac{\phi(x_\alpha+e_i)-\phi(x_\alpha)}{a_\sigma}\right)^2 + m^2\phi^2(x_\alpha)\right] \\ &\equiv -a_\sigma^3 a_\tau \sum_\alpha \mathcal{L}. \end{aligned} \tag{11}$$

At this point the first problem occurs in evaluating the partition function. If one considers massless bosons, the last term in S vanishes. The S itself vanishes on lines of constant field configurations, i.e. $\phi(x_\alpha) = \text{const.}$ for all lattice points x_α in the multi-dimensional integration volume. Consequently the integrand is equal to 1 on these lines and the integral diverges, because one has to integrate over all values of $\phi(x_\alpha)$, including infinitly large values.

In the case of massive bosons, however, the mass term has the effect of a Gaussian convergence factor, i.e. it suppresses large values of $\phi(x_\alpha)$ in the multi-dimensional integral (10) and makes sure that the partition function converges. First we consider the case $m \neq 0$ and discuss later a possibility to avoid the divergence at $m = 0$.

But first of all we want to explain the similarities between quantum field theories on a lattice and spin models known from statistical physics. To that end we evaluate the quotients of differences in S (see (11)). Due to the periodic boundary conditions one obtains

$$\begin{aligned} S[\phi] &= \frac{1}{2} a_\sigma^3 a_\tau \sum_\alpha \left[\left(\frac{6}{a_\sigma^2}+\frac{2}{a_\tau^2}+m^2\right)\phi^2(x_\alpha) \right. \\ &\quad \left. - \frac{2}{a_\sigma^2}\sum_{i=1}^{3}\phi(x_\alpha+e_i)\phi(x_\alpha) - \frac{2}{a_\tau^2}\phi(x_\alpha+e_0)\phi(x_\alpha)\right] \quad . \end{aligned} \tag{12}$$

The bracket in the first term can be rewritten as

$$\begin{aligned} \frac{6}{a_\sigma^2}+\frac{2}{a_\tau^2}+m^2 &= \frac{2}{a_\sigma a_\tau}\left(\frac{3a_\tau}{a_\sigma}+\frac{a_\sigma}{a_\tau}+\frac{m^2 a_\sigma a_\tau}{2}\right) \\ &\equiv \frac{2}{a_\sigma a_\tau k} \quad , \end{aligned} \tag{13}$$

with

$$\begin{aligned} k^{-1} &\equiv 3\xi^{-1}+\xi+\frac{(m a_\sigma)^2}{2\xi} \quad , \\ \xi &\equiv a_\sigma/a_\tau \quad . \end{aligned} \tag{14}$$

Example 7.8.

It follows that

$$S[\phi]=\sum_{\alpha}\left[\frac{a_\sigma^2}{k}\phi^2(x_\alpha)-\frac{a_\sigma^2}{\xi}\sum_{i=1}^{3}\phi(x_\alpha+e_i)\phi(x_\alpha)-a_\sigma^2\xi\phi(x_\alpha+e_0)\phi(x_\alpha)\right]$$

$$\equiv-\sum_{\alpha}\left[\sigma^2(x_\alpha)+\frac{k}{\xi}\sum_{i=1}^{3}\sigma(x_\alpha+e_i)\sigma(x_\alpha)\right.$$

$$\left.+k\xi\sigma(x_\alpha+e_0)\phi(x_\alpha)\right]\equiv S[\phi]\quad,\tag{15}$$

where the fields have been renamed according to

$$\sigma(x_\alpha)\equiv a_\sigma k^{-1/2}\phi(x_\alpha)\quad.\tag{16}$$

However, the action (15) now is identical with the action of a spin system, namely the *non-isotropic four-dimensional Gauss model*,

$$S[\sigma]\equiv-\sum_{\alpha}\left[-\sigma^2(x_\alpha)+K_\sigma\sum_{i=1}^{3}\sigma(x_\alpha+e_i)\sigma(x_\alpha)+K_\tau\sigma(x_\alpha+e_0)\phi(x_\alpha)\right]\quad,\tag{17}$$

where we defined $K_\sigma\equiv k/\xi$ and $K_\tau\equiv k\xi$. The partition function then reads

$$\mathcal{Z}(T,V;m)\sim K_\tau^{N_\sigma^3N_\tau/2}\int\prod_{\alpha}d\sigma(x_\alpha)\mathrm{e}^{-S[\sigma]}\quad.\tag{18}$$

For isotropic lattices ($\xi=1\Rightarrow K_\sigma=K_\tau\equiv K$) one can transform (17) into a better-known form of the action, i.e. the action for spin systems. Then one can summarise the last two terms to

$$K\sum_{n.n.(\alpha)}\sigma(x_{n.n.(\alpha)})\sigma(x_\alpha)$$

where "$n.n.(\alpha)$" denotes the lattices index of a *nearest neighbour* of the spin $\sigma(x_\alpha)$ in the four-dimensional lattice. Since one has to sum over all lattice points, the action assumes the form

$$S[\sigma]=-K\sum_{\{ij\}}\sigma_i\sigma_j+\sum_i\sigma_i^2\quad.\tag{19}$$

Here "$\{ij\}$" denote all pairs of nearest neighbours in the four-dimensional lattice. We have abbreviated the points x_α in the four-dimensional lattice by i or j, respectively. Equation (11) is nothing else but the action for a system of spins $\{\sigma_i\}$ with a Gaussian distribution of the *spin lengths* $|\sigma_i|$. The case $m=0$ corresponds to the critical coupling $K=1/4$ in the Gauss model. The partition function assumes the well-known form

$$\mathcal{Z}\sim\int\prod_i d\sigma_i e^{-H[\sigma]}\tag{20}$$

for spin systems. The "energy" of the system $H[\sigma]$ is just the action $S[\sigma]$.

Example 7.8.

Now we further evaluate the partition function (18). The only problem in doing this is the occurrence of terms which contain products of fields at different lattice points. These terms, however, can be removed by Fourier transformation

$$\sigma(x_\alpha) = (N_\sigma^3 N_\tau)^{-1/2} \sum_q \mathrm{e}^{\mathrm{i}q\cdot x_\alpha} \Phi(q) \quad . \tag{21}$$

The momenta q are those of the first Brillouin zone of the reciprocal lattice,

$$\begin{aligned} q &= \left(\frac{2\pi}{N_\tau a_\tau} j_0, \frac{2\pi}{N_\sigma a_\sigma} \boldsymbol{j}\right) \quad , \\ j_0 &= 0, \pm 1, \pm 2, \ldots, \pm\left(\frac{N_\tau}{2} - 1\right), \frac{N_\tau}{2}, \\ j_i &= 0, \pm 1, \pm 2, \ldots, \pm\left(\frac{N_\sigma}{2} - 1\right), \frac{N_\sigma}{2}, \quad i = 1, 2, 3 \quad . \end{aligned}$$

Now we insert (21) into the action (17) and make use of the disctrete form of the completeness relation

$$\sum_\alpha \mathrm{e}^{\mathrm{i}q\cdot x_\alpha} \equiv N_\sigma^3 N_\tau \delta^{(4)}_{q,0} \equiv N_\sigma^3 N_\tau \delta_{q_0,0}\delta_{q_1,0}\delta_{q_2,0}\delta_{q_3,0} \quad . \tag{22}$$

Hence

$$\begin{aligned} S[\Phi] &= -\sum_q \Phi(-q)\left(-1 + K_\sigma \sum_{i=1}^{3} \mathrm{e}^{-\mathrm{i}q\cdot e_i} + K_\tau\, \mathrm{e}^{-\mathrm{i}q\cdot e_0}\right)\Phi(q) \\ &= \frac{k}{2\xi}\sum_q \Phi(-q)\left(6 + 2\xi^2 + [ma_\sigma]^2 - 2\sum_{i=1}^{3} \mathrm{e}^{-\mathrm{i}q\cdot e_i} - 2\xi^2\, \mathrm{e}^{-\mathrm{i}q\cdot e_0}\right)\Phi(q) \\ &= \frac{k}{2\xi}\sum_q \Phi(-q)\left([ma_\sigma]^2 - 2\sum_{i=1}^{3}\left[\mathrm{e}^{-\mathrm{i}q\cdot e_i} - 1\right] - 2\xi^2\left[\mathrm{e}^{-\mathrm{i}q\cdot e_0} - 1\right]\right)\Phi(q) \\ &= \frac{k}{2\xi}\sum_q \Phi(-q)\Bigg([ma_\sigma]^2 - 2\sum_{i=1}^{3}\left[\cos(q\cdot e_i) - 1\right] \\ &\qquad - 2\xi^2\left[\cos(q\cdot e_0) - 1\right]\Bigg)\Phi(q) \quad . \end{aligned} \tag{23}$$

Here we have used the definition of k and the expansion of the exponential function into trigonometric functions. Since the sum extends over all q, the sines in this expansion vanishes. Employing in addition $q \cdot e_i \equiv q_i a_\sigma$, $q \cdot e_0 \equiv q_i a_\tau$, the unit vectors on the lattice just have the lengths a_σ and a_τ, and we obtain

$$S[\Phi] = \frac{k}{2\xi}\sum_q \Phi(-q)\mathcal{D}_0^{-1}(a_\sigma, \xi, q)\Phi(q) \quad , \tag{24}$$

$$\mathcal{D}_0^{-1} \equiv [ma_\sigma]^2 + 4\sum_{i=1}^{3} \sin^2\left[\frac{q_i a_\sigma}{2}\right] + 4\xi^2 \sin^2\left[\frac{q_0 a_\tau}{2}\right] \quad . \tag{25}$$

Example 7.8.

The quantity $\mathcal{D}_0^{-1}$ is the *inverse propagator for spin-0 particles on the lattice.* Taking into account the relation

$$\int \left[\prod \frac{\mathrm{d}x_i}{\sqrt{2\pi}}\right] \exp\left\{-\frac{1}{2}\sum_{ij} x_i A_{ij} x_j\right\} = (\det A)^{-1/2} \quad , \tag{26}$$

we immediately arrive at the result for the partition function of the system. Just like in the case of gauge fields (massless spin-1 particles) one obtains (neglecting irrelevant constant factors)

$$\begin{aligned} \mathcal{Z}(T,V;m) &= (k\xi)^{N_\sigma^3 N_\tau/2}\left(\det\left[\frac{k}{\xi}\mathcal{D}_0^{-1}\right]\right)^{-1/2} \\ &= \xi^{N_\sigma^3 N_\tau}\prod_q \left[\mathcal{D}_0^{-1}(a_\sigma,\xi,q)\right]^{-1/2} \end{aligned} \tag{27}$$

$$\begin{aligned} \Leftrightarrow \ln \mathcal{Z}(T,V;m) &= N_\sigma^3 N_\tau \ln\xi - \frac{1}{2}\sum_q \ln\left[\mathcal{D}_0^{-1}(a_\sigma,\xi,q)\right] \\ &\equiv N_\sigma^3 N_\tau \ln\xi - \frac{1}{2}\sum_j \ln\left\{[ma_\sigma]^2 + 4\sum_{i=1}^{3}\sin^2\left[\frac{\pi j_i}{N_\sigma}\right]\right. \\ &\quad \left. + 4\xi^2\sin^2\left[\frac{\pi j_0}{N_\tau}\right]\right\} \quad . \end{aligned} \tag{28}$$

With the help of

$$\begin{aligned} p &= T\left.\frac{\partial \ln \mathcal{Z}}{\partial V}\right|_T \quad , \\ \varepsilon &= \frac{T^2}{V}\left.\frac{\partial \ln \mathcal{Z}}{\partial T}\right|_V \quad , \end{aligned} \tag{29}$$

and the lattice relations

$$T\left.\frac{\partial}{\partial V}\right|_T = \frac{\xi}{3N_\sigma^3 N_\tau a_\sigma^4}\left(a_\sigma\left.\frac{\partial}{\partial a_\sigma}\right|_\xi + \xi\left.\frac{\partial}{\partial \xi}\right|_{\alpha_\sigma}\right) \tag{30}$$

$$\frac{T^2}{V}\left.\frac{\partial}{\partial T}\right|_V = \frac{\xi^2}{N_\sigma^3 N_\tau a_\sigma^4}\left.\frac{\partial}{\partial \xi}\right|_{\alpha\sigma} \quad , \tag{31}$$

one can easily calculate

$$p\, a_\sigma^4 = \frac{\xi}{3} - \frac{\xi}{3N_\sigma^3 N_\tau}\sum_j \frac{[ma_\sigma]^2 + 4\xi^2\sin^2(\pi j_0/N_\tau)}{[ma_\sigma]^2 + 4b^2(j) + 4\xi^2\sin^2(\pi j_0/N_\tau)} \tag{32}$$

$$\varepsilon\, a_\sigma^4 = \xi - \frac{\xi}{N_\sigma^3 N_\tau}\sum_j \frac{4\xi^2\sin^2(\pi j_0/N_\tau)}{[ma_\sigma]^2 + 4b^2(j) + 4\xi^2\sin^2(\pi j_0/N_\tau)} \quad , \tag{33}$$

where we have used the abbreviation

$$b^2(j) = \sum_{i=1}^{3}\sin^2(\pi j_i/N_\sigma) \quad . \tag{34}$$

Example 7.8.

As mentioned earlier, one has to avoid the integration over constant field configurations in the evaluation of the partition function. These configurations correspond on the reciprocal lattice to fields without momentum, $q \equiv 0$. In the massless case $m = 0$ the term $j_0 = j_1 = j_2 = j_3 = 0$ must therefore be excluded from the sums (32) and (33).

The partition function (28) still contains the vacuum energy. One can approximately eliminate this energy by subtracting the corresponding terms for pressure and energy density on a lattice with $N_\tau \equiv N_\sigma$ (to evaluate thermodynamical quantities one usually chooses lattices with $N_\sigma \gg N_\tau$; because of $T = 1/aN_\tau$ a lattice with $N_\tau \equiv N_\sigma$ represents a system with much lower temperature and can therefore serve as an approximation for the "cold" vacuum).

Energy density and pressure for a non-interacting system of massless bosons on a $(N_\sigma^3 \times N_\tau)$ lattice with $\xi = 1$ then read

$$\begin{aligned} \varepsilon\, a^4 &= -\frac{1}{N_\sigma^3} \sum_{j}{}' \left(\frac{1}{N_\tau} \frac{\sin^2(\pi j_0/N_\tau)}{b^2(j) + \sin^2(\pi j_0/N_\tau)} \right. \\ &\qquad \left. - \frac{1}{N_\sigma} \frac{\sin^2(\pi j_0/N_\sigma)}{b^2(j) + \sin^2(\pi j_0/N_\sigma)} \right) \quad , \qquad (35) \\ p\, a^4 &\equiv \frac{1}{3}\, \varepsilon\, a^4 \quad , \qquad (36) \end{aligned}$$

where the prime on the sum indicates the omission of the term $j = 0$. The final evaluation of these quantities now has to be carried out numerically, which is no problem, because the sum includes only a finite number of terms.

Since a massless Bose gas is known in the literature as a *Stefan-Boltzmann gas*, we denote the energy density (35) by $\varepsilon_{\mathrm{SB}}^{L}$ (the index L indicates that the energy density has been evaluated on a lattice).

7.3 QCD Sum Rules

In this section we discuss the techniques of QCD sum rules, which allow an approximate phenomenological treatment of nonperturbative effects in QCD. It turns out that this method is extremely useful in calculating the lowest-mass hadronic bound states or determining effective coupling constants. In addition, QCD sum rules offer some surprising insights into the internal wave functions of nucleons and pions.

The basic ideas of QCD sum rules are met frequently in QCD. As the full QCD interaction is strongly nonperturbative and can be solved exactly only in special cases (e.g., using lattice gauge theories; see Sect. 7.1), one attempts to separate perturbative and nonperturbative contributions, describing the latter by a set of phenomenologically effective Feyman rules.

The starting point for QCD sum rules, first formulated by M.A. Shifmann, A.I. Vainshtein, and V.I. Zakharow in 1979[12], is the operator-product expansion we met in Sect. 5.4. The OPE gives a general form for the quantities of interest, and QCD sum rules are a phenomenological procedure for evaluating the matrix elements of the operators that occur. This is a general procedure with many applications. We shall introduce it by discussing a specific application: the determination of hadron masses. For this purpose the operator to be expanded is the general time-ordered product of two currents:

$$\mathrm{i}\int \mathrm{d}x\ \mathrm{e}^{\mathrm{i}q\cdot x}\,T[j_\Gamma(x)j_\Gamma(0)] = C_I^\Gamma \mathcal{I} + \sum_n C_n^\Gamma(q)\mathcal{O}_n \quad . \tag{7.149}$$

The operators $\mathcal{O}_n$ can be ordered by their naive dimension ($\dim[q] = 3/2$, $\dim[G_{\mu\nu}] = 2$, $\dim[m] = 1$). As we shall only be interested in vacuum expectation values, we can restrict ourselves to spin-0 operators, which are

$$\begin{array}{rcl}
\mathcal{I} & , & d=0 \quad , \\
\mathcal{O}_m = m\bar{q}q & , & d=4 \quad , \\
\mathcal{O}_G = G^a_{\mu\nu}G_{a\mu\nu} & , & d=4 \quad , \\
\mathcal{O}_\Gamma = \bar{q}\Gamma q\bar{q}\Gamma q & , & d=6 \quad , \\
\mathcal{O}_\sigma = m'\bar{q}\sigma_{\mu\nu}\dfrac{\lambda^a}{2}qG^a_{\mu\nu} & , & d=6 \quad , \\
\mathcal{O}_f = f_{abc}G^a_{\mu\nu}G^b_{\nu\gamma}G^c_{\gamma\mu} & , & d=6 \quad , \quad \text{etc.}
\end{array} \tag{7.150}$$

To evaluate (7.136) one must know the vacuum expectation values of all operators and the corresponding Wilson coefficients, which can be calculated perturbatively and will be discussed in detail below. But first we shall present the general technique of QCD sum rules.

QCD sum rules start from the fact that the vacuum polarization tensor can be described at the hadronic level, i.e., in terms of hadrons and hadronic resonances. The optical theorem states that the total cross section of a certain reaction equals the imaginary part of its forward-scattering amplitude. This can be understood from the fact that the cross section is proportional to the square of some amplitude, graphically depicted by doubling the corresponding graph.

The forward-scattering amplitude corresponds to the vacuum polarization tensor. Hence for a vector coupling we can write

$$\mathrm{Im}\Pi^V\left(s=q^2\right) = \frac{9}{64\pi^2\alpha^2}s\sigma\left(\mathrm{e}^+ + \mathrm{e}^- \to \gamma^* \to \text{hadrons}\right) \quad . \tag{7.151}$$

The cross section of virtual photons into hadrons has been extensively studied in electron positron annihilation, e.g., at DESY in Hamburg and at SLAC in Stanford. For low virtuality of the photon there is a contribution only when q^2 is just the mass squared of a hadronic resonance with photon quantum numbers, i.e., a 1^{--} resonance. (Here 1 is the spin, the first "−" represents the internal parity, and the second "−" C parity. Only neutral particles have a C parity.) Therefore the vacuum

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[12] See M.A. Shifmann, A.I. Vainshtein, and V.I. Zakharow: Nucl. Phys. **B147**, 385 and 448 (1979).

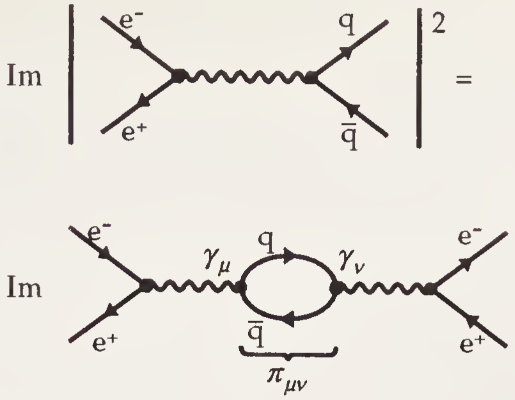


Fig. 7.20. A graphical demonstration of the optical theorem

polarization tensor can be thought of as a sum of δ functions of the type $\delta(q^2 - m_R^2)$. In reality the hadronic states are rather broad (see Fig. 7.21), but here they can be approximated by delta functions. Bear in mind that, in principle, we can imagine inserting a Breit Wigner form instead; however, the precission achieved is such that it does not really make a difference. As Π^V is dimensionless, this must be multiplied by the only dimensional quantity $s = m_R^2$. All other constants appearing here are absorbed in a parameter g_R, and we have

$$\mathrm{Im}\left[\Pi^V(s = q^2)\right] = \sum_q \frac{\pi}{e_q^2} \sum_R \frac{m_R^2}{g_{qR}^2}\, \delta(s - m_R^2) \quad . \tag{7.152}$$

Here the sum over resonances has been split into sums of resonances that require the creation of a q$\bar{\mathrm{q}}$ pair of flavor q. For $q = c$ these are, e.g., J/ψ, ψ', ψ'', $\cdots$. For sufficiently high virtualities of the photon, more than one particle can be created, and discrete resonances are replaced by continua of states. For virtualities just above the continuum threshold, the cross section cannot be expressed in a general simple form. Above the continuum threshold this is possible, since just the elementary $\mathrm{e}^+ + \mathrm{e}^- \to \mathrm{q} + \bar{\mathrm{q}}$ cross section can be inserted.

$$\sigma(\mathrm{e}^+ + \mathrm{e}^- \to \text{hadrons}) = \frac{4\pi\alpha^2}{3s} \sum_q e_q^2 \tag{7.153}$$

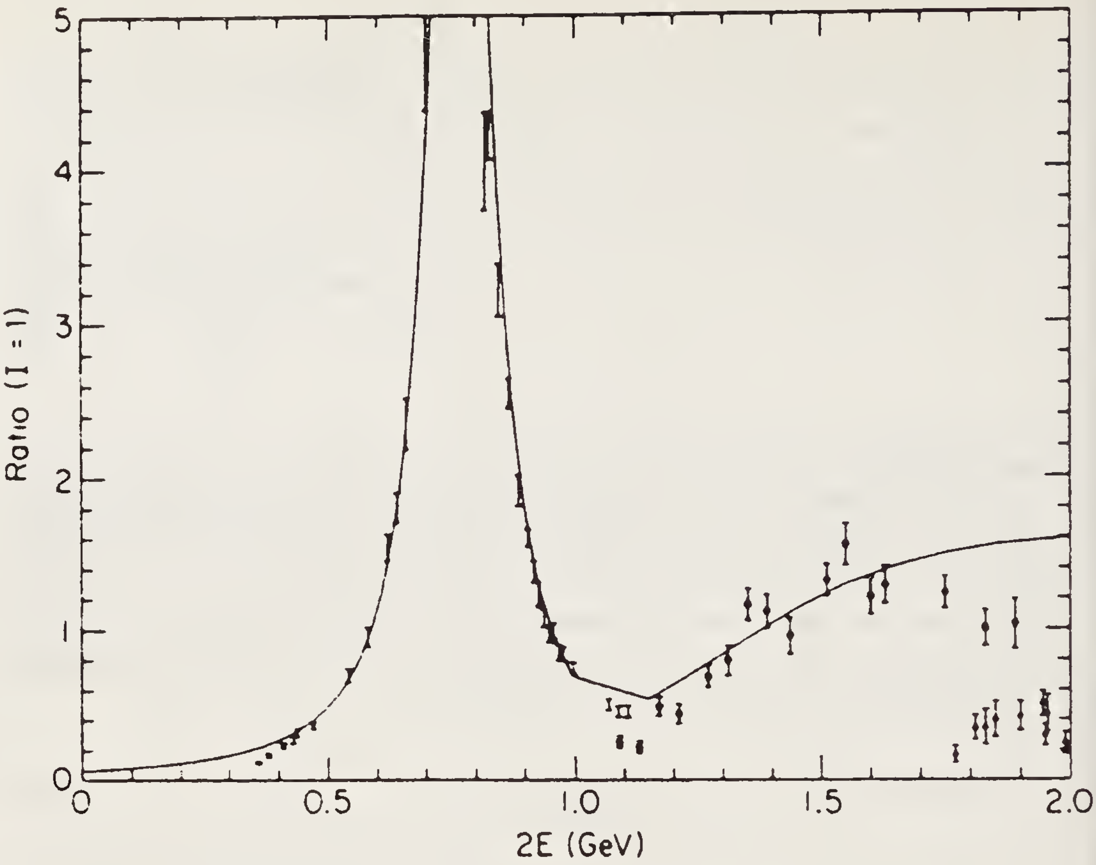
The ratio of the cross sections for multi pion production and (7.153) is plotted in Fig. 7.21.

The rho resonance at 770 MeV is very prominent, while the higher resonances merge with the continuum. The continuum threshold s_0 is at about 1.3 GeV.

$$\mathrm{Im}\left[\Pi^V(s = q^2)\right] = \frac{3\pi}{e_f^2} \sum_R \frac{m_R^2}{g_R^2}\, \delta(s - m_R^2) + \sum_f \frac{9}{16\pi} e_f^2\, \Theta(s - s_{0f}) \; . \tag{7.154}$$

Here we have counted each flavor f three times, since there are three colors. The continuum thresholds s_{0f} for the different flavors f can be taken to be free-fitting parameters. In our case we take only up and down quarks into account and choose $s_0 = s_{0u} = s_{0d} \approx 1.3$ GeV and neglect all except the ρ resonance in the first term on the rhs of (7.154). As we have already mentioned, its approximation by the

Fig. 7.21. The ratio $\sigma(e^+e^- \rightarrow n\pi) / \sigma(e^+e^- \rightarrow \mu^+\mu^-)$



free-quark cross section is poor near the continuum threshold, and one attempts to minimize the errors resulting from this by suitably adjusting s_{0f}. In the following, we shall expand the correlator in (7.149) in the framework of OPE and in this way obtain a representation of Π^V as a function of expectation values of certain operators. On the other hand, the imaginary part of Π^V can, as discussed above, be parametrized by discrete resonances and a continuum contribution. This will enable us to express the mass of the lightest resonance in terms of the expectation values mentioned above. This approach can be repeated for each J^{PC} channel so that many masses (and coupling constants) are fitted essentially by three expectation values. Our trust in this procedure is based primarily on its success. QCD sum rules consistently describe a plethora of hadronic data, all of them with a typical accuracy of about 20%. On the other hand, the fitted expectation values

$$\begin{aligned}
\left\langle 0 \left| \frac{\alpha_s}{\pi} G^a_{\mu\nu} G^{a\mu\nu} \right| 0 \right\rangle &= (360 \pm 20 \text{ MeV})^4 \quad , \\
\langle 0 | \bar{u}u | 0 \rangle = \langle 0 | \bar{d}d | 0 \rangle &= -(225 \pm 25 \text{ MeV})^3 \quad , \\
\langle 0 | m_s \bar{s}s | 0 \rangle &= -(210 \pm 5 \text{ MeV})^4 \quad , \\
\langle 0 | m \bar{q}q | 0 \rangle &= -(100 \text{ MeV})^4 \quad , \quad m = \frac{1}{2}(m_u + m_d) \quad , \\
\langle 0 | \bar{q} g \sigma_{\mu\nu} G^{a\mu\nu} q | 0 \rangle &= 0.436 \text{ GeV}^5
\end{aligned} \qquad (7.155)$$

have far-reaching physical implications. They describe physical properties of the nonperturbative QCD ground state and give, for example the vacuum energy density. To illustrate this let us discuss the trace of the energy momentum tensor of

QCD,

$$\Theta^{\mu}_{\mu} = \frac{\beta(g)}{2g} G^{a}_{\mu\nu} G^{a\mu\nu} + \sum m\bar{q}q \quad , \tag{7.156}$$

with the QCD beta function

$$\beta(g) = -\frac{g^3}{(4\pi)^2}\left(11 - \frac{2}{3}N_F\right) + \dots \tag{7.157}$$

This gives, on inserting (7.142),

$$\Theta^{\mu}_{\mu} = -\frac{1}{8}\left(11 - \frac{2}{3}N_F\right)\frac{\alpha_s}{\pi} G^{a}_{\mu\nu} G^{a\mu\nu} + \sum_q m_q \bar{q}q \quad , \tag{7.158}$$

$$\langle \Theta^{\mu}_{\mu} \rangle \approx -(370\ \mathrm{MeV})^4 \quad . \tag{7.159}$$

In the bag model Θ^{μ}_{μ} is simply (ρ is the energy density, p is the pressure)

$$\Theta^{\mu}_{\mu}\Big|_{\mathrm{bag}} = \rho - 3p = -4B \quad , \tag{7.160}$$

which suggests that the bag constant is ($N_F = 3$)

$$B = \frac{9}{32}\left\langle \frac{\alpha_s}{\pi} G^{a}_{\mu\nu} G^{a\mu\nu} \right\rangle \approx (260\ \mathrm{MeV})^4 \quad , \tag{7.161}$$

which is substantially larger than the value fitted in the simple bag model (see (3.143)) $B = (146\ \mathrm{MeV})^4$. There do exist a number of modified bag models with a smaller bag radius and correspondingly larger B values. Some of them use B's as large as $(250\ \mathrm{MeV})^4$ and their authors claim that the sum rule value (7.161) would support their model. Such claims are, however, not very sensible. In each of these models additional interactions are introduced. The energy and pressure associated with these additional interactions and that due to the QCD vacuum state are interwoven, such that is not so clear which quantity should be compared to (7.159). Most probably B should only be regarded as an effective parameter and the bag model as a simple model with limited connections to QCD.

On the other hand (7.159) is a clear statement of the fact that the vacuum of particle physics is far from being a simple empty state but is instead a complicated structure with a definite energy. To illustrate this fact let us calculate the total energy content of a cubic meter of QCD vacuum.

$$\begin{aligned} 1\ \mathrm{m}^3 \times (260\ \mathrm{MeV})^4 &= \mathrm{MeV} \times 260 \times \left(10^{15}\right)^3 \left(\frac{260\ \mathrm{MeV\ fm}}{197\ \mathrm{MeV\ fm}}\right)^3 \\ &= 6.0 \times 10^{47}\ \mathrm{MeV} = 10^{18}\ \mathrm{kg} = 10^{15}\ \mathrm{t} \end{aligned} \tag{7.162}$$

This corresponds to the mass energy of a million billions tonnes of matter. These numbers become much more exotic if one studies astronomical distances.

Chiral symmetry, i.e. SU(2), plays an important and still not fully understood role in full QCD. The normal isospin symmetry transformation is

$$\psi(x) \to \mathrm{e}^{-\frac{\mathrm{i}}{2}\alpha_a(x)\tau^a}\psi(x) \quad . \tag{7.163}$$

In the absence of quark mass terms a second symmetry follows from this, namely

$$\psi(x) \to \mathrm{e}^{-\frac{\mathrm{i}}{2}\alpha_a(x)\tau^a\gamma_5}\psi(x) \quad . \tag{7.164}$$

The reason that $\bar{\psi}(x)\gamma_\mu\psi(x)$ is invariant is that γ_5 anticommutes with every γ matrix, and thus any bilinear form with an even number of γ matrices between $\psi^\dagger(x)$ and $\psi(x)$ is chirally symmetric, and all bilinear forms with an odd number are not:

$$\begin{aligned}\psi^\dagger(x)\gamma_0\gamma_\mu\psi(x) \to\ & \psi^\dagger(x)\mathrm{e}^{\frac{\mathrm{i}}{2}\alpha_a(x)\tau^a\gamma_5}\gamma_0\gamma_\mu\mathrm{e}^{-\frac{\mathrm{i}}{2}\alpha_a\tau^a\gamma_5}\psi(x)\\ &= \psi^\dagger(x)\gamma_0\mathrm{e}^{-\frac{\mathrm{i}}{2}\alpha_a(x)\tau^a\gamma_5}\gamma_\mu\mathrm{e}^{-\frac{\mathrm{i}}{2}\alpha_a\tau^a\gamma_5}\psi(x)\\ &= \psi^\dagger(x)\gamma_0\gamma_\mu\mathrm{e}^{\frac{\mathrm{i}}{2}\alpha_a(x)\tau^a\gamma_5}\mathrm{e}^{-\frac{\mathrm{i}}{2}\alpha_a\tau^a\gamma_5}\psi(x)\\ &= \psi^\dagger(x)\gamma_0\gamma_\mu\psi(x) \quad .\end{aligned} \tag{7.165}$$

Consequently the quark-mass term $m_q\bar{\psi}\psi$ violates the symmetry (7.163) but since the QCD up and down quark masses are small, their symmetry-breaking effects should be weak, such that a symmetric theory should give a good description. Surprisingly this is not the case. While the up and down quark masses are only about $m_\mathrm{u} = 5.6 \pm 1.1$ MeV and $m_\mathrm{d} \approx 9.9 \pm 1.1$ MeV, the chiral partners ρ and a_1 have vastly different masses

$$m_{\mathrm{a}_1} - m_\rho = 1260\ \mathrm{MeV} - 770\ \mathrm{MeV} = 490\ \mathrm{MeV} \quad , \tag{7.166}$$

which implies that in addition to the small explicit breaking of chiral symmetry there must also be a spontaneous symmetry violation. "Spontaneous symmetry breaking" (SSB) designates all cases in which a symmetry present in the Lagrangian is not present in the actual physical ground state. Let us add that ρ has the J^{PC} quantum numbers 1^{--}, while a_1 has 1^{++}. Both are the lightest mesons with these quantum numbers. As (7.164) mixes vector states with axial-vector states, ρ and a_1 would have the same mass if (7.164) were a symmetry operation. Similarily $\omega(m_\omega = 782\ \mathrm{MeV}, J^{PC} = 1^{--})$ and $\mathrm{f}_1(m_{\mathrm{f}_1} = 1282\ \mathrm{MeV}, J^{PC} = 1^{++})$ should be degenerate, etc. The pion, however, should play a special role; see below. Let us also note that in lattice calculations (see Sect. 7.1) the restoration of chiral symmetry at the phase transition manifests itself by m_ρ and m_{a_1} converging to the same value. If (7.164) is a symmetry operation, it follows that

$$Q_5^a = \int \mathrm{d}^3x\ \bar{\psi}(x)\gamma_0\gamma_5\frac{1}{2}\tau^a\psi(x) \tag{7.167}$$

is an additional conserved quantity, similar to the total isospin

$$Q^a = \int \mathrm{d}^3x\ \bar{\psi}(x)\gamma_0\frac{1}{2}\tau^a\psi(x) \quad . \tag{7.168}$$

As (7.164) is a symmetry of the Lagrangian (neglecting quark masses for the time being), $|0\rangle, \hat{Q}_5|0\rangle, \hat{Q}_5^2|0\rangle \ldots$ all have the same energy. On the other hand we just argued that this symmetry is spontaneously broken for the ground state; thus

$$\hat{Q}_5|0\rangle = |v_1\rangle \neq |0\rangle \quad , \quad \hat{Q}_5^2|0\rangle = |v_2\rangle \neq |0\rangle \quad . \tag{7.169}$$

Owing to all the other symmetries of the theory, $|0\rangle, |v_1\rangle, |v_2\rangle, \ldots$ can differ only by the presence of $\bar{\mathrm{q}}\mathrm{q}$ pairs. To lowest order

$$\left\langle 0 \left| \hat{\bar{\psi}} \gamma_\mu \gamma_5 \frac{\tau^a}{2} \psi \right| v_1 \right\rangle \neq 0 \quad . \tag{7.170}$$

We substitute $\phi_\mu^a(x) = \hat{\bar{\psi}}(x) \gamma_\mu \gamma_5 \frac{\tau^a}{2} \psi(x)$. As $\phi_\mu^a(x)$ transforms as an isospin vector we find that

$$\begin{aligned}
&\phi_\mu^a(x) \rightarrow \phi_\mu^a(x) - \mathrm{i}\delta\alpha^A(x)(T^A)_{ab}\phi_\mu^b(x) \quad , \\
&\phi_\mu^a(x) \rightarrow \mathrm{e}^{\mathrm{i}\delta\alpha^b(x) Q_5^b} \phi_\mu^a(x) \mathrm{e}^{-\mathrm{i}\delta\alpha^c(x) Q_5^c} \\
&\qquad \Rightarrow \phi_\mu^a(x) + \mathrm{i}\delta_\alpha^A(x) \left[Q_5^A, \phi_\mu^a(x) \right] \\
&\qquad \Rightarrow \left(T^A \right)_{ab} \phi_\mu^b(x) = - \left[Q_5^A, \phi_\mu^a \right] \quad .
\end{aligned} \tag{7.171}$$

Taking the vacuum excpectation value of (7.157) yields

$$\begin{aligned}
\langle 0 | \left(T^A \right)_{ab} \phi_\mu^b(x) | 0 \rangle &= - \langle 0 | Q_5^A \underbrace{\phi_\mu^a | 0 \rangle}_{=0} + \langle 0 | \phi_\mu^a Q_5^A | 0 \rangle \\
&= \langle 0 | \phi_\mu^a | v_1 \rangle \neq 0 \quad .
\end{aligned} \tag{7.172}$$

Obviously there must be at least one linear combination of the $\phi^b(x)$'s with a nonvanishing vacuum expectation value. For SU(2) the number of generators is three, just like the number of vector components. In fact the pions are a regular representation of the SU(2). Therefore we shall not distinguish bestween A and a indices. Let us define

$$\delta^{ab} \sigma = \mathrm{i} \left[Q_5^a, \phi^b \right] \quad . \tag{7.173}$$

Equation (7.172) then implies that

$$\mathrm{i} \langle 0 | \left[Q_5^a, \phi^b \right] | 0 \rangle = \langle 0 | \sigma | 0 \rangle \neq 0 \quad . \tag{7.174}$$

To describe SSB one usually introduces an effective field σ with a potential that is minimal for a finite value σ, here we have shown that even without specifying an explicit model one must always have an effective σ term. Next we introduce the pion in our scheme. Fourier transforming $\phi_\mu^a(x)$, we get

$$\left\langle 0 \left| \bar{\psi}(p') \gamma_\mu \gamma_5 \frac{\tau^a}{2} \psi(p) \right| v_1 \right\rangle \neq 0 \quad . \tag{7.175}$$

We assume that $\bar{\psi}, \psi$ are incoming or outgoing on-shell particles (either quarks or nucleons). We then multiply this expression by $q_\mu = p'_\mu - p_\mu$. On the other hand we know that $|v_1\rangle$ should be identified with some field. In principle

$$Q_5 |0\rangle = |v_1\rangle$$

could be related to any pseudoscalar, but for $[H_0, Q_5] = 0$ it follows that

$$\hat{H}_0 |v_1\rangle = + Q_5 \hat{H}_0 |0\rangle = 0 \quad . \tag{7.176}$$

such that in the absence of chiral symmetry breaking the particle must be mass-less (it is the corresponding, so-called Goldstone boson). The pion with its extremely small mass (compared with other hadrons) is the only natural candidate. Equation (7.174) thus implies that

$$\begin{aligned}&\left\langle 0\left|\bar{\psi}(p')(\not{p}'-\not{p})\gamma_5\frac{\tau^a}{2}\psi(p)\right|\pi^b\right\rangle\\&=\left\langle 0\left|\bar{\psi}(p')(m\gamma_5+\gamma_5\not{p})\frac{\tau^a}{2}\psi(p)\right|\pi^b\right\rangle\\&=-\mathrm{i}m\left\langle 0\left|\bar{\psi}\mathrm{i}\gamma_5\tau^a\psi\right|\pi^b\right\rangle\sim\delta_{ab}\neq 0 \quad .\end{aligned}\tag{7.177}$$

Transforming this back into coordinate space yields

$$\begin{aligned}&\left\langle 0\left|\partial^\mu\left(\bar{\psi}(x)\gamma_\mu\gamma_5\frac{\tau^a}{2}\psi\right)\right|\pi^b\right\rangle=\left\langle 0\left|\partial^\mu A^a_\mu\right|\pi^b\right\rangle\\&=\delta^{ab}\cdot\text{const.}=\delta^{ab}f_\pi m_\pi^2 \quad ,\end{aligned}\tag{7.178}$$

with the usual definition of the pion decay constant f_π. This relationship is postulated to hold at the operator level to give what is called a PCAC (partially conserved axial vector current)

$$\partial_\mu A^{a\mu}=f_\pi m_\pi^2\pi^a \quad .\tag{7.179}$$

This equation implies that the violation of axial-vector current conservation can be described as being due to the coupling to pions. From this starting point many interesting relations can be deduced. We are especially interested in the following:

$$\begin{aligned}I&=\int \mathrm{d}^4x\ \mathrm{e}^{-\mathrm{i}k\cdot x}\left\langle 0\left|T\left(\partial^\mu A^a_\mu(0)\partial^\nu A^b_\nu(x)\right)\right|0\right\rangle\\&=f_\pi^2m_\pi^4\int \mathrm{d}^4x\ \mathrm{e}^{-\mathrm{i}k\cdot x}\left\langle 0\left|T\left(\pi^a(0)\pi^b(x)\right)\right|0\right\rangle\\&=f_\pi^2m_\pi^4\delta^{ab}\cdot\mathrm{i}D_\pi(k) \quad .\end{aligned}\tag{7.180}$$

We look at this in the limit $k\rightarrow 0$ ($D_\pi(k)=1/(k^2-m_\pi^2)\rightarrow -1/m_\pi^2$) and obtain

$$I=-\mathrm{i}f_\pi^2m_\pi^2\delta^{ab} \quad .\tag{7.181}$$

On the other hand by partial integration we have

$$\begin{aligned}I(k\rightarrow 0)=&\int \mathrm{d}^4x\ \left(-\partial^\nu\mathrm{e}^{-\mathrm{i}k\cdot x}\right)\left\langle 0\left|T\left(\partial^\mu A^a_\mu(0)A^b_\nu(x)\right)\right|0\right\rangle\\&-\int \mathrm{d}^4x\ \mathrm{e}^{-\mathrm{i}k\cdot x}\left\langle 0\left|\left[\partial^\nu\Theta(x_0)A^b_\nu(x)\partial^\mu A^a_\mu(0)\right.\right.\right.\\&\left.\left.\left.+\partial^\nu\Theta(-x_0)\partial^\mu A^a_\mu(0)A^b_\nu(x)\right]\right|0\right\rangle \quad ,\end{aligned}\tag{7.182}$$

$$\begin{aligned}k\rightarrow 0\qquad&=0-\int \mathrm{d}^4x\ \delta(x_0)\left\langle 0\left|\left[A^b_0(x),\partial^\mu A^a_\mu(0)\right]\right|0\right\rangle\\&=-\int \mathrm{d}^4x\ \delta(x_0)\left\langle 0\left|\left[A^b_0(0),\partial^\mu A^b_\mu(-x)\right]\right|0\right\rangle\\&=\int \mathrm{d}^3x\ \left\langle 0\left|\left[A^b_0(0,\boldsymbol{x}),\partial^0A^a_0(0,0)\right]\right|0\right\rangle \quad ,\end{aligned}\tag{7.183}$$

because only terms even in $\boldsymbol{x}$ contribute. As A^b_0 is not explicitly time dependent, this becomes

$$=\mathrm{i}\int \mathrm{d}^3x\ \left\langle 0\left|\left[A^b_0(0,\boldsymbol{x}),\left[A^a_0(0,0),H(0)\right]\right]\right|0\right\rangle \quad .\tag{7.184}$$

This we shall evaluate for

$$\hat{H} = \hat{H}_0 + \hat{H}_{\rm int} = +\hat{H}_0 + \int \mathrm{d}^3y \;\left[m_u \bar{u}(0,\boldsymbol{y})u(0,\boldsymbol{y}) + m_d \bar{d}(0,\boldsymbol{y})d(0,\boldsymbol{y})\right] \quad (7.185)$$

with

$$\left[q^+_\alpha(\boldsymbol{x},t), q_\beta(\boldsymbol{y},t)\right] = \delta_{\alpha\beta}\delta^3(\boldsymbol{x}-\boldsymbol{y}) \quad . \qquad (7.186)$$

By straightforward calculation (see Exercise 7.9) we get

$$\langle 0 \,|\, [A^b_0(0,\boldsymbol{x}), [A^b_0(0,0), H_{\rm int}(0)]] \,|\, 0\rangle = \left\langle 0 \left| \frac{m_{\rm u}+m_{\rm d}}{2}(\bar{u}u + \bar{d}d)\right| 0\right\rangle \qquad (7.187)$$

and thus

$$(m_{\rm u} + m_{\rm d})\langle 0|\bar{u}u + \bar{d}d|0\rangle = -f_\pi^2 m_\pi^2 \approx -2 \times (100\ \mathrm{MeV})^4 \quad . \qquad (7.188)$$

We have seen how the non perturbative effects leading to the breaking of isosymmetry and thus to hadronic mass correction lead to the introduction of a quark condensate (7.174). QCD sumrules invoke similar condensates to describe the non-perturbative effects of QCD. In general here one has to allow for the appearance of all possible condensates, of all the operators O_n appearing in (7.149). Let us now return to this equation.

EXERCISE

7.9 PCAC and the Quark Condensate

Problem. Calculate

$$\int\int \langle 0 \,|\, [A^b_0(0,\boldsymbol{x}), [A^a_0(0,\boldsymbol{0}), m_{\rm u}\bar{u}(y)u(y) + m_{\rm d}\bar{d}(y)d(y)]] \,|\, 0\rangle \,\mathrm{d}^3x\,\mathrm{d}^3y \qquad (1)$$

with

$$A^b_0(x) = \bar{q}(x)\gamma_0\gamma_5\frac{\tau^b}{2}q(x) \quad , \quad q(x) = \begin{pmatrix} u(x) \\ d(x)\end{pmatrix} \quad . \qquad (2)$$

Solution. First we write

$$m_{\rm u}\bar{u}(y)u(y) + m_{\rm d}\bar{d}(y)d(y) = \frac{m_{\rm u}+m_{\rm d}}{2}\bar{q}(y)q(y) + \frac{m_{\rm u}-m_{\rm d}}{2}\bar{q}(y)\tau^3 q(y) \qquad (3)$$

and then we write with simultaneous Lorentz and isospin indices $\alpha, \beta, \gamma, \delta$

$$\begin{aligned}
&\left[\bar{q}(0)\frac{\tau^a}{2}\gamma_0\gamma_5 q(0), \bar{q}(y)q(y)\right] \\
&\quad = \left[q^+_\alpha(0)q_\beta(0), q^+_\gamma(y)q_\delta(y)\right]\left(\frac{\tau^a}{2}\gamma_5\right)_{\alpha\beta}(\gamma_0)_{\gamma\delta} \\
&\quad = \left(-q^+_\alpha q^+_\gamma\left\{q_\beta, q_\delta\right\} + q^+_\alpha\left\{q_\beta q^+_\gamma\right\}q_\delta - q^+_\gamma\left\{q^+_\alpha, q_\delta\right\}q_\beta\right. \\
&\qquad \left. + \left\{q^+_\alpha, q^+_\gamma\right\}q_\delta q_\beta\right)\left(\frac{\tau^a}{2}\gamma_5\right)_{\alpha\beta}(\gamma_0)_{\gamma\delta} \qquad (4)
\end{aligned}$$

Exercise 7.9.

as

$$\begin{aligned}(\ldots) &= -q_\alpha^+ q_\gamma^+ q_\beta q_\delta - q_\alpha^+ q_\gamma^+ q_\delta q_\beta + q_\alpha^+ q_\beta q_\gamma^+ q_\delta + q_\alpha^+ q_\gamma^+ q_\beta q_\delta \\ &\quad - q_\gamma^+ q_\alpha^+ q_\delta q_\beta - q_\gamma^+ q_\delta q_\alpha^+ q_\beta + q_\alpha^+ q_\gamma^+ q_\delta q_\beta + q_\gamma^+ q_\alpha^+ q_\delta q_\beta \\ &= q_\alpha^+ q_\gamma^+ q_\delta q_\beta(+1-1-1+1+1-1) + \left[q_\alpha^+ q_\beta, q_\gamma^+ q_\delta\right] \quad .\end{aligned} \tag{5}$$

Now we use

$$\begin{aligned}&\{q_\beta(0,\mathbf{0}), q_\delta(0,\boldsymbol{y})\} = 0 = \left\{q_\alpha^+(0,\mathbf{0}), q_\gamma^+(0,\boldsymbol{y})\right\} \quad , \\ &\left\{q_\beta(0,\mathbf{0}), q_\gamma^+(0,\boldsymbol{y})\right\} = \delta_{\beta\gamma}\delta^3(\boldsymbol{y}) \quad , \\ &\left\{q_\alpha^+(0,\mathbf{0}), q_\delta(0,\boldsymbol{y})\right\} = \delta_{\alpha\delta}\delta^3(\boldsymbol{y})\end{aligned} \tag{6}$$

to get

$$\begin{aligned}[\ldots,\ldots] &= \delta^3(\boldsymbol{y})\left[q_\alpha^+(0)q_\delta(0)\left(\frac{\tau^a}{2}\gamma_5\gamma_0\right)_{\alpha\delta} - q_\gamma^+(0)q_\beta(0)\left(\gamma_0\frac{\tau^a}{2}\gamma_5\right)_{\gamma\beta}\right] \\ &= \delta^3(\boldsymbol{y})q_\alpha^+(0)q_\delta(0)\left(\tau^a\gamma_5\gamma_0\right)_{\alpha\delta} \quad .\end{aligned} \tag{7}$$

Similarly we obtain

$$\begin{aligned}&\left[\bar{q}(0)\frac{\tau^a}{2}\gamma_0\gamma_5 q(0), \bar{q}(0,\boldsymbol{y})\tau^3 q(0,\boldsymbol{y})\right] \\ &\quad = \delta^3(\boldsymbol{y})\left[q_\alpha^+(0)q_\delta(0)\left(\frac{\tau^a}{2}\gamma_5\gamma_0\tau^3\right)_{\alpha\delta} - q_\gamma^+(0)q_\beta(0)\left(\gamma_0\tau^3\frac{\tau^a}{2}\gamma_5\right)_{\gamma\beta}\right] \\ &\quad = \delta^3(\boldsymbol{y})q_\alpha^+(0)q_\delta(0)\left(\gamma_5\gamma_0\underbrace{\frac{1}{2}\left\{\tau^a,\tau^3\right\}}_{\delta_{a3}}\right)_{\alpha\delta}\end{aligned} \tag{8}$$

and thus

$$\begin{aligned}&\int \mathrm{d}^3y\,\left[A_0^a(0,\mathbf{0}), \frac{m_\mathrm{u}+m_\mathrm{d}}{2}\bar{q}(0,\boldsymbol{y})\,\mathbb{1}\,q(0,\boldsymbol{y}) + \frac{m_\mathrm{u}-m_\mathrm{d}}{2}\bar{q}(0,\boldsymbol{y})\tau^3 q(0,\boldsymbol{y})\right] \\ &\quad = \frac{m_\mathrm{u}+m_\mathrm{d}}{2}q_\alpha^+(0)q_\delta(0)\left(\gamma_5\gamma_0\tau^a\right)_{\alpha\delta} \\ &\qquad + \frac{m_\mathrm{u}-m_\mathrm{d}}{2}q_\alpha^+(0)q_\delta(0)\left(\gamma_5\gamma_0\right)_{\alpha\delta}\,\delta_{a3} = J(0) \quad .\end{aligned} \tag{9}$$

The next step just repeats the preceding one.

$$\begin{aligned}&\int \mathrm{d}^3x\,\left[A_0^b(0,\boldsymbol{x}), J(0)\right] \\ &\quad = \int \mathrm{d}^3r\;\delta^3(\boldsymbol{x})\left[\frac{m_\mathrm{u}+m_\mathrm{d}}{2}q_\alpha^\dagger(0)q_\delta(0)\left(\frac{\tau^b}{2}\gamma_5\gamma_5\gamma_0\tau^a - \gamma_5\gamma_0\tau^a\frac{\tau^b}{2}\gamma_5\right)_{\alpha\delta}\right. \\ &\qquad \left. + \frac{m_\mathrm{u}-m_\mathrm{d}}{2}q_\alpha^\dagger(0)q_\delta(0)\left(\frac{\tau^b}{2}\gamma_5\gamma_5\gamma_0\delta_{a3} - \gamma_5\gamma_0\delta_{a3}\frac{\tau^b}{2}\gamma_5\right)_{\alpha\delta}\right] \\ &\quad = q_\alpha^\dagger(0)q_\delta(0)\left[\frac{m_\mathrm{u}+m_\mathrm{d}}{2}\left(\gamma_0\frac{1}{2}\left\{\tau^b,\tau^a\right\}\right)_{\alpha\delta} + \frac{m_\mathrm{u}-m_\mathrm{d}}{2}\left(\gamma_0\tau^b\delta_{a3}\right)_{\alpha\delta}\right] \\ &\quad = \frac{m_\mathrm{u}+m_\mathrm{d}}{2}\bar{q}(0)\,q(0)\,\delta_{ab} + \frac{m_\mathrm{u}-m_\mathrm{d}}{2}\bar{q}(0)\,\tau^b\,q(0)\,\delta_{a3} \quad .\end{aligned} \tag{10}$$

Exercise 7.9.

Finally we take the vacuum expectation value of this expression. As the vacuum cannot carry any quantum number such as isospin, the second term does not contribute:

$$\begin{aligned}\int \mathrm{d}^3x\ \mathrm{d}^3y\ \Big\langle 0\Big|\Big[A_0^b(0,\boldsymbol{x}),\ \big[A_0^a(0,\boldsymbol{O}), m_\mathrm{u}\bar{u}(0,\boldsymbol{y})u(0,\boldsymbol{y}) + m_\mathrm{d}\bar{d}(0,\boldsymbol{y})d(0,\boldsymbol{y})\big]\Big]\Big|0\Big\rangle \\ = \frac{m_\mathrm{u}+m_\mathrm{d}}{2}\langle 0|\bar{u}(0)u(0)+\bar{d}(0)d(0)|0\rangle\,\delta_{ab} \quad . \qquad (11)\end{aligned}$$

Let us now return to QCD sum-rule calculation. We illustrate this method by presenting all the detailed steps for one specific example: the calculation of the ρ mass. We wish to evaluate the left-hand side of (7.149) at the quark level, using the condensates (7.150). The result will be compared with (7.154) and from this comparison we shall extract an estimate for the ρ mass.

The simplest contribution to (7.149) is just the perturbative graph. The flavor structure of ρ is simply

$$\begin{aligned}\rho^+ &= \frac{1}{\sqrt{2}}(u_1\bar{d}_2 - \bar{d}_1u_2) \quad , \\ \rho^- &= -\frac{1}{\sqrt{2}}(d_1\bar{u}_2 - \bar{u}_1d_2) \quad , \\ \rho^0 &= \frac{1}{2}\Big[\big(d_1\bar{d}_2 - u_1\bar{u}_2\big) - \big(\bar{d}_1d_2 - \bar{u}_1u_2\big)\Big] \quad ,\end{aligned} \qquad (7.189)$$

implying that

$$\begin{aligned}\big\langle\big|j_\mu(x)j_\nu(0)\big|\big\rangle_{\rho^+} &= -\frac{1}{2}\langle u_1(x)\gamma_\mu\bar{d}_2(x)\big(d_1\gamma_\nu\bar{u}_2 - \bar{u}_1\gamma_\nu d_2\big)(0)\rangle\cdot 2 \\ &= \big\langle u_1(x)\gamma_\mu\bar{d}_2(x)\bar{u}_1(0)\gamma_\nu d_2(0)\big\rangle \\ &= \big\langle u(x)\gamma_\mu\bar{d}(x)\bar{u}(0)\gamma_\nu d(0)\big\rangle \quad , \\ \big\langle\big|j_\mu(x)j_\nu(0)\big|\big\rangle_{\rho^0} &= \big\langle\bar{q}(x)\gamma_\mu q(x)\bar{q}(0)\gamma_\nu q(0)\big\rangle \quad .\end{aligned} \qquad (7.190)$$

Thus we do not have to distinguish between up and down quarks, at least not in the usual limit $m_\mathrm{u}, m_\mathrm{d} \to 0$. In the following we discuss the ρ^0 case. For the perturbative part (without any additive interactions or vacuum insertions) we get

$$\Pi_{\mu\nu}(q) = \mathrm{i}\int \mathrm{d}^4x\ \mathrm{e}^{\mathrm{i}qx}\langle 0|T\{\bar{q}(x)\gamma_\mu q(x)\bar{q}(0)\gamma_\nu q(0)\}|0\rangle \quad . \qquad (7.191)$$

Wick's theorem gives simply

$$\begin{aligned}&\langle 0|T\{\bar{q}(x)\gamma_\mu q(x)\bar{q}(0)\gamma_\nu q(0)\}|0\rangle \\ &\quad = (\gamma_\mu)_{i_1i_2}(\gamma_\nu)_{i_3i_4}\langle 0|T\{\bar{q}_{i_1}(x)q_{i_2}(x)\bar{q}_{i_3}(0)q_{i_4}(0)\}|0\rangle \\ &\quad = (\gamma_\mu)_{i_1i_2}(\gamma_\nu)_{i_3i_4}\langle 0|T\{\bar{q}_{i_1}(x)q_{i_4}(0)\}|0\rangle\langle 0|T\{q_{i_2}(x)\bar{q}_{i_3}(0)\}|0\rangle \\ &\quad = (\gamma_\mu)_{i_1i_2}(\gamma_\nu)_{i_3i_4}\ \mathrm{i}S_{\mathrm{F},i_4,i_1}(-x)\ \mathrm{i}S_{\mathrm{F},i_2,i_3}(x) \\ &\quad = \mathrm{tr}\left(\gamma_\mu\frac{\not{x}}{2\pi^2(x^2)^2}\,\gamma_\nu\frac{-\not{x}}{2\pi^2(x^2)^2}\right) \\ &\quad = \frac{1}{\pi^4}\frac{x^2g_{\mu\nu} - 2x_\mu x_\nu}{(x^2)^4} \quad .\end{aligned} \qquad (7.192)$$

Using Equation (1) from Exercise 4.5,

$$\int \mathrm{d}^d x \, \frac{\mathrm{e}^{\mathrm{i}p\cdot x}}{(-x^2)^\nu} = -\mathrm{i}\pi^2 \frac{\Gamma(2-\nu+\varepsilon)}{\Gamma(\nu)} \left(\frac{-4\pi^2\mu^2}{p^2}\right)^\varepsilon \left(\frac{-p^2}{4}\right)^{\nu-2} \quad , \tag{7.193}$$

it is easy to calculate the Fourier transformation of (7.192):

$$\begin{aligned}
&\mathrm{i} \int \mathrm{d}^4 x \; \mathrm{e}^{\mathrm{i}p\cdot x} \langle 0 | T\{\bar{q}(x)\gamma_\mu q(x) \bar{q}(0)\gamma_\nu q(0)\} | 0 \rangle \\
&\quad = \mathrm{i} \int \mathrm{d}^4 x_\mathrm{E} \; \mathrm{e}^{-\mathrm{i}p_\mathrm{E}\cdot x_\mathrm{E}} \frac{-1}{\pi^4} \frac{x_\mathrm{E}^2 g_{\mu\nu} + 2x_{\mathrm{E}\mu}x_{\mathrm{E}\nu}}{(x_\mathrm{E}^2)^4} \\
&\quad = \frac{-\mathrm{i}}{\pi^4} g_{\mu\nu} \left(+\mathrm{i}\pi^2\right) \frac{\Gamma(\varepsilon-1)}{\Gamma(3)} \left(\frac{-4\pi^2\mu^2}{p_\mathrm{E}^2}\right)^\varepsilon \frac{p_\mathrm{E}^2}{4} \\
&\qquad - \frac{\mathrm{i}}{\pi^4} 2 \left(-\frac{\partial}{\partial p_{\mathrm{E}\mu}} \frac{\partial}{\partial p_{\mathrm{E}\nu}}\right) \int \mathrm{d}^d x_\mathrm{E} \; \mathrm{e}^{-\mathrm{i}p_\mathrm{E}\cdot x_\mathrm{E}} \; \frac{1}{(x_\mathrm{E}^2)^4} \\
&\quad = \frac{1}{8\pi^2} \Gamma(\varepsilon-1) \, p_\mathrm{E}^2 \left(\frac{-4\pi^2\mu^2}{p_\mathrm{E}^2}\right)^\varepsilon g_{\mu\nu} \\
&\qquad + \frac{2\mathrm{i}}{\pi^4} (-\mathrm{i}\pi^2) \frac{\Gamma(\varepsilon-2)}{\Gamma(4)} \left(-4\pi^2\mu^2\right)^\varepsilon \; \frac{1}{16} \; \frac{\partial}{\partial p_{\mathrm{E}\mu}} \frac{\partial}{\partial p_{\mathrm{E}\nu}} \left(p_\mathrm{E}^2\right)^{2-\varepsilon} \\
&\quad = \frac{1}{8\pi^2} \Gamma(\varepsilon-1) \, p_\mathrm{E}^2 \, g_{\mu\nu} \left(\frac{-4\pi^2\mu^2}{p_\mathrm{E}^2}\right)^\varepsilon \\
&\qquad + \frac{1}{8\pi^2} \frac{\Gamma(\varepsilon-2)}{6} \left(\frac{-4\pi^2\mu^2}{p_\mathrm{E}^2}\right)^\varepsilon \left[(2-\varepsilon)(2-\varepsilon)4 \, p_{\mathrm{E}\mu} \, p_{\mathrm{E}\nu} + (2-\varepsilon) \, 2 \, g_{\mu\nu} \, p_\mathrm{E}^2\right] \\
&\quad = \frac{1}{8\pi^2} \, \Gamma(\varepsilon-1) \left(\frac{-4\pi^2\mu^2}{p_\mathrm{E}^2}\right)^\varepsilon \left[\frac{2}{3} p_\mathrm{E}^2 \, g_{\mu\nu} + (1-\varepsilon) \, \frac{2}{3} \, p_{\mathrm{E}\mu} \, p_{\mathrm{E}\nu}\right] \\
&\quad = \frac{1}{8\pi^2} \left(\frac{1}{\varepsilon} + \Gamma'(1)\right) \left(\frac{4\pi^2\mu^2}{p^2}\right)^\varepsilon \frac{2}{3} \left(p^2 g_{\mu\nu} - p_\mu p_\nu\right) + \dots \\
&\quad = \frac{1}{8\pi^2} \ln\left(\frac{4\pi^2\mu^2}{p^2}\right) \frac{2}{3} \left(p^2 g_{\mu\nu} - p_\mu p_\nu\right) \\
&\qquad + \text{terms without an imaginary part} \quad .
\end{aligned} \tag{7.194}$$

Every quark color gives the same contribution such that we get finally

$$\Pi_{\mu\nu} = \frac{1}{4\pi^2} \left(p^2 g_{\mu\nu} - p_\mu p_\nu\right) \ln\left(\frac{4\pi^2\mu^2}{p^2}\right) + \dots \tag{7.195}$$

Thus we have calculated the perturbative contribution to lowest order of α_s. Clearly these perturbative contributions alone cannot be sufficient to completely obtain non perturbative quantities such as hadron masses. The fundamental assumption of the QCD sum-rule approach is that these non perturbative effects can to a good approximation be described by vacuum properties. As the vacuum is supposed to be homogeneous those fields coupling to it must have zero momentum.

In the presence of background particles the propagator of, for example a Dirac field is

$$S_\mathrm{F}(k) = (\not{k} + m) \left[\frac{1}{k^2 - m^2 + \mathrm{i}\varepsilon} + 2\pi\mathrm{i} \; \delta(k^2 - m^2) \, \Theta(k_0) \, n(\boldsymbol{k})\right] \quad . \tag{7.196}$$

Assuming that $n(\boldsymbol{k}) = c\, m\, \delta^3(\boldsymbol{k})$, in coordinate space for $m \longrightarrow 0$ this becomes

$$\begin{aligned} S_{\mathrm{F}}(x) &= \frac{1}{(2\pi)^4} \int \mathrm{d}^4k\; \mathrm{e}^{\mathrm{i}k\cdot x} S(k) \\ &= \frac{-\mathrm{i}}{2\pi^2} \frac{\not{x}}{(x^2)^2} + \frac{1}{(2\pi)^3} \int \mathrm{d}^4k\; \frac{c\,m}{2m}\, \delta^4(k)\, \mathrm{e}^{\mathrm{i}k\cdot x} \\ &= \frac{-\mathrm{i}}{2\pi^2} \frac{\not{x}}{(x^2)^2} + \frac{c}{2(2\pi)^3} \quad . \end{aligned} \tag{7.197}$$

Thus it would seem natural to describe the complex vacuum structure by adding constant real numbers to the field propagators:

$$\left[S_{\mathrm{F}}(x)\right]_{ij}^{ab} = \frac{-\mathrm{i}}{2\pi^2} \frac{(\not{x})_{ij}}{(x^2)^2}\, \delta_{ab} - \frac{1}{12} \langle \bar{q} q \rangle\, \delta_{ab}\, \delta_{ij} \quad . \tag{7.198}$$

This form is also suggested by Wick's theorem if the expectation values of normal ordered products are interpreted as condensates.

$$T\left[\psi_i^a(x) \bar{\psi}_j^b(x')\right] = \,:\psi_i^a(x) \bar{\psi}_j^b(x'): + \left\langle 0 \middle| T\left[\psi_i^a(x) \bar{\psi}_j^b(x')\right] \middle| 0 \right\rangle \quad , \tag{7.199}$$

$$\begin{aligned} &\left\langle \mathrm{vac} \middle| T\left\{\psi_i^a(x) \bar{\psi}_j^b(x')\right\} \middle| \mathrm{vac} \right\rangle \\ &= \left\langle \mathrm{vac} \right| :\psi_i^a(x) \bar{\psi}_j^b(x'): \left| \mathrm{vac} \right\rangle + \left\langle 0 \middle| T\left\{\psi_i^a(x) \bar{\psi}_j^b(x')\right\} \middle| 0 \right\rangle \\ &= -\frac{1}{12}\, \delta_{ab}\, \delta_{ij} \langle \bar{q} q \rangle + \mathrm{i} S_{\mathrm{F}ij}(x - x')\, \delta_{ab} \quad , \end{aligned} \tag{7.200}$$

Using (7.186) we shall next calculate the quark-condensate contribution to the ρ sum rule. It is represented by Fig. 7.22, while Fig. 7.23 shows the lowest-order perturbative contribution.

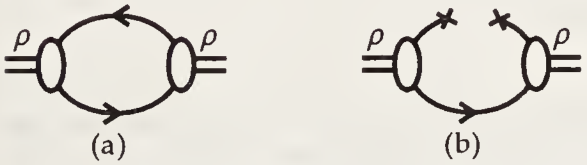


Fig. 7.22a,b. A graphical representation of (**a**) the lowest-order, purely perturbative contribution, and (**b**) the lowest quark condensate contribution

With (7.200) we get simply

$$\begin{aligned} \Pi_{\mu\nu} &= \sum_{a,b} \mathrm{i} \int \mathrm{d}^4x\; \mathrm{e}^{\mathrm{i}qx}\, (\gamma_\mu)_{i_1 i_2} (\gamma_\nu)_{i_3 i_4}\, \mathrm{i} S_{\mathrm{F}i_4 i_1}(-x) \left(-\frac{1}{12} \langle \bar{q} q \rangle\right)\, \delta_{ab}\, \delta_{i_2 i_3} \\ &= \frac{-3\mathrm{i}}{12} \langle \bar{q} q \rangle \int \mathrm{d}^4x\; \mathrm{e}^{\mathrm{i}qx}\, \mathrm{tr}\left(\gamma_\mu \gamma_\nu S_{\mathrm{F}}(-x)\right) \quad . \end{aligned} \tag{7.201}$$

For massless quarks the trace vanishes and we get strictly zero. Therefore we expand S_{F} in m and also keep the term linear in m:

$$
\begin{aligned}
-\mathrm{i}S_\mathrm{F}(-x) &= \frac{1}{(2\pi)^4}\int \mathrm{d}^4k\ \mathrm{e}^{\mathrm{i}kx}\,\frac{\not k + m}{k^2 - m^2 + \mathrm{i}\varepsilon}\\
&= \frac{-\mathrm{i}}{2\pi^2}\,\frac{\not x}{(x^2)^2} + \frac{m}{4\pi^2}\,\frac{1}{x^2} + \dots
\end{aligned}
\tag{7.202}
$$

We thus get

$$
\begin{aligned}
\Pi_{\mu\nu} &= -\frac{\mathrm{i}}{4}\,\langle\bar{q}q\rangle\, m \int \mathrm{d}^4x\ \mathrm{e}^{\mathrm{i}qx}\,\frac{1}{4\pi^2}\,4g_{\mu\nu}\,\frac{1}{x^2}\\
&= \frac{\pi^2}{4\pi^2}\frac{\Gamma(1)}{\Gamma(1)}\left(\frac{q^2}{4}\right)^{-1}\left(\frac{-4\pi^2\mu^2}{q^2}\right)^{\varepsilon} m\,\langle\bar{q}q\rangle\, g_{\mu\nu}\\
&= \frac{1}{q^2}\, m\,\langle\bar{q}q\rangle\, g_{\mu\nu}\\
&= \frac{1}{q^2 + \mathrm{i}\eta}\, m\,\langle\bar{q}q\rangle\, g_{\mu\nu} \quad .
\end{aligned}
\tag{7.203}
$$

In the last step we have reintroduced the usual $\mathrm{i}\eta$ prescription, which we did not write out explicitly during the calculation. As two graphs contribute, we have to multiply this by a factor 2. Also we know that $\Pi_{\mu\nu}$ is transverse, and we can make this explicit by adding the $q_\mu q_\nu$ term:

$$
\Pi_{\mu\nu} = \left(q^2 g_{\mu\nu} - q_\mu q_\nu\right) 2\,\frac{m\langle\bar{q}q\rangle}{q^2 + \mathrm{i}\eta} \quad .
\tag{7.204}
$$

Next we calculate the lowest-order gluon condensate contributions, corresponding to the graphs in Fig. 7.23.

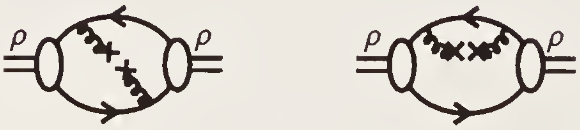


Fig. 7.23. Lowest-order couplings to the gluon condensate

The fields coupling into condensates all have zero momentum, since the vacuum condensates are time and space independent. Thus the gluon field can be easily expressed in terms of the constant gluon field strengths $G^a_{\mu\nu}$,

$$
A^a_\mu(z) = \frac{1}{2} z_\rho G^a_{\rho\mu}(0)
\tag{7.205}
$$

$$
\begin{aligned}
\Rightarrow\quad & \partial_\nu A^a_\mu(z) - \partial_\mu A^a_\nu(z) + g f_{abc} A^b_\mu(z) A^c_\nu(z)\\
&= \frac{1}{2} G^a_{\nu\mu}(0) - \frac{1}{2} G^a_{\mu\nu}(0) + g\frac{1}{4} z_\alpha z_\beta f_{abc}\, G^b_{\alpha\mu}(0)\, G^c_{\beta\nu}(0)\\
&= G^a_{\nu\mu}(0) \quad ,
\end{aligned}
\tag{7.206}
$$

where we have used the fact that $G^a_{\nu\mu}(0)$ is the same for all colors, since the vacuum does not distinguish any color state $G^b_{\alpha\mu}(0) = G^c_{\beta\nu}(0)$.

The quark propagator in coordinate space in the gluon background fields of the vacuum condensates is then

$$
\begin{aligned}
-\mathrm{i}S(x,y;A) = {} & \frac{1}{2\pi^2}\,\frac{(x-y)\cdot\gamma}{(x-y)^4} - \frac{gG_{\rho\mu}(0)}{4\pi^4}\,\frac{1}{2}\int \mathrm{d}^4z\,\frac{\not{x}-\not{z}}{(x-z)^4}\,z_\rho\gamma_\mu\frac{\not{z}-\not{y}}{(z-y)^4} \\
& + \frac{g^2G_{\rho\mu}(0)G_{\rho'\mu'}(0)}{8\pi^6}\,\frac{1}{4}\int \mathrm{d}^4z_1\,\mathrm{d}^4z_2\,\frac{\not{x}-\not{z}_1}{(x-z_1)^4}\,(z_1)_\rho\gamma_\mu\frac{\not{z}_1-\not{z}_2}{(z_1-z_2)^4} \\
& \times (z_2)_{\rho'}\gamma_{\mu'}\frac{\not{z}_2-\not{y}}{(z_2-y)^4} \\
& + \dots \qquad (7.207)
\end{aligned}
$$

As we calculate in Euclidean space we do not distinguish between covariant and contravariant indices.

We have to perform these integrals using dimensional regularization. In Exercise 7.10 we do this for the first integral on the right-hand side. The double integral and all higher ones vanish. The result is then

$$
\begin{aligned}
S(x,y;A) = \mathrm{i}\Bigg[& \frac{1}{2\pi^2}\,\frac{\not{r}}{(r^2)^2} + \frac{\mathrm{i}g}{32\pi^2r^2}\left(\not{r}\gamma_\mu\gamma_\rho - \gamma_\rho\gamma_\mu\not{r}\right)G_{\rho\mu}(0) \\
& + \frac{\mathrm{i}g}{4\pi^2r^4}\not{r}y_\rho x_\mu G_{\rho\mu}(0)\Bigg] \qquad (7.208)
\end{aligned}
$$

with

$$
r_\mu = x_\mu - y_\mu \quad . \qquad (7.209)
$$

EXERCISE

7.10 Calculation of QCD-Sumrule Graphs with Dimensional Regularization

Problem. Calculate the first integral in (7.207) using the techniques of dimensional regularization introduced in Sect. 4.3.

Solution. We first introduce the usual Feynman parameters:

$$
\begin{aligned}
I &= \int \mathrm{d}^4z\,\frac{(\not{x}-\not{z})\gamma_\mu(\not{z}-\not{y})z_\rho}{(x-z)^4(z-y)^4} \\
&= \Gamma(4)\int du\; u(1-u)\int \mathrm{d}^4z\,\frac{(\not{x}-\not{z})\gamma_\mu(\not{z}-\not{y})z_\rho}{\left[u(x-z)^2+(1-u)(z-y)^2\right]^4} \quad . \qquad (1)
\end{aligned}
$$

Then, again as usual, we bring the denominator into the quadratic normal form:

$$
\begin{aligned}
& u(x-z)^2+(1-u)(z-y)^2 \\
&\quad = z^2 - 2z\left[ux+(1-u)y\right] + ux^2 + (1-u)y^2 \\
&\quad = \left\{z - \left[ux+(1-u)y\right]\right\}^2 + u(1-u)x^2 + u(1-u)y^2 - 2u(1-u)x\cdot y \\
&\quad = \left\{z - \left[ux+(1-u)y\right]\right\}^2 + u(1-u)(x-y)^2 \quad . \qquad (2)
\end{aligned}
$$

Exercise 7.10.

Substituting $z \to z + ux + (1-u)y$ gives

$$I = \Gamma(4) \int du\, u(1-u) \int \mathrm{d}^4 z \, \frac{\left[(1-u)(\not{x} - \not{y}) - \not{z}\right]\gamma_\mu}{\left[z^2 + u(1-u)(x-y)^2\right]^4} \times \left[\not{z} + u(\not{x} - \not{y})\right]\left[z_\rho + ux_\rho + (1-u)y_\rho\right] \quad . \tag{3}$$

Only the terms even in z contribute to the integral, and $z_\rho z_\alpha$ contribute $z^2 g_{\rho\alpha}/4$:

$$\begin{aligned} I = {} & 6 \int_0^1 \mathrm{d}u\, u(1-u) \int \mathrm{d}^4 z \, \frac{(1-u)u(\not{x} - \not{y})\gamma_\mu(\not{x} - \not{y})}{\left[z^2 + u(1-u)(x-y)^2\right]^4} \\ & \times \left[ux_\rho + (1-u)y_\rho\right] + 6 \int_0^1 \mathrm{d}u\, u(1-u) \\ & \times \int \frac{\mathrm{d}^4 z}{\left[z^2 + u(1-u)(x-y)^2\right]^4} \left\{ \frac{1}{2} z^2 \gamma_\mu \left[ux_\rho + (1-u)y_\rho\right] \right. \\ & \left. + \frac{1}{4} z^2 (1-u)(\not{x} - \not{y})\gamma_\mu\gamma_\rho - \frac{1}{4} z^2 \gamma_\rho\gamma_\mu u(\not{x} - \not{y}) \right\} \quad . \end{aligned} \tag{4}$$

The nominator of the second integral is abbreviated by writing $z^2 \cdot f(u)$. We now integrate using (4.88) and (4.90):

$$\begin{aligned} I = {} & 6 \int_0^1 \mathrm{d}u\, u(1-u) \left\{ \left[u(1-u)(x-y)^2\right]^{-2} \right. \\ & \times u(1-u)(\not{x} - \not{y})\gamma_\mu(\not{x} - \not{y})\left[ux_\rho + (1-u)y_\rho\right] \frac{\mathrm{i}\pi^2 \Gamma(2)}{\Gamma(4)} \\ & \left. + f(u) \frac{\mathrm{i}\pi^2 \Gamma(1)}{\Gamma(4)} 2\left[u(1-u)(x-y)^2\right]^{-1} \right\} \\ = {} & \mathrm{i}\pi^2 \frac{1}{(x-y)^4} \left[\int_0^1 \mathrm{d}u\, (\not{x} - \not{y})\gamma_\mu(\not{x} - \not{y})\left[ux_\rho + (1-u)y_\rho\right] \right. \\ & + 2 \int_0^1 \mathrm{d}u \left\{ \frac{1}{2}\gamma_\mu \left[ux_\rho + (1-u)y_\rho\right] + \frac{1}{4}(1-u)(\not{x} - \not{y})\gamma_\mu\gamma_\rho \right. \\ & \left. \left. - \frac{1}{4}\gamma_\rho\gamma_\mu u(\not{x} - \not{y}) \right\}(x-y)^2 \right] \quad . \end{aligned} \tag{5}$$

Both the integral $\int u \, \mathrm{d}u$ and $\int (1-u)\, \mathrm{d}u$ give $1/2$

$$\begin{aligned} I = {} & \frac{\mathrm{i}\pi^2}{2} \frac{1}{(x-y)^4} \left\{ (\not{x} - \not{y})\gamma_\mu(\not{x} - \not{y})(x+y)_\rho + \gamma_\mu(x_\rho + y_\rho)(x-y)^2 \right. \\ & \left. + \frac{1}{2}\left[(\not{x} - \not{y})\gamma_\mu\gamma_\rho - \gamma_\rho\gamma_\mu(\not{x} - \not{y})\right](x-y)^2 \right\} \end{aligned}$$

Exercise 7.10.

$$
\begin{aligned}
&= \frac{i\pi^2}{4}\frac{1}{(x-y)^2}\left[(\not{x}-\not{y})\gamma_\mu\gamma_\rho - \gamma_\rho\gamma_\mu(\not{x}-\not{y})\right] \\
&\quad + \frac{i\pi^2}{2}\frac{1}{(x-y)^4}\left[2(x-y)_\mu(\not{x}-\not{y})(x+y)_\rho\right] \\
&= \frac{i\pi^2}{4}\frac{1}{(x-y)^2}\left[(\not{x}-\not{y})\gamma_\mu\gamma_\rho - \gamma_\rho\gamma_\mu(\not{x}-\not{y})\right] \\
&\quad + i\pi^2\frac{1}{(x-y)^4}\left[x_\mu x_\rho + x_\mu y_\rho - y_\mu x_\rho - y_\mu y_\rho\right](\not{x}-\not{y}) \quad . \qquad (6)
\end{aligned}
$$

This still has to be multiplied by $G_{\rho\mu}(0)$, which is antisymmetric in ρ and μ, leading to

$$
\begin{aligned}
\Delta S(x,y;A) &= -\frac{ig}{32\pi^2}\frac{1}{(x-y)^2}\left[(\not{x}-\not{y})\gamma_\mu\gamma_\rho - \gamma_\rho\gamma_\mu(\not{x}-\not{y})\right]G_{\rho\mu}(0) \\
&\quad - \frac{ig}{8\pi^2}\frac{1}{(x-y)^2}2x_\mu y_\rho(\not{x}-\not{y})G_{\rho\mu}(0) \quad . \qquad (7)
\end{aligned}
$$

Only the first graphs in Fig. 7.23 have to be calculated:

$$
\begin{aligned}
&\langle 0|T\{\bar{q}(x)\gamma_\mu q(x)\bar{q}(0)\gamma_\nu q(0)\}|0\rangle \\
&\Rightarrow (\gamma_\mu)_{i_1 i_2}(\gamma_\nu)_{i_3 i_4}\mathrm{i}\left(-\frac{\mathrm{i}}{32\pi^2}\right)\left[\not{x}\gamma_{\mu''}\gamma_{\rho''} - \gamma_{\rho''}\gamma_{\mu''}\not{x}\right]_{i_4 i_1}\frac{G_{\rho''\mu''}(0)}{(x-y)^2} \\
&\quad \times \mathrm{i}\left(+\frac{\mathrm{i}}{32\pi^2}\right)\left[\not{x}\gamma_{\mu'}\gamma_{\rho'} - \gamma_{\rho'}\gamma_{\mu'}\not{x}\right]_{i_2 i_3}\frac{G_{\rho'\mu'}(0)}{(x-y)^2} \\
&= -G_{\rho''\mu''}(0)G_{\rho'\mu'}(0)\frac{g^2}{32\cdot 32\pi^4}\frac{1}{(x-y)^4} \\
&\quad \times \mathrm{tr}\{[\not{x}\gamma_{\mu''}\gamma_{\rho''} - \gamma_{\rho''}\gamma_{\mu''}\not{x}]\gamma_\mu[\not{x}\gamma_{\mu'}\gamma_{\rho'} - \gamma_{\rho'}\gamma_{\mu'}\not{x}]\gamma_\nu\} =: T_{\mu\nu} \quad . \qquad (7.210)
\end{aligned}
$$

Now we have to insert also the SU(3) matrices to express $G_{\rho''\mu''}G_{\rho'\mu'}$ by the vacuum condensate:

$$
\begin{aligned}
&\frac{\alpha_s}{\pi}G_{\rho''\mu''}G_{\rho'\mu'} \to \mathrm{tr}\left\{\frac{\lambda^a}{2}\frac{\lambda^b}{2}\right\}\frac{\alpha_s}{\pi}G^a_{\rho''\mu''}G^b_{\rho'\mu'} \\
&= \frac{1}{2}\frac{\alpha_s}{\pi}G^a_{\rho''\mu''}G^b_{\rho'\mu'} \qquad (7.211) \\
&= \mathrm{const}\cdot\left\langle 0\left|\frac{\alpha_s}{\pi}G^a_{\alpha\beta}G^{a\alpha\beta}\right|0\right\rangle\left(g_{\rho''\rho'}g_{\mu''\mu'} - g_{\rho''\mu'}g_{\mu''\rho'}\right) \quad .
\end{aligned}
$$

To determine the constant we contract with $g^{\rho''\rho'}g^{\mu''\mu'}$:

$$
\begin{aligned}
&\frac{1}{2}\frac{\alpha_s}{\pi}G^a_{\rho'\mu'}G^{a\rho'\mu'} = \mathrm{const}\cdot\left\langle 0\left|\frac{\alpha_s}{\pi}G^a_{\alpha\beta}G^{a\alpha\beta}\right|0\right\rangle(16-4) \qquad (7.212) \\
&\Rightarrow \mathrm{const} = \frac{1}{24} \quad .
\end{aligned}
$$

Inserting this into (7.210) yields

$$
\begin{aligned}
&-\left\langle 0\left|\frac{\alpha}{\pi}G^a_{\alpha\beta}G^{a\alpha\beta}\right|0\right\rangle \frac{1}{32\cdot 8\pi^2}\cdot\frac{1}{24}\cdot\frac{1}{\left(x^2\right)^7}\\
&\quad\times\left[\mathrm{tr}\left\{\left[\not{x}\gamma_{\mu'}\gamma_{\rho'}-\gamma_{\rho'}\gamma_{\mu'}\not{x}\right]\gamma_\mu\left[\not{x}\gamma_{\mu'}\gamma_{\rho'}-\gamma_{\rho'}\gamma_{\mu'}\not{x}\right]\gamma_\nu\right\}\right.\\
&\quad\left.-\mathrm{tr}\left\{\left[\not{x}\gamma_{\rho'}\gamma_{\mu'}-\gamma_{\mu'}\gamma_{\rho'}\not{x}\right]\gamma_\mu\left[\not{x}\gamma_{\mu'}\gamma_{\rho'}-\gamma_{\rho'}\gamma_{\mu'}\not{x}\right]\gamma_\nu\right\}\right] \quad . \qquad (7.213)
\end{aligned}
$$

The traces give

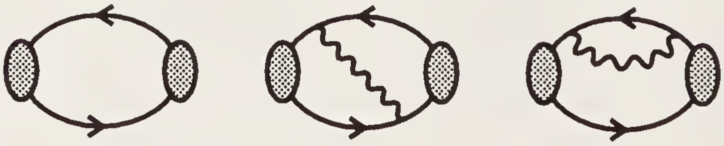
$$
\begin{aligned}
&\mathrm{tr}\{-2\not{x}\gamma_{\mu'}\gamma_{\mu'}\not{x}\gamma_\mu\gamma_\nu+2\not{x}\gamma_{\mu'}\gamma_\mu\gamma_{\mu'}\not{x}\gamma_\nu+2\gamma_{\mu'}\not{x}\gamma_\mu\not{x}\gamma_{\mu'}\gamma_\nu-2\gamma_\mu\not{x}\gamma_{\mu'}\gamma_{\mu'}\not{x}\gamma_\nu\\
&\qquad-4x_\nu\gamma_{\mu'}\gamma_\mu\not{x}\gamma_{\mu'}+4g_{\mu'\nu}\not{x}\gamma_\mu\not{x}\gamma_{\mu'}+4g_{\mu\mu'}\gamma_{\mu'}\not{x}\gamma_\nu\not{x}-4x_\mu\gamma_{\mu'}\gamma_{\mu'}\not{x}\gamma_\nu\}\\
&\quad=\mathrm{tr}\{-8x^2\gamma_\mu\gamma_\nu-8\not{x}\gamma_\mu\not{x}\gamma_\nu-8\gamma_\mu\gamma_\nu x^2\\
&\qquad-16x_\nu x_\mu+8\gamma_\mu\not{x}\gamma_\nu\not{x}-16x_\mu\not{x}\gamma_\nu\} \qquad (7.214)\\
&\quad=-16x^2\cdot 4g_{\mu\nu}-32\cdot 4x_\nu x_\mu=-64\left(x^2g_{\mu\nu}+2x_\mu x_\nu\right) \quad ,\\
&T_{\mu\nu}=\frac{1}{96\pi^2}\left\langle 0\left|\frac{\alpha}{\pi}G^a_{\alpha\beta}G^{a\alpha\beta}\right|0\right\rangle\frac{x^2g_{\mu\nu}+2x_\mu x_\nu}{\left(x^2\right)^2} \quad .
\end{aligned}
$$

This leads to the same integral we encountered in (7.203) repeating the same calculations therefore gives

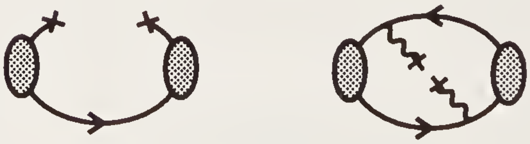
$$
\begin{aligned}
\Pi_{\mu\nu}&=\frac{1}{96\pi^2}\left\langle 0\left|\frac{\alpha}{\pi}G^a_{\alpha\beta}G^{a\alpha\beta}\right|0\right\rangle\frac{4\pi^2}{\left(q^2\right)^2}\left(q^2g_{\mu\nu}-q_\mu q_\nu\right)\cdot 2\\
&=\frac{1}{2}(q^2g_{\mu\nu}-q_\mu q_\nu)\frac{1}{(q^2+\mathrm{i}\varepsilon)^2}\left\langle\frac{\alpha_s}{\pi}G^a_{\alpha\beta}G^{a\alpha\beta}\right\rangle \quad . \qquad (7.215)
\end{aligned}
$$

It should now be obvious how to proceed further. Taking into account the following graphs we get

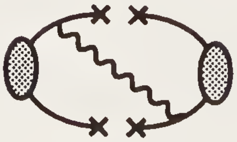
$$
\Pi_{\mu\nu}(q)=(q^2g_{\mu\nu}-q_\mu q_\nu)\left\{-\frac{1}{4\pi^2}\left(1+\frac{\alpha_s}{\pi}\right)\ln\frac{-q^2}{\mu^2}-\frac{6m^2}{q^2+\mathrm{i}\varepsilon}\right.
$$



$$
+\frac{2}{(q^2+\mathrm{i}\varepsilon)^2}m\langle\bar{q}q\rangle+\frac{1}{12}\frac{1}{(q^2+\mathrm{i}\varepsilon)^2}\left\langle\frac{\alpha_s}{\pi}G^a_{\alpha\beta}G^{a\alpha\beta}\right\rangle
$$



$$
\left.+\frac{112}{91}\frac{\pi\alpha_s}{(q^2+\mathrm{i}\varepsilon)^3}\langle\bar{q}q\rangle^2\right\} \quad . \qquad (7.216)
$$



$$\begin{aligned}\frac{1}{3q^2}\mathrm{Im}\,\Pi_{\mu\nu}(q)g^{\mu\nu} = \Bigg\{ & \frac{1}{4\pi}\left(1+\frac{\alpha_s}{\pi}\right) - m\langle\bar{q}q\rangle\pi\left(-\frac{\partial}{\partial q^2}\delta\left(q^2\right)\right) \\ & -\frac{\pi}{12}\left(-\frac{\partial}{\partial q^2}\delta\left(q^2\right)\right)\left\langle\frac{\alpha_s}{\pi}G^a_{\alpha\beta}G^{a\alpha\beta}\right\rangle \\ & -\left[\frac{1}{2}\left(\frac{\partial}{\partial q^2}\right)^2\delta\left(q^2\right)\right]2\pi\cdot\frac{112}{91}\langle\bar{q}q\rangle^2\pi\alpha_s\Bigg\} \quad . \qquad (7.217)\end{aligned}$$

We have so far calculated the partonic $\Pi_{\mu\nu}(q^2)$ tensor. Next we have to relate our result to the hadron description of $\Pi_{\mu\nu}(q^2)$. The quark description used a current with the quantum numbers of the rho. However, several resonances exist with such quantum numbers as well as continuum states. The unique property of the rho is that it is the lightest state and this property can be used to project it out from both the hadron and parton descriptions. There exist various methods to do this; we wish to discuss here only one, the Borel transformation, defined by

$$I = \frac{1}{\pi M^2}\int\limits_0^\infty \mathrm{e}^{-s/M^2}\,\mathrm{Im}\,[\Pi(s)]\,\mathrm{d}s \quad . \qquad (7.218)$$

Inserting (7.154) into this expression and replacing the upper integration bound by s_0 gives

$$\begin{aligned} I &= \frac{1}{\pi M^2}\int\limits_0^{s_0} \mathrm{e}^{-s/M^2}\,\mathrm{Im}\,(\Pi(s))\,\mathrm{d}s \\ &= \frac{1}{\pi M^2}\sum_f \frac{3\pi}{e_f^2}\sum_R \frac{m_R^2}{g_R^2}\mathrm{e}^{-m_R^2/M^2} \quad . \qquad (7.219)\end{aligned}$$

Obviously for small enough M^2 only the lowest-mass state survives. Here, however, we run into problems, since M^2 cannot be arbitrarily small, or otherwise the highest-twist contributions to the partonic description would become arbitrarily large. To understand this one has to know that the Borel transformation is equivalent to the following mathematical operation:

$$\begin{aligned} & \frac{1}{\pi M^2}\int\limits_0^{s_0} \mathrm{e}^{-s/M^2}\,\mathrm{Im}\,[\Pi(s)]\,\mathrm{d}s \\ & = \lim_{Q^2,n\to\infty,\;Q^2/n=M^2=\mathrm{const}} \frac{1}{(n-1)!}Q^{2n}\left(-\frac{\mathrm{d}}{\mathrm{d}Q^2}\right)^n \Pi(Q^2) \\ & = \lim_{Q^2,n\to\infty,\;Q^2/n=M^2} \frac{nQ^{2n}}{\pi}\int\limits_0^{s_0}\frac{\mathrm{Im}\,\Pi(s)}{\left(s+Q^2\right)^{n+1}}\,\mathrm{d}s \\ & = \lim_{Q^2,n\to\infty,\;Q^2/n=M^2} \frac{n}{\pi Q^2}\int\limits_0^{s_0}\frac{\mathrm{Im}\,\Pi(s)}{\left(1+\frac{s}{Q^2}\right)^{n+1}}\,\mathrm{d}s \end{aligned}$$

$$= \lim_{Q^2, n \to \infty,\, Q^2/n = M^2} \frac{1}{\pi M^2} \int_0^{s_0} \mathrm{e}^{-(n+1)\ln(1+s/Q^2)} \mathrm{Im}\, \Pi(s)\, \mathrm{d}s$$

$$= \frac{1}{\pi M^2} \int_0^{s_0} \mathrm{e}^{-s/M^2} \mathrm{Im}\, \Pi(s)\, \mathrm{d}s \quad . \tag{7.220}$$

where we have started from the well-known dispersion relation

$$\Pi\left(q^2\right) = \frac{1}{\pi} \int_0^{s_0} \frac{\mathrm{Im}\, \Pi(s)}{\left(s + Q^2\right)}\, \mathrm{d}s \quad . \tag{7.221}$$

Now the higher twist corrections to the partonic calculations, or the higher-dimension condensates that parametrize them, are suppressed by powers of Q^2. Q^2 must be large in order to allow for perturbative calculations in α. Thus to make M^2 small would require very large values of n. But such high derivatives increase the importance of the higher twist effects as

$$\frac{Q^{2n}}{(n-1)!} \left(-\frac{\mathrm{d}}{\mathrm{d}Q^2}\right)^n \frac{1}{\left(Q^2\right)^N} = \frac{(N+n-1)!}{(N-1)!} \frac{1}{(n-1)!} \frac{1}{\left(Q^2\right)^N} \sim \frac{(N+n-1)!}{(n-1)!} \tag{7.222}$$

becomes very large with increasing N. Thus QCD sum rules are for most applications an expansion with a limited domain of applicability. If M^2 becomes too small the higher twist effects on the partonic side become uncontrollable; if it becomes too large, the hadronic side becomes a complicated mixture of hadronic states. Luckily for most problems one finds an intermediate domain around $M = 1$ GeV in which the result depends only slightly on M, indicating that the approximations made were acceptable. Inserting (7.154), we obtain for the Borel transform of the hadronic part simply

$$\frac{1}{\pi M} \int_0^{s_0} \mathrm{e}^{-s/M^2} \mathrm{Im}\, \Pi(s)\, \mathrm{d}s = \mathrm{e}^{-m_\rho^2/M^2} \frac{3\pi m_\rho^2}{e_f^2 g_R^2} \frac{1}{\pi M^2} \quad . \tag{7.223}$$

This integral is now easy, yielding

$$\begin{aligned} &\frac{1}{\pi M^2} \int_0^{s_0} \mathrm{e}^{-s/M^2} \Pi_{\mu\nu}(q') g^{\mu\nu}\, \mathrm{d}s \\ &= \frac{1}{4\pi^2} \left[\left(1 + \frac{\alpha_s}{\pi}\right) \left(1 - \mathrm{e}^{-s_0/M^2}\right) + \frac{8\pi^2}{M^4} \langle 0 | m \bar{q} q | 0 \rangle \right. \\ &\quad \left. + \frac{\pi^2}{3M^4} \left\langle 0 \left| \frac{\alpha_s}{\pi} G^a_{\alpha\beta} G^{a\alpha\beta} \right| 0 \right\rangle - \frac{448}{81} \frac{\pi^2 \alpha_s}{M^6} |\langle 0 | \bar{q} q | 0 \rangle|^2 \right] \quad . \end{aligned} \tag{7.224}$$

This expression can now be equated with (7.223). However, there is still one last problem to be solved. The current substituted in (7.190) to describe vector mesons can still be arbitrarily normalized. Thus we introduce a factor λ_ρ and write

$$\begin{aligned}
&\frac{1}{4\pi^2}\left[\left(1+\frac{\alpha_s(\lambda)}{\pi}\right)\left(1-\mathrm{e}^{-s_0/M^2}\right)+\frac{8\pi^2}{M^4}\langle 0|m\bar{q}q|0\rangle\right.\\
&\quad\left.+\frac{\pi^2}{3M^4}\left\langle 0\left|\frac{\alpha_s}{\pi}G^a_{\alpha\beta}G^{a\alpha\beta}\right|0\right\rangle-\frac{448}{81}\frac{\pi^2\alpha_s}{M^6}|\langle 0|\bar{q}q|0\rangle|^2\right]\\
&=\lambda_\rho\frac{1}{\pi M^2}\frac{3\pi m_\rho^2}{e_f^2 g_R^2}\mathrm{e}^{-m_\rho^2/M^2}=\frac{1}{\pi M^2}J \quad . \qquad (7.225)
\end{aligned}$$

We get rid of λ_ρ and all other constants by dividing $\mathrm{d}J/\mathrm{d}M^2$ by J

$$\begin{aligned}
\frac{\mathrm{d}J}{\mathrm{d}M^2}/J=&\left[\left(1+\frac{\alpha_s}{\pi}\right)\left(1-\mathrm{e}^{-s_0/M^2}\right)-M^2\left(1+\frac{\alpha_s}{\pi}\right)\frac{s_0}{M^2}\mathrm{e}^{-s_0/M^2}\right.\\
&-\frac{8\pi^2}{M^4}\langle 0|m\bar{q}q|0\rangle-\frac{\pi^2}{3M^4}\left\langle 0\left|\frac{\alpha_s}{\pi}G^a_{\alpha\beta}G^{a\alpha\beta}\right|0\right\rangle\\
&\left.+2\cdot\frac{448}{81}\frac{\pi^2\alpha_s}{M^6}\langle 0|\bar{q}q|0\rangle^2\right]\\
&\times\left[M^2\left(1+\frac{\alpha_s}{\pi}\right)\left(1-\mathrm{e}^{-s_0/M^2}\right)+\frac{8\pi^2}{M^2}\langle 0|m\bar{q}q|0\rangle\right.\\
&\left.+\frac{\pi^3}{3M^2}\left\langle 0\left|\frac{\alpha_s}{\pi}G^a_{\alpha\beta}G^{a\alpha\beta}\right|0\right\rangle-\frac{448}{81}\frac{\pi^2\alpha_s}{M^4}\langle 0|\bar{q}q|0\rangle^2\right]^{-1}=\frac{m_\rho^2}{M^4} \quad .
\end{aligned}$$
$$(7.226)$$

$$\begin{aligned}
m_\rho^2=M^2&\left[\left(1+\frac{\alpha_s}{\pi}\right)\left(1-\left(1+\frac{s_0}{M^2}\right)\mathrm{e}^{-s_0/M^2}\right)-\frac{1}{M^4}A+\frac{2}{M^6}B\right]\\
&\times\left[\left(1+\frac{\alpha_s}{\pi}\right)\left(1-\mathrm{e}^{-s_0/M^2}\right)+\frac{1}{M^4}A-\frac{1}{M^6}B\right]^{-1} \quad . \qquad (7.227)
\end{aligned}$$

With the condensates from (7.155) the numerical constants are

$$\begin{aligned}
A&=-8\pi^2(0.10)^4\ \mathrm{GeV}^4+\frac{\pi^3}{3}(0.36)^4\ \mathrm{GeV}^4\\
&=(-0.007+0.055)\ \mathrm{GeV}^4=0.05\ \mathrm{GeV}^4 \quad , \qquad (7.228)\\
B&=(0.225)^8\cdot\pi^2\cdot 0.25\cdot\frac{448}{81}\ \mathrm{GeV}^6=0.035\ \mathrm{GeV}^6 \quad .
\end{aligned}$$

The resulting form of m_ρ^2 as a function of M^2 is plotted in Fig. 7.24.

Similar calculations can be repeated for any set of parameters, and the resulting masses agree quite well with the real values except for the pion. However, the QCD sum rule technique is not limited to the calculation of masses. It is also possible to calculate vertices, such as any coupling to the quark in the proton according to Fig. 7.25.

The proton current can be written in many different forms. The simplest is the so-called Ioffe current

$$\eta_I(x)=\varepsilon_{abc}\left(u^{ta}(x)\gamma_0\gamma_2\gamma_\mu u^b(x)\right)\gamma_5\gamma^\mu d^c(x) \quad . \qquad (7.229)$$

Fig. 7.24. The rho mass as a function of M^2, as given by (7.227) for $s_0 = 1.5\ \mathrm{GeV}^2$. The convergence is clearly convincing in the range $M^2 = 0.5 - 0.8\ \mathrm{GeV}^2$

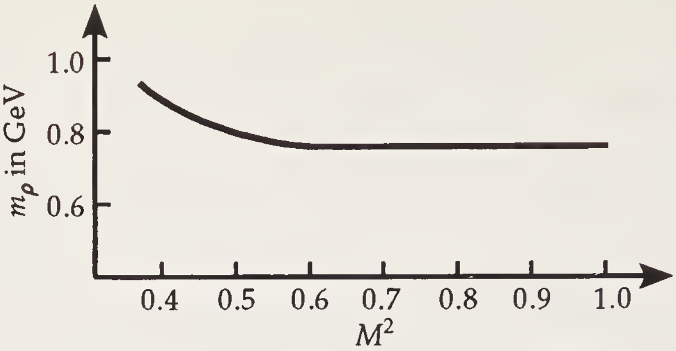
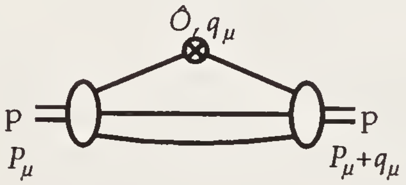


Fig. 7.25. The general form of a proton vertex $\hat{O}$ at which momentum q_μ is transferred



Here, the first $u(x)$ field is transposed so that the bracket is just a number as far as the spinor indices are concerned and a Lorentz vector with respect to the index of γ_μ. It is easy to show that this combination indeed has isospin 1/2 and not 3/2, which is the second possibility in the coupling of 3 isospin-1/2 objects

$$\begin{aligned} I^2 &= I_3^2 + \frac{1}{2}(I_+ I_- + I_- I_+) \quad , \\ I_j &= I_j(1) + I_j(2) + I_j(3) \quad , \end{aligned} \tag{7.230}$$

where the $I_j(k)$ act on quark number k. To show this one has to use the so called Fierz-transformation:

$$\begin{aligned} I^2 \eta_I(x) = {} & \frac{1}{4}\eta_I(x) + \frac{1}{2} I_+ \left[\varepsilon_{abc} \left(d^{ta}(x)\gamma_0\gamma_2\gamma_\mu u^b(x)\right) \gamma_5 \gamma^\mu d^c(x) \right. \\ & \left. + \varepsilon_{abc} \left(u^{ta}(x)\gamma_0\gamma_2\gamma_\mu d^b(x)\right) \gamma_5 \gamma^\mu d^c(x)\right] \\ & + \frac{1}{2} I_- \left[\varepsilon_{abc} \left(u^{ta}(x)\gamma_0\gamma_2\gamma_\mu u^b(x)\right) \gamma_5 \gamma^\mu u^c(x)\right] \\ = {} & \frac{1}{4}\eta_I(x) + \frac{3}{2}\eta_I(x) + \varepsilon_{abc} \left(d^{ta}(x)\gamma_0\gamma_2\gamma_\mu u^b(x)\right) \gamma_5 \gamma^\mu u^c(x) \\ & + \varepsilon_{abc} \left(u^{ta}(x)\gamma_0\gamma_2\gamma_\mu d^b(x)\right) \gamma_5 \gamma^\mu u^c(x) \quad . \end{aligned} \tag{7.231}$$

The last two terms are equal (please remember that the bracket is just a number).

$$\begin{aligned} d^{ta}(1)\gamma_0\gamma_2\gamma_\mu u^b(2) &= \left[d^{ta}(1)\gamma_0\gamma_2\gamma_\mu u^b(2)\right]^t \\ &= u^{bt}(2)\gamma_\mu^t \gamma_2^t \gamma_0^t d^a(1) = -u^{bt}(2)\gamma_2\gamma_0\gamma_\mu d^a(1) \\ &= u^{bt}(2)\gamma_0\gamma_2\gamma_\mu d^a(1) \to u^{at}(1)\gamma_0\gamma_2\gamma_\mu d^b(2) \quad . \end{aligned} \tag{7.232}$$

In the last step we have used the antisymmetry of ε_{abc} and the fact that the total wave function is odd under the exchange of any two particles. The Fierz transformation in our case is

$$\gamma_5\Big(\left(u^{at}\gamma_0\gamma_2\right)\gamma_\mu d^b\Big)\gamma^\mu u^c$$
$$= -\frac{1}{4}\gamma_5\left\{\left(u^{at}\gamma_0\gamma_2 u^c\right)d^b - 2\left(u^{at}\gamma_0\gamma_2\gamma_\mu u^c\right)\gamma^\mu d^b\right.$$
$$\left. -2\left(u^{at}\gamma_0\gamma_2\gamma_\mu\gamma_5 u^c\right)\gamma^\mu\gamma_5 d^b - \left(u^{at}\gamma_0\gamma_2\gamma_5 u^c\right)\gamma_5 d^b\right\} \quad . \qquad (7.233)$$

Now one can see from (7.232) that all combinations for which the transposed bracket has the opposite sign vanish owing to the symmetry operation ($a \to c, 1 \to 2$). But

$$\begin{aligned}\left(u^{at}\gamma_0\gamma_2 u^c\right)^t &= u^{ct}\gamma_2^t\gamma_0^t u^a = -u^{ct}\gamma_0\gamma_2 u^a \quad ,\\ \left(u^{at}\gamma_0\gamma_2\gamma_\mu\gamma_5 u^c\right)^t &= u^{ct}\gamma_5^t\gamma_0\gamma_2\gamma_\mu u^a = -u^{ct}\gamma_0\gamma_2\gamma_\mu\gamma_5 u^a \quad ,\\ \left(u^{at}\gamma_0\gamma_2\gamma_5 u^c\right)^t &= u^{ct}\gamma_5\gamma_2\gamma_0 u^a = -u^{ct}\gamma_0\gamma_2\gamma_5 u^a \quad .\end{aligned} \qquad (7.234)$$

Thus only the second term in (7.233) survives:

$$\begin{aligned}\left(u^{at}\gamma_0\gamma_2\gamma_\mu d^b\right)\gamma_5\gamma^\mu u^c &= -\frac{1}{2}\left(u^{at}(1)\gamma_0\gamma_2\gamma_\mu u^c(3)\right)\gamma^\mu\gamma_5 d^b(2)\\ &\Rightarrow -\frac{1}{2}\left(u^{at}(1)\gamma_0\gamma_2\gamma_\mu u^b(2)\right)\gamma^\mu\gamma_5 d^c(3) \quad ,\end{aligned} \qquad (7.235)$$

and we have

$$I^2\eta_I(x) = \frac{1}{4}\eta_I(x) + \frac{3}{2}\eta_I(x) - 2\cdot\frac{1}{2}\eta_I(x) = \frac{3}{4}\eta_I(x) = \frac{1}{2}\left(\frac{1}{2}+1\right)\eta_I(x) \, , \qquad (7.236)$$

which demonstrates that $\eta_I(x)$ indeed has isospin 1/2. In calculating the triangle graph in Fig. 7.25 basically all techniques are the same except that now we have to perform a double Borel transformation to make sure that both the incoming and the outgoing baryons are projected onto the proton state:

$$s_1 = p^2 \quad , \quad s_2 = (p+q)^2 \quad ,$$
$$\int_0^{s_0} \mathrm{d}s_1 \, \mathrm{e}^{-s_1/2M^2} \int_0^{s_0} \mathrm{d}s_2 \, \mathrm{e}^{-s_2/2M^2} \ldots \quad . \qquad (7.237)$$

In fact the applicability of QCD sum rules for calculations of quantities other than masses is not trivial. The Borel transformation projects out the lowest-mass eigenstate, which is the correct state only if it is unique, i.e., if any state is uniquely determined by the quantum numbers contained in the current and by the mass. For fully relativistic problems this is ascertained by general theorems. With these remarks we wish to end our introduction to QCD sum rules. Let us summarize what we have done by noting that with the few parameters for the condensates one is able to calculate with a typical accuracy of 20%:

- the masses of $\rho, \phi, \delta, \mathrm{a}_1, \mathrm{N}, \Sigma, \Lambda, \Xi, \Delta, \ldots$;
- the vertex constants such as $g_{\pi\mathrm{NN}}, g_{\pi\mathrm{N}\Delta}, g_{\omega\rho\pi}, \ldots$;
- the heavier quarkonia states ${}^1S_0, {}^3S_1, {}^3P_2, \ldots$ for $\bar{\mathrm{c}}\mathrm{c}$ and $\bar{\mathrm{b}}\mathrm{b}$;
- quantities such as the total momentum carried by quarks or gluons at the scale M^2 which is $\sum_q \int_0^1 xq(x)\,\mathrm{d}x$;
- the form of the 3-quark component of the nucleon wave function and of the quark antiquark component of the pion wave function (the so-called Chernyak–Zhitnitsky wave functions), and many more quantities.

8. Phenomenological Models for Non-Perturbative QCD Problems

As the complete calculation of many, especially dynamical nonperturbative problems, for example, using lattice calculations, is still impossible, theoreticants have tried to develop simple physical models for these problems. Using suitable assumptions and parameter choices, one then attempts to reproduce as many properties of QCD as possible. We shall consider two such problems here: the ground state of QCD and the quark–gluon plasma. In both cases we shall restrict ourselves to a few remarks only.

8.1 The Ground State of QCD

Many semiphenomenological models have been developed for the QCD ground state. We shall consider here only the so-called "Spaghetti vacuum" which is one of the most promising candidates and has the advantage of being easily visualized. As a starting point, let us ask what the fate is of a hypothetical monopole–antimonopole pair in a superconductor. The Meissner–Ochsenfeld effect teaches us that the flux lines of the magnetic field cannot enter any superconducting region. Consequently a region of normal conduction is created in between the pair (see Fig. 8.1).

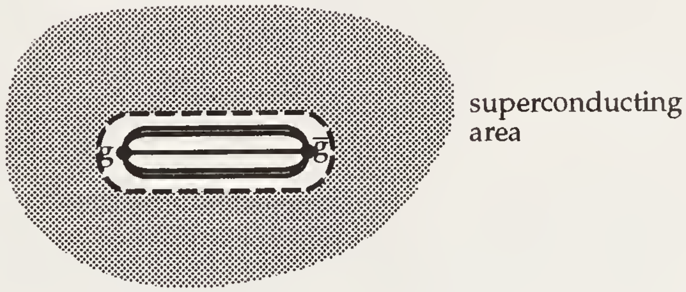


Fig. 8.1. A magnetic monopole(g)–antimonopole($\bar{g}$) pair creates a string of normal conductance in a superconductor

Thus a string is created, and, since the energy density in the region of normal conduction is larger than in the superconductivity region, the magnetic charges are confined.

Our model system therefore shows the same properties as QCD, with magnetic charge and magnetic field playing the role of color charges and color electric fields. This leads us to the dual superconductor picture, i.e., the assumption that the QCD ground state has the properties of a superconductor in which the roles of magnetic and electric fields are interchanged. To put this on a firm foundation, one must show that QCD contains some objects that can play the role of Cooper pairs. As Cooper pairs are bound by the electric field, one expects that their dual partners in QCD are bound by color magnetic fields.

From the many ideas about what these objects might be, we shall discuss only the "Spaghetti vacuum", in which the Cooper pairs are strings of color magnetic fields. To obtain this picture we start with the simplest question and ask how spontaneous color magnetization can arise in a pure gauge theory. To this end, we introduce a color magnetic background field H, determine the corresponding vacuum energy, and calculate the total energy with respect to H. The color electric and color magnetic fields are defined as

$$E^{aj} = F^a_{0j} = \nabla_0 A^a_j - \nabla_j A^a_0 + g f^{abc} A^b_0 A^c_j \quad , \tag{8.1}$$

$$H^{aj} = -\varepsilon_{jik} F^a_{ik} = -\varepsilon_{jik}\left(\nabla_i A^a_k - \nabla_k A^a_i + g f^{abc} A^b_i A^c_k\right) \quad . \tag{8.2}$$

To see whether a homogeneous magnetic field can lower the total energy, we set

$$H^{aj} = H\delta_{a3}\delta_{j3} \Rightarrow A^{0a}_\mu = -Hx_1\delta_{\mu 2}\delta_{a3} \quad . \tag{8.3}$$

For simplicity we shall keep to an SU(2) gauge theory. We consider a small variation A^a_ν around A^{0a}_μ:

$$\begin{aligned} F^a_{\mu\nu}F^{a\mu\nu} = {} & \left\{\nabla_\mu A^1_\nu - \nabla_\nu A^1_\mu + g\left[A^2_\mu\left(A^3_\nu - Hx_1\delta_{\nu 2}\right) - \left(A^3_\mu - Hx_1\delta_{\mu 2}\right)A^2_\nu\right]\right\}^2 \\ & + \left\{\nabla_\mu A^2_\nu - \nabla_\nu A^2_\mu + g\left[\left(A^3_\mu - Hx_1\delta_{\mu 2}\right)A^1_\nu - A^1_\mu\left(A^3_\nu - Hx_1\delta_{\nu 2}\right)\right]\right\}^2 \\ & + 2H^2 + \left[\nabla_\mu A^3_\nu - \nabla_\nu A^3_\mu + g\left(A^1_\mu A^2_\nu - A^2_\mu A^1_\nu\right)\right]^2 \\ & + 2\left[\nabla_\mu A^3_\nu - \nabla_\nu A^3_\mu + g\left(A^1_\mu A^2_\nu - A^2_\mu A^1_\nu\right)\right] \\ & \times (-H)\left(\delta_{\mu 1}\delta_{\nu 2} - \delta_{\mu 2}\delta_{\nu 1}\right) \quad . \end{aligned} \tag{8.4}$$

We consider small variations around $H^{a\mu}$ and therefore neglect higher powers in A:

$$\begin{aligned} F^a_{\mu\nu}F^{a\mu\nu} \approx {} & \left[\nabla_\mu A^1_\nu - \nabla_\nu A^1_\mu - gHx_1\left(A^2_\mu\delta_{\nu 2} - A^2_\nu\delta_{\mu 2}\right)\right]^2 \\ & + \left[\nabla_\mu A^2_\nu - \nabla_\nu A^2_\mu - gHx_1\left(A^1_\nu\delta_{\mu 2} - A^1_\mu\delta_{\nu 1}\right)\right]^2 \\ & + 2H^2 + \left(\nabla_\mu A^3_\nu - \nabla_\nu A^3_\mu\right)^2 \\ & - 4H\left[\nabla_1 A^3_2 - \nabla_2 A^3_1 + g\left(A^1_1 A^2_2 - A^2_1 A^1_2\right)\right] \quad . \end{aligned} \tag{8.5}$$

The source terms for A^3_μ that appear in the last term vanish by partial integration of the action integral if we demand that $A^3_2(x_1 \to \pm\infty) = A^3_1(x_2 \to \pm\infty) = 0$. This is equivalent to demanding a fixed background field. Indeed the appearance of this source term is an indication that A^3_μ is not a physical degree of freedom, and it can be shown that A^1_μ and A^2_μ are the only physical degrees of freedom. We shall content ourselves here with the observation that the source terms vanish by partial integration and that the only remaining term in A^3_μ is a free Lagrange density and is therefore not of interest to us. Dropping these terms we have

$$\begin{aligned} F^a_{\mu\nu}F^{a\;\mu\nu} = {} & \left|\nabla_\mu\left(A^1_\nu + \mathrm{i}A^2_\nu\right) - \nabla_\nu\left(A^1_\mu + \mathrm{i}A^2_\mu\right) - gHx_1\left[\left(A^2_\mu - \mathrm{i}A^1_\mu\right)\delta_{\nu 2}\right.\right. \\ & \left.\left. - \left(A^2_\nu - \mathrm{i}A^1_\nu\right)\delta_{\mu 2}\right]\right|^2 + 2H^2 - \mathrm{i}2gH\left[\left(A^1_1 + \mathrm{i}A^2_1\right)\left(A^1_2 - \mathrm{i}A^2_2\right)\right. \\ & \left. - \left(A^1_2 + \mathrm{i}A^2_2\right)\left(A^1_1 - \mathrm{i}A^2_1\right)\right] \quad . \end{aligned} \tag{8.6}$$

In the last step, we have rewritten $F^a_{\mu\nu}F^{a\mu\nu}$ in a way to show that it is useful to replace A^1_μ and A^2_μ by

$$W_\mu = \frac{1}{\sqrt{2}}\left(A^1_\mu + \mathrm{i}A^2_\mu\right) \quad , \quad W^+_\mu = \frac{1}{\sqrt{2}}\left(A^1_\mu - \mathrm{i}A^2_\mu\right) \quad . \tag{8.7}$$

With these we obtain

$$\begin{aligned} F^a_{\mu\nu}F^{a\mu\nu} &= 2\left|\left(\nabla_\mu - \mathrm{i}gHx_1\delta_{\mu2}\right)W_\nu - \left(\nabla_\nu - \mathrm{i}gHx_1\delta_{\nu2}\right)W_\mu\right|^2 \\ &\quad + 4\mathrm{i}gH\left(W_1^+W_2 - W_2^+W_1\right) + 2H^2 \quad . \end{aligned} \tag{8.8}$$

The Lagrange density to be investigated is thus

$$\begin{aligned} L = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} &= -\frac{1}{2}\left|\left(\nabla_\mu - \mathrm{i}gHx_1\delta_{\mu2}\right)W_\nu - \left(\nabla_\nu - \mathrm{i}gHx_1\delta_{\nu2}\right)W_\mu\right|^2 \\ &- \mathrm{i}gH\left(W_1^+W_2 - W_2^+W_1\right) - \frac{1}{2}H^2 \quad , \end{aligned} \tag{8.9}$$

and the resulting equation of motion is

$$\begin{aligned} &-\left(-\nabla^\mu - \mathrm{i}gHx_1\delta_{\mu2}\right)\left[\left(\nabla_\mu - \mathrm{i}gHx_1\delta_{\mu2}\right)W_\nu - \left(\nabla_\nu - \mathrm{i}gHx_1\delta_{\nu2}\right)W_\mu\right] \\ &- \mathrm{i}gH\left(\delta_{\nu1}W_2 - \delta_{\nu2}W_1\right) = 0 \quad . \end{aligned} \tag{8.10}$$

To make sure that W^+_μ, W_μ are indeed the physical degrees of freedom, it is useful to choose the so-called background gauge:

$$\left(\partial^\mu - \mathrm{i}gHx_1\delta_{\mu2}\right)W_\mu = 0 \quad . \tag{8.11}$$

(For $H \longrightarrow 0$, this is the usual transverse gauge.)

That this gauge singles out the physical degrees of freedom is formally demonstrated by choosing the gauge (8.11), determining the corresponding ghost fields, and then showing that they cancel the contribution of A^3_μ and those components of W_μ for which $\left(\partial^\mu - \mathrm{i}gHx_1\delta_{\mu2}\right)W_\mu \neq 0$.

From (8.11) it follows that

$$\begin{aligned} &\left(\nabla^\mu + \mathrm{i}gHx_1\delta_{\mu2}\right)\left(\nabla_\mu - \mathrm{i}gHx_1\delta_{\mu2}\right)W_\nu \\ &\quad = \left(\nabla^\mu - \mathrm{i}gHx_1\delta^{\mu2}\right)\left(\nabla_\mu - \mathrm{i}gHx_1\delta_{\mu2}\right)W_\nu \\ &\quad = \left(\nabla_\nu - \mathrm{i}gHx_1\delta_{\nu2}\right)\left(\nabla^\mu + \mathrm{i}gHx_1\delta_{\mu2}\right)W_\mu \\ &\qquad - \mathrm{i}gH(-\delta_{\mu1})\delta_{\nu2}W_\mu - \mathrm{i}gH\delta_{\nu1}\delta_{\mu2}W_\mu \\ &\quad = -\mathrm{i}gH(W_2\delta_{\nu1} - W_1\delta_{\nu2}) \quad , \end{aligned} \tag{8.12}$$

or

$$\left(\nabla^\mu - \mathrm{i}gHx_1\delta^{\mu2}\right)^2 W_\nu - 2gH\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \mathrm{i} & 0 \\ 0 & -\mathrm{i} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{\nu\mu} W_\mu = 0 \quad . \tag{8.13}$$

The matrix in (8.13) has the eigenvalues ± 1 and a double eigenvalue at zero. Its eigenvectors are $(0, 1, \mp\mathrm{i}, 0)$, $(0, 0, 0, 1)$, $(1, 0, 0, 0)$. The last is excluded by the gauge condition (8.11), while the others correspond to spin ± 1 and 0. The matrix in (8.13) contains the submatrix

$$(S_3)_{jk} = \mathrm{i}\varepsilon_{3jk} = \begin{pmatrix} 0 & \mathrm{i} & 0 \\ -\mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad , \tag{8.14}$$

i.e., the third component of the spin matrix for spin 1. The factor 2 is analogous to the g factor. Since gluons are massless, $S = 0$ is excluded (formally one finds that ghost fields cancel the contribution of $S = 0$ gluons). Thus the two physical fields are those with $S = \pm 1$, and they satisfy

$$\begin{aligned} &\left(\nabla^\mu - \mathrm{i}gHx_1\delta^{\mu 2}\right)^2 W_\nu - 2gH \cdot (\pm 1) W_\nu \\ &= \left\{ -\frac{\partial^2}{\partial x_0^2} + \frac{\partial^2}{\partial x_1^2} - \left(\mathrm{i}\frac{\partial}{\partial x_2} + gHx_1 \right)^2 + \frac{\partial^2}{\partial x_3^2} \pm 2gH \right\} W_\nu = 0 \quad . \end{aligned} \tag{8.15}$$

Continuing to the Fourier transform $\tilde{W}_\nu(E, x_1, k_2, k_3)$ we have

$$\left(E^2 + \frac{\partial^2}{\partial x_1^2} - (k_2 + gHx_1)^2 - k_3^2 \mp 2gH \right) \tilde{W}_\nu = 0 \quad , \tag{8.16}$$

and we substitute $x_1 = y - k_2/gH$:

$$\left(E^2 - k_3^2 \mp 2gH + \frac{\partial^2}{\partial y^2} - g^2H^2y^2 \right) \tilde{W}_\nu(E, y, k_2, k_3) = 0 \quad . \tag{8.17}$$

The last two terms correspond to the equation of the harmonic oscillator:

$$\omega \Psi(y) = -\left(\frac{1}{2}\frac{\partial^2}{\partial y^2} - \frac{1}{2}g^2H^2y^2 \right) \Psi(y) \tag{8.18}$$

with energy eigenvalues

$$\omega_n = \left(n + \frac{1}{2} \right) gH \quad . \tag{8.19}$$

With the ansatz

$$\tilde{W}_\nu = \Psi(y)\, \phi(E, k_2, k_3) \tag{8.20}$$

it now follows that

$$\left(E^2 - k_3^2 \mp 2gH - 2\omega_n \right) \tilde{W}_\nu = 0 \tag{8.21}$$

$$\Rightarrow \quad E = \sqrt{2gH \left(n + \frac{1}{2} \pm 1 \right) + k_3^2} = E' + \mathrm{i}\Gamma \quad . \tag{8.22}$$

Note that E becomes imaginary for $n = 0$ and $k_3^2 < gH$. Since an imaginary energy corresponds to a decay width, this state is unstable (see Fig. 8.2).

The appearance of an unstable mode indicates that the model assumptions of a homogeneous color magnetic field are unrealistic. In spite of this, we shall investigate what result would be obtained if this state were not unstable, since this will help us to understand the role of the unstable mode.

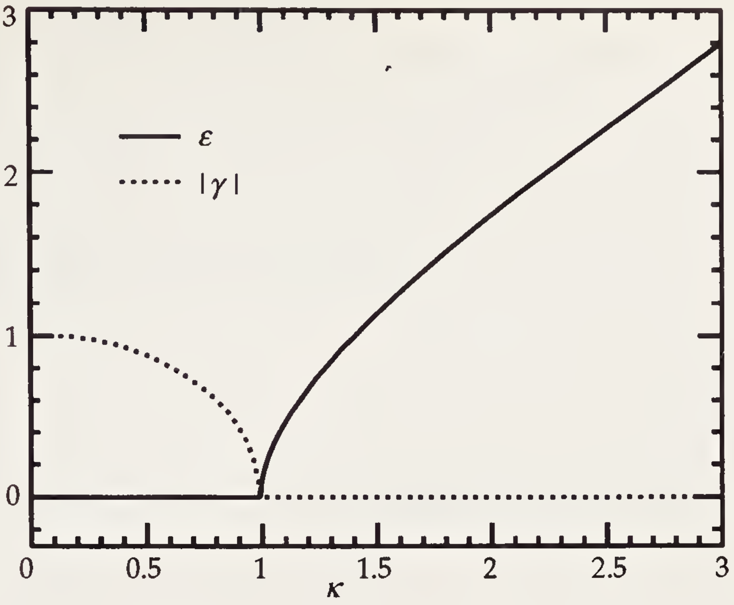


Fig. 8.2. The unstable mode, with $\varepsilon = E_1'/\sqrt{gH}$, $\gamma = \Gamma/\sqrt{gH}$, and $\kappa = k_3/\sqrt{gH}$. For $\kappa < 1$ the energy is zero and the state decays or grows exponentially depending on the sign of γ

From the single-particle energies (8.22) follows the energy density

$$\varepsilon = 2\,\frac{1}{2}\,C\int\frac{\mathrm{d}k_3}{2\pi}\left(\sum_{n=0}^{\infty}\sqrt{2gH\left(n-\frac{1}{2}\right)+k_3^2}+\sum_{n=0}^{\infty}\sqrt{2gH\left(n+\frac{3}{2}\right)+k_3^2}\right)\ . \tag{8.23}$$

The factor 2 stands for the two degrees of freedom W_μ and W_μ^+; the factor 1/2 appears because we are calculating the vacuum energy. The factor C comes from changing $\int \mathrm{d}k_1\mathrm{d}k_2$ to $\sum_{n=0}^{\infty}$. Since for the nth Landau state it holds that

$$k_1^2 + k_2^2 \sim ngH \quad , \tag{8.24}$$

we expect that

$$\frac{\int \mathrm{d}k_1\mathrm{d}k_2}{(2\pi)^2} \Rightarrow \frac{\int k\mathrm{d}k}{2\pi} \Rightarrow \frac{gH}{4\pi}\sum_n \quad . \tag{8.25}$$

A precise calculation yields

$$C = \frac{gH}{2\pi} \quad , \tag{8.26}$$

$$\begin{aligned}\varepsilon - \varepsilon(H=0) = \frac{gH}{2\pi}\int\frac{\mathrm{d}k_3}{2\pi}&\left[\sum_{n=1}^{\infty}\sqrt{2gH\left(n-\frac{1}{2}\right)+k_3^2}\right.\\ &\left.+\sum_{n=0}^{\infty}\sqrt{2gH\left(n+\frac{3}{2}\right)+k_3^2}\right]\\ &-2\times2\times\frac{1}{2}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\sqrt{k^2}+\frac{gH}{2\pi}\int\frac{\mathrm{d}k_3}{2\pi}\sqrt{k_3^2-gH}\ .\end{aligned} \tag{8.27}$$

The factors in front of the last-but-one term are: spin degrees of freedom × color degrees of freedom $\times\frac{1}{2}$. To perform the sums we introduce parameters δ and η:

$$\begin{aligned}\varepsilon-\varepsilon(H=0)=\lim_{\eta,\delta\to 0}\Bigg[&\frac{gH}{\pi}\int\limits_0^\infty\frac{\mathrm{d}k_3}{2\pi}\left\{\sum_{n=1}^{\infty}\left[2gH\left(n-\frac{1}{2}\right)+k_3^2-\mathrm{i}\delta\right]^{1/2+\eta}\right.\\&\left.+\left[2gH\left(n+\frac{3}{2}\right)+k_3^2-\mathrm{i}\delta\right]^{1/2-\eta}\right\}\\&-\frac{1}{\pi^2}\int\limits_0^\infty\mathrm{d}k\,k^2(k^2-\mathrm{i}\delta)^{1/2-\eta}+\frac{gH}{2\pi^2}\int\limits_0^\infty\mathrm{d}k_3(k_3^2-gH-\mathrm{i}\delta)^{1/2-\eta}\Bigg]\end{aligned}\tag{8.28}$$

and use

$$(\omega^2-\mathrm{i}\delta)^{\nu-\mu}=\frac{\mathrm{i}^{-\nu+\eta}}{\Gamma(-\nu+\eta)}\int\limits_0^\infty\mathrm{d}\tau\,\tau^{-\nu+\eta-1}\,\mathrm{e}^{-\mathrm{i}\tau(\omega^2-\mathrm{i}\delta)}\tag{8.29}$$

$$\begin{aligned}\varepsilon-\varepsilon(H=0)=\lim_{\eta,\delta\to 0}\Bigg[&\frac{gH}{2\pi^2}\frac{\mathrm{i}^{-1/2+\eta}}{\Gamma\left(-\frac{1}{2}+\eta\right)}\int\limits_0^\infty\mathrm{d}k\int\limits_0^\infty\mathrm{d}\tau\tau^{-3/2+\eta}\\&\times\mathrm{e}^{-\mathrm{i}\tau(k^2-\mathrm{i}\delta)}\left\{\sum_{n=1}^{\infty}\mathrm{e}^{-\mathrm{i}\tau 2gH\left(n-\frac{1}{2}\right)}+\sum_{n=0}^{\infty}\mathrm{e}^{-\mathrm{i}\tau 2gH\left(n+\frac{3}{2}\right)}\right\}\\&-\frac{\mathrm{i}^{-\frac{1}{2}+\eta}}{\Gamma\left(-\frac{1}{2}+\eta\right)}\frac{1}{\pi^2}\int\limits_0^\infty\mathrm{d}k\int\limits_0^\infty\mathrm{d}\tau k^2\tau^{-\frac{3}{2}+\eta}\,\mathrm{e}^{-\mathrm{i}\tau(k^2-\mathrm{i}\delta)}\\&+\frac{gH\mathrm{i}^{-\frac{1}{2}+\eta}}{2\pi^2\Gamma\left(-\frac{1}{2}+\eta\right)}\int\limits_0^\infty\mathrm{d}k\int\limits_0^\infty\mathrm{d}\tau\,\tau^{-\frac{3}{2}+\eta}\,\mathrm{e}^{-\mathrm{i}\tau(k^2-\mathrm{i}\delta-gH)}\Bigg]\,.\end{aligned}\tag{8.30}$$

The calculation of the integrals is demonstrated in Problem 8.1. The result is

$$\varepsilon-\varepsilon(H=0)=\frac{11(gH)^2}{48\pi^2}\ln\frac{gH}{\mu^2}-\mathrm{i}\frac{(gH)^2}{8\pi}\quad .\tag{8.31}$$

EXERCISE

8.1 The QCD Vacuum Energy-Density

Problem. Derive (8.31) from (8.30).

Solution. The sums in (8.30) can be directly evaluated:

$$\begin{aligned}&\sum_{n=1}^{\infty}\mathrm{e}^{-\mathrm{i}\tau 2gH\left(n-\frac{1}{2}\right)}+\sum_{n=0}^{\infty}\mathrm{e}^{-\mathrm{i}\tau 2gH\left(n+\frac{3}{2}\right)}\\&=\frac{\mathrm{e}^{\mathrm{i}\tau gH}}{1-\mathrm{e}^{-\mathrm{i}\tau 2gH}}-\mathrm{e}^{\mathrm{i}\tau gH}+\frac{\mathrm{e}^{-3\mathrm{i}\tau gH}}{1-\mathrm{e}^{-2\mathrm{i}\tau gH}}\\&=\frac{\mathrm{e}^{2\mathrm{i}\tau gH}+\mathrm{e}^{-2\mathrm{i}\tau gH}}{\mathrm{e}^{\mathrm{i}\tau gH}-\mathrm{e}^{-\mathrm{i}\tau gH}}-\mathrm{e}^{\mathrm{i}\tau gH}=\frac{\cos(2gH\tau)}{\mathrm{i}\sin(gH\tau)}-\mathrm{e}^{\mathrm{i}\tau gH}\quad .\end{aligned}\tag{1}$$

Exercise 8.1.

Also the k integrations can be directly performed:

$$\int_0^\infty dk\, e^{-i\tau k^2} = \frac{1}{2}\sqrt{\frac{\pi}{\tau}}\, e^{-i\frac{\pi}{4}}$$
$$\int_0^\infty dk\, k^2\, e^{-i\tau k^2} = i\frac{\partial}{\partial\tau}\int_0^\infty dk\, e^{-i\tau k^2} = -\frac{i}{4}\frac{\sqrt{\pi}}{\tau^{\frac{3}{2}}}\, e^{-i\frac{\pi}{4}} . \tag{2}$$

Therefore we get

$$\begin{aligned}
&\varepsilon - \varepsilon(H=0) \\
&\quad = \lim_{\eta,\delta\to 0}\left[\frac{gH}{4\pi^{\frac{3}{2}}}\frac{e^{-i\frac{\pi}{2}+i\frac{\pi}{2}\eta}}{\Gamma\left(-\frac{1}{2}+\eta\right)}\int_0^\infty d\tau\, \tau^{-2+\eta}e^{-\delta\tau}\left(\frac{\cos(2gH\tau)}{i\sin(gH\tau)} - e^{i\tau gH}\right)\right. \\
&\qquad + \frac{i}{4}\frac{e^{-i\frac{\pi}{2}+i\frac{\pi}{2}\eta}}{\pi^{\frac{3}{2}}\Gamma\left(-\frac{1}{2}+\eta\right)}\int_0^\infty d\tau\, \tau^{-3+\eta}e^{-\delta\tau} \\
&\qquad \left. + \frac{gH}{4}\frac{e^{-i\frac{\pi}{2}+i\frac{\pi}{2}\eta}}{\pi^{\frac{3}{2}}\Gamma\left(-\frac{1}{2}+\eta\right)}\int_0^\infty d\tau\, \tau^{-2+\eta}e^{i\tau gH-\delta\tau}\right] .
\end{aligned} \tag{3}$$

Now we employ the relation

$$\cos(2gH\tau) = 1 - 2\sin^2(gH\tau) \quad , \tag{4}$$

which yields

$$\begin{aligned}
&\frac{\cos(2gH\tau)}{i\sin(gH\tau)} - e^{i\tau gH} \\
&\quad = \frac{1}{i\sin(gH\tau)} - \frac{2}{i}\frac{1}{2i}\left(e^{i\tau gH} - e^{-i\tau gH}\right) - e^{i\tau gH} \\
&\quad = \frac{1}{i\sin(gH\tau)} - e^{-i\tau gH} \\
&\quad \to \frac{1}{i\sin(gH\tau - i\tilde{\eta})} - e^{-i\tau gH} \quad .
\end{aligned} \tag{5}$$

Here we have defined the singularities by inserting $i\tilde{\eta}$ ($\tilde{\eta} > 0$). The integral exists provided $\eta > 2$. We calculate it in this region and determine its value for $\eta \to 0$ by analytic continuation. This very procedure has already been used in dimensional regularization. Identity (5) ensures that the integrand vanishes for $\tau = R\ e^{i\phi}$, $-\frac{\pi}{2} < \phi < 0$, $R \to \infty$. Therefore the integral surrounding the fourth quadrant vanishes:

$$\oint d\tau \ldots = \int_0^\infty d\tau \ldots + \int_{-i\infty}^0 d\tau \ldots = 0 \quad , \tag{6}$$

Since the integrand has no singularities we substitute $\tau = -is$ and evaluate the last integral with the help of (8.29):

Exercise 8.1.

$$
\begin{aligned}
&\varepsilon-\varepsilon(H=0)\\
&\quad=\lim_{\eta,\delta\to 0}\left[\frac{gH}{4\pi^{\frac{3}{2}}}\frac{-\mathrm{i}\,\mathrm{e}^{\mathrm{i}\frac{\pi}{2}\eta}}{\Gamma\left(-\frac{1}{2}+\eta\right)}\,(-\mathrm{i})^{-1+\eta}\right.\\
&\qquad\times\int_0^\infty \mathrm{d}s\; s^{-2+\eta}\,\mathrm{e}^{-\delta r}\left(\frac{1}{\sinh(gHs)}-\mathrm{e}^{-sgH}\right)\\
&\qquad+\frac{\mathrm{e}^{\mathrm{i}\frac{\pi}{2}\eta}(-\mathrm{i})^{-2+\eta}}{4\pi^{\frac{3}{2}}\Gamma\left(-\frac{1}{2}+\eta\right)}\int_0^\infty \mathrm{d}s\; s^{-3+\eta}\,\frac{gH}{4}\,\frac{\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}+\mathrm{i}\frac{\pi}{2}\eta}}{\pi^{\frac{3}{2}}\Gamma\left(-\frac{1}{2}+\eta\right)}\\
&\qquad\left.\times\frac{(-1+\eta)}{\mathrm{e}^{\mathrm{i}\frac{\pi}{2}(-1+\eta)}}(-gH+\mathrm{i}\delta)^{1-\eta}\right]\quad. \qquad (7)
\end{aligned}
$$

Finally we substitute $s=v/gH$:

$$
\begin{aligned}
\varepsilon-\varepsilon(H=0)&=\lim_{\eta\to 0}\frac{(gH)^{2-\eta}}{4\pi^{\frac{3}{2}}}\frac{1}{\Gamma\left(-\frac{1}{2}+\eta\right)}\\
&\quad\times\left[\int_0^\infty \mathrm{d}v\; v^{-2+\eta}\left(\frac{1}{\sinh(v)}-\mathrm{e}^{-v}-\frac{1}{v}\right)-(-1+\eta)(-)^{n}\right]\quad. \qquad (8)
\end{aligned}
$$

The integral of e^{-v} leads to a gamma function:

$$
\int_0^\infty \mathrm{d}v\; v^{-2+\eta}\,\mathrm{e}^{-v}=\Gamma(-1+\eta)\quad, \qquad (9)
$$

while the integral over $1/\sinh v-1/v$ can be split into a finite part and one that diverges in the limit $\eta\to 0$:

$$
\begin{aligned}
&\int_0^\infty \mathrm{d}v\; v^{-2+\eta}\left(\frac{1}{\sinh(v)}-\frac{1}{v}\right)\\
&\quad=\int_0^\infty \mathrm{d}v\; v^{-2+\eta}\left(\frac{1}{\sinh(v)}-\frac{1}{v}+\frac{1}{6}v\right)-\frac{1}{6}\int_0^\infty \mathrm{d}v\; v^{-1+\eta}\\
&\quad=C-\frac{1}{6}\frac{1}{\eta}\quad, \qquad (10)
\end{aligned}
$$

$$
\begin{aligned}
\varepsilon-\varepsilon(H=0)&=\lim_{\eta\to 0}\frac{(gH)^{2-\eta}}{4\pi^{\frac{3}{2}}\Gamma\left(-\frac{1}{2}+\eta\right)}\\
&\quad\times\left\{C-\frac{1}{6}\frac{1}{\eta}-\Gamma\left[(-1+\eta)(1+(-)^{\eta})\right]\right\}\quad. \qquad (11)
\end{aligned}
$$

With

$$
\Gamma(-1+\eta)=\frac{1}{-1+\eta}\Gamma(\eta)=\frac{\Gamma(1+\eta)}{(-1+\eta)\eta} \qquad (12)
$$

we are now able to calculate the limiting case $\eta\to 0$:

Exercise 8.1.

$$(gH)^{2-\eta} = (gH)^2(1-\eta\ln(gH)+\cdots) \quad ,$$

$$\begin{aligned} \Gamma\left(-\frac{1}{2}+\eta\right) &= \Gamma\left(-\frac{1}{2}\right)\left[1+\eta\Psi\left(-\frac{1}{2}\right)\right]+\cdots \quad , \\ &= -2\sqrt{\pi}\left[1+\eta\Psi\left(-\frac{1}{2}\right)\right]+\cdots \quad , \end{aligned} \tag{13}$$

$$\Gamma(-1+\eta) = \frac{-1-\eta}{\eta}\,[1+\eta\Psi(1)]+\cdots \quad ,$$

$$(-)^\eta = 1+\eta\mathrm{i}\pi$$

$$\Rightarrow \varepsilon-\varepsilon(H=0) = \lim_{\eta\to 0}\left\{\frac{(gH)^2}{8\pi^2\eta}+\frac{(gH)^2}{8\pi^2}\left[\frac{11}{6}\ln(gH)+C'-\mathrm{i}\pi\right]\right\} \quad . \tag{14}$$

Here all constants have been absorbed into C', which does not depend on H. The divergent first part can be renormalized. The renormalized, i.e., the physical, energy density is then

$$\varepsilon-\varepsilon(H=0) = \frac{11}{6}\frac{(gH)^2}{8\pi^2}\ln(gH)+C'-\mathrm{i}\frac{(gH)^2}{8\pi} \quad . \tag{15}$$

In the transition to (8.28), however, we should have introduced a factor $m^{2\eta}$ in order to conserve the dimension (m is supposed to be an energy). Then we obtain

$$\frac{11(gH)^2}{48\pi^2}\left[\ln\left(\frac{gH}{m^2}\right)+\frac{6}{\pi}C'\right] = \frac{11(gH)^2}{48\pi^2}\ln\left(\frac{gH}{\mu^2}\right) \quad , \tag{16}$$

where C' has been absorbed by the definition of μ

$$\mu^2 = m^2\mathrm{e}^{-6C'/11} \quad , \tag{17}$$

$$\varepsilon-\varepsilon(H=0) = \frac{11(gH)^2}{48\pi^2}\ln\left(\frac{gH}{\mu^2}\right)-\mathrm{i}\frac{(gH)^2}{8\pi} \quad . \tag{18}$$

It should be noticed that the techniques used in this exercise to calculate (8.31) are practically the same as those introduced systematically in Chapter 4.3 in the context of dimensional regularization.

The presence of the imaginary part is again an indication of the instability. If we minimize the real part of the vacuum energy $H^2/2$ (see the last term in (8.9)), we obtain a minimum at $H \neq 0$:

$$\begin{aligned} \frac{\partial}{\partial H}\left[\frac{1}{2}H^2+\frac{11(gH)^2}{48\pi^2}\ln\left(\frac{gH}{\mu^2}\right)\right] &= H+\frac{11g^2H}{24\pi^2}\left[\ln\left(\frac{gH}{\mu^2}\right)+\frac{1}{2}\right] = 0 \\ \Rightarrow \quad gH &= \mu^2\exp\left[-\left(\frac{24\pi^2}{11g^2}+\frac{1}{2}\right)\right] \quad , \end{aligned} \tag{8.32}$$

which does not fix the value of gH since μ^2 was arbitrary up to now. As we have effectively performed a dimensional regularization, μ^2 is proportional to the renormalization scale. Indeed, the real part of $\varepsilon-\varepsilon(H=0)$ can be deduced from renormalization group properties in a very elegant fashion. The prefactor then comes from the β function:

$$\frac{1}{2g}\beta \longleftrightarrow \frac{g^2}{32\pi^2}\frac{11\times 2}{3} = \frac{11g^2}{48\pi^2} \qquad \text{for SU(2)} \quad . \tag{8.33}$$

We demonstrate this in Example 8.2 to show the strength of renormalization group arguments. However, this example will also show the weakness of these techniques, since one tacitly uses a number of assumptions without being able to check their validity. So the renormalization group treatment gives no indication of the existence of unstable states and predicts a constant color magnetic field; but the presence of unstable modes shows that this is not a physical solution.

In order to have any hope of obtaining a realistic ground state a consistent treatment of the unstable modes has to be developed. We shall not perform these calculations explicitly but only illustrate the ideas. There is a well-known method of treating unstable modes from the problem of spontaneous symmetry breaking. Here, too, the usual vacuum modes are not stable, and the field drifts into a finite vacuum expectation value. If the Higgs field is expanded instead around this vacuum expectation value, only stable modes are seen. By analogy to this, one is led to the following procedure for treating unstable modes.

1. Isolate the unstable modes and rewrite the Lagrange density to have them appear in the same way as Higgs fields.
2. Insert nonvanishing vacuum expectation values and determine the energetically optimal gauge field configuration.

The first step can indeed be performed. The second step is very difficult and only possible in the framework of certain ansätze. We therefore show just one of the results[1] (see Fig. 8.3). Other ansätze yield slightly different results but a domain structure at a length scale $\sqrt{gH_0}$ in the xy plane is always found, and compensating positive and negative fields in H in large spatial regions. One can immediately imagine why no unstable modes appear in these solutions.

The lowest Landau state ($n = 0$) extends over large spatial domains. Therefore the H fields average out and the term $\pm 2g\langle H\rangle \approx 0$ no longer leads to instabilities. The higher Landau states are localized in the xy plane up to $\sqrt{gM_0}/n$ and thus experience a more or less constant H field which as has just been calculated, leads to a lowering of the energy. Figure 8.3 shows a system of parallel tubes made of color magnetic fields that are completely analogous to vortices in type II superconductors. This parallel supports the picture of the QCD vacuum as a dual superconductor. Figure 8.3 has been calculated as a solution of the classical field equations. In quantum mechanics, fields will oscillate around this configuration. In particular, the magnetic flux tubes will no longer be strictly parallel but change their orientation over large spatial regions. This property caused the model to be baptized the "Spaghetti vacuum" and guarantees Lorentz invariance after averaging over sufficiently large spatial domains.

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[1] See J. Ambjørn and P. Olesen: Nucl. Phys. **B170**, 60 (1980).

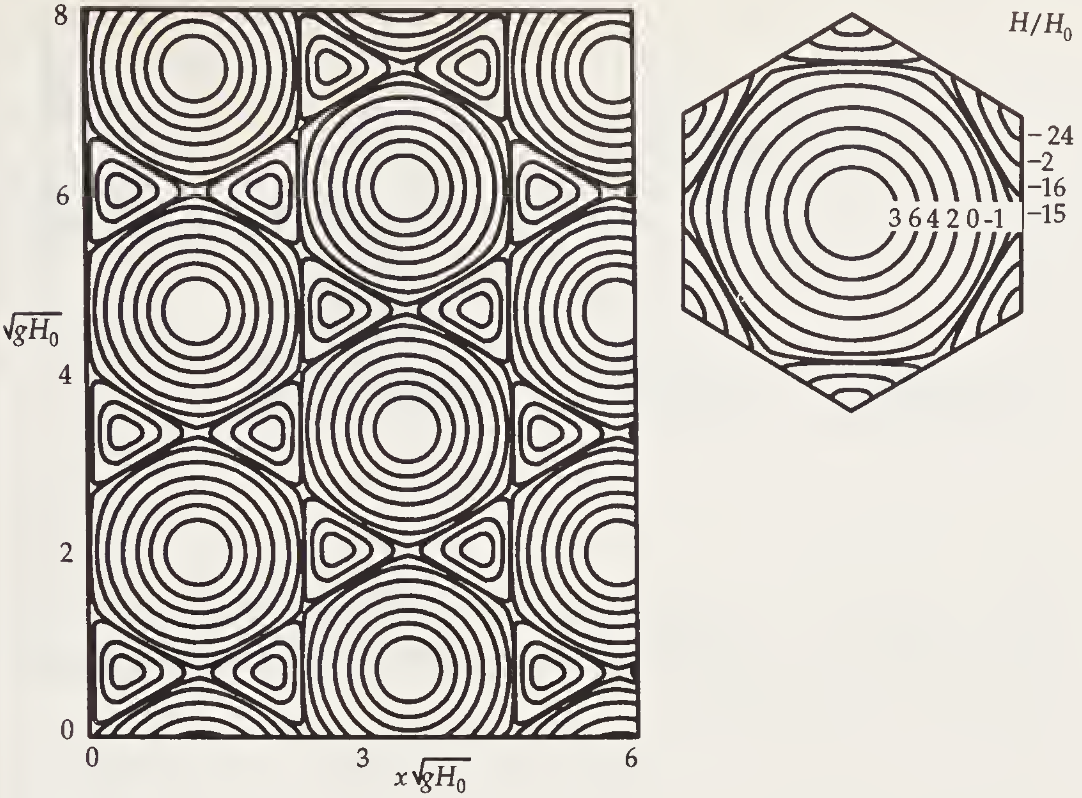

Fig. 8.3. The Ambjørn–Olesen solution for the QCD ground state. The H field is parallel to the z axis. Contour lines are at $0.8H_0$, $0.6H_0$, $0.4H_0$, $0.2H_0$, $0.0H_0$, $-0.1H_0$, $-0.15H_0$, $-0.16H_0$, $-0.2H_0$, $-0.24H_0$. (J. Ambjørn and P. Olesen: Nucl. Phys. **B170**, 60 (1980).)

The picture of a dual superconductor is further supported by lattice calculations for some simple systems. One can, for example, calculate the field distribution around a static quark–antiquark pair. If the dual-superconductor picture is correct, the color electric field cannot extend into the QCD vacuum, so that a string is created outside of which E vanishes. Figure 8.4 shows this schematically as well as the results of a Monte Carlo calculation.

In conclusion there are detailed and quasi-phenomenological models for the QCD ground state. The “Spaghetti vacuum” is just one of them. All have in common that they lead to highly complicated nonperturbative field configurations and that these have a constant negative energy density compared to the perturbative vacuum. This energy density cannot simply be identified with the bag constant. The fine structure of the QCD ground state should exhibit a length scale of $1/\lambda_{\mathrm{QCD}} \approx 1$ fm. Thus the average energy density is not relevant for a hadron; and the precise microscopic structure must be known.

EXAMPLE

8.2 The QCD Ground State and the Renormalization Group

In this example we wish to show how (8.31) (or more accurately the real part of (8.31)) can be derived from renormalization arguments in a very elegant manner. We want to investigate the effective Lagrangian of an SU(2) gauge theory for a constant color magnetic field. In lowest order this is simply $-1/2H^2$. Consequently the next order can be written as

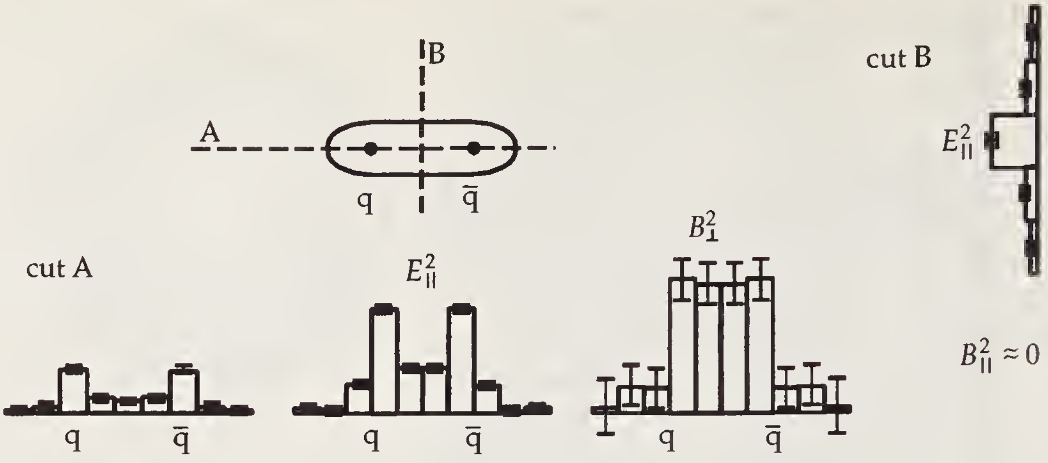

Fig. 8.4. The result of a Monte Carlo calculation for color electric and color magnetic fields around a static quark–antiquark pair || denotes the direction parallel to the A axis (From J.W. Flower and S.W. Otto: Phys. Lett. **B160**, 128 (1985).) The color magnetic fields are enlarged by a factor of 10

$$L_{\mathrm{eff}} = F(H, \mu, g) \quad , \tag{1}$$

where μ denotes the renormalization in such a way that L_{eff} becomes the free Lagrangian for $H = \mu^2$. Since the effective Lagrangian must not depend on μ, the renormalization group equation

$$\left[\mu^2 \frac{\partial}{\partial \mu^2} + \beta(g)\frac{\partial}{\partial g} + 2\gamma(g)H^2 \frac{\partial}{\partial H^2}\right] F(H, \mu, \gamma) = 0 \tag{2}$$

must be fulfilled. The derivative of (2) with respect to H^2 is

$$\left[\mu^2 \frac{\partial}{\partial \mu^2} + \beta(g)\frac{\partial}{\partial g} + 2\gamma(g) + 2\gamma(g)H^2 \frac{\partial}{\partial H^2}\right] \frac{\partial}{\partial H^2} F(H, \mu, g) = 0 \quad . \tag{3}$$

Since $\partial F / \partial H^2$ is a dimensionless quantity, it can only depend on H / μ^2. Therefore we define

$$t = \ln \frac{H}{\mu^2} \quad , \quad \frac{\partial}{\partial H^2} F(H, \mu, g) = G(t, g) \quad , \tag{4}$$

$$\left[-(1-\gamma)\frac{\partial}{\partial t} + \beta \frac{\partial}{\partial g} + 2\gamma\right] G(t, g) = 0 \quad . \tag{5}$$

This is the typical form of a renormalization group equation. Taking into account the boundary condition

$$G\big|_{H=\mu^2} = G\big|_{t=0} = -\frac{1}{2} \tag{6}$$

yields

$$G(g, t) = \exp\left(2 \int_0^t \bar{\gamma}(g, x)\mathrm{d}x\right) G'(g, t) \quad , \tag{7}$$

$$G'(g, 0) = -\frac{1}{2} \quad , \tag{8}$$

Example 8.2.

and

$$\left[-(1-\gamma)\frac{\partial}{\partial t}+\beta\frac{\partial}{\partial g}\right] G'(g,t)=0 \quad . \tag{9}$$

We assign the new name g_0 to the constant g, because we want to introduce a function $g(t,g_0)$ in the following:

$$G(g_0,t)=\exp\left(2\int\limits_0^t \bar{\gamma}(g_0)\mathrm{d}x\right) G'(g_0,t) \quad , \tag{10}$$

$$G'(G_0,0)=-\frac{1}{2} \quad , \tag{11}$$

and

$$\left[-(1-\gamma)\frac{\partial}{\partial t}+\beta\frac{\partial}{\partial g_0}\right] G'(g_0,t)=0 \quad . \tag{12}$$

Clearly (12) is solved by every function $G'\big(g(g_0,t)\big)$ if

$$\frac{\partial}{\partial t}g(g_0,t)=\frac{\beta}{1-\gamma}\frac{\partial}{\partial g_0}g(g_0,t) \quad . \tag{13}$$

It is always possible to define $G'\big(g(g_0,t)\big)$ and $g(g_0,t)$ in such a way that $\partial g(g_0,t)/\partial g_0=1$ holds, such that (13) simplifies to

$$\frac{\partial}{\partial t}g(g_0,t)=\frac{\beta}{1-\gamma} \quad . \tag{14}$$

Now (7) is evaluated by perturbative methods. In the case of small t we have $G'(g)\approx -1/2$, and only the result for the *anomalous dimension* γ in the limit of $t\to 0$ is needed. Since we have not discussed the renormalization of gauge theories in this volume, we can only cite the result. It turns out for a pure SU(2) gauge theory that $\overline{\gamma}$ and β are in lowest order proportional to each other:

$$\begin{aligned}\overline{\gamma}=\frac{\beta}{2g_0}&=\frac{\mu^2}{g_0}\frac{\partial g}{\partial\mu^2}\bigg|_{\mu^2=H}\\ &=-\frac{1}{g_0}\frac{\partial g}{\partial t}\bigg|_{t\to 0}=-\frac{\partial \ln g}{\partial t}\bigg|_{t\to 0} \quad .\end{aligned} \tag{15}$$

This relation allows us to evaluate (7):

$$\begin{aligned}G&\approx\frac{1}{2}\mathrm{e}^{-2\{\ln[g(g_0,t)]-\ln(g_0)\}}=-\frac{1}{2}\left(\frac{g(g_0,t)}{g_0}\right)^{-2}\\ &\approx-\frac{1}{2}\left(1+\frac{\beta}{g_0}t\right)=-\frac{1}{2}-\frac{1}{2}\frac{g_0^2}{(4\pi)^2}\frac{11\times 2}{3}t \quad ,\end{aligned} \tag{16}$$

$$G\approx-\frac{1}{2}-\frac{11g_0^2}{48\pi^2}\ln\left(\frac{H}{\mu^2}\right)+O(g^4) \quad . \tag{17}$$

Therefore F assumes the form

Example 8.2.

$$F = -\frac{11g^2H^2}{48\pi^2}\left[\ln\left(\frac{H}{\mu^2}\right) - \frac{1}{2}\right] = -\frac{11g^2H^2}{48\pi^2}\ln\left(\frac{H}{H_0}\right) \quad . \tag{18}$$

Savvidy first found this derivation.[2] Later Nielsen and Olesen discovered in a lengthier calculation the non-stable states, which are overlooked by the renormalization group method. This indicates that one has to be very cautious when employing such abstract principles.

Without color electric fields the energy density is

$$\varepsilon - \varepsilon(H=0) = -F = \frac{11g^2H^2}{48\pi^2}\ln\frac{H}{H_0} \quad .$$

EXAMPLE

8.3 The QGP as a Free Gas

In order to allow for simple calculations the QGP is usually described as a free gas consisting of quarks and gluons. As we already discussed this is theoretically not well founded at $T \approx T_c$. However, those calculations frequently yield results which are qualitatively correct. Thus we simply add the gas pressures of a free gluon gas

$$p_G = 16 \times \frac{\pi^2}{90}T^4 \tag{1}$$

and of a free quark gas

$$p_Q = 12 \times \frac{T^4}{12}\left[\frac{7\pi^2}{30} + \left(\frac{\mu}{T}\right)^2 + \frac{1}{2\pi^2}\left(\frac{\mu}{T}\right)^4\right] \tag{2}$$

and identify the result with the bag pressure B. In the case $p_G + p_Q > B$ the QGP region is supposed to expand and one should be able to derive the critical temperature from

$$\begin{aligned} B &= p_G + p_Q \\ &= T_{cr}^4\left[\frac{37\pi^2}{90} + \left(\frac{\mu}{T_{cr}}\right)^2 + \frac{1}{2\pi^2}\left(\frac{\mu}{T_{cr}}\right)^4\right] \end{aligned} \tag{3}$$

for every value of μ. Assuming $B = (145\ \mathrm{MeV})^4$ yields the result shown in Fig. 8.9. This now has to be compared with Fig. 8.5. Apparenty there is a rough qualitative but no quantitative agreement for the phase boundaries. One has

$$\frac{T(\mu=0)}{T\left(\mu=\frac{4}{3}T_{\max}\right)} = \begin{cases} \frac{155}{113} = 1.4 & \text{for lattice calculations} \\ \frac{102\ \mathrm{MeV}}{91\ \mathrm{MeV}} = 1.1 & \text{for the free gas.} \end{cases} \tag{4}$$

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[2] See S.G. Matinyan and G.K Savvidy: Yad. Fiz. **25**, 218 (1977).

Since the position of the phase boundaries is quite insensitive to theoretical subtleties, one can expect that the free gas treatment of QCD leads for more sensitive quantities to results which are wrong by much more than 30 per cent. Examples of such quantities are the total number of kaons or lambda-particles created.

Example 8.3.

8.2 The Quark–Gluon Plasma

In Sect. 7.2 we discussed that lattice calculations show a phase transition at a critical temperature $T_c \sim 100 - 200\,\mathrm{MeV}$ (Figs. 7.3 and 7.19). Such a phase transition is typical for non–abelian gauge theories. It has been studied extensively for SU(2) and SU(3) but should exist for all SU(N) groups. If it could be experimentally investigated in detail such studies would definitely improve our understanding of some of the basic properties of QCD. The hope is that in high-energy collisions of heavy ions this new high-temperature phase can indeed be produced for sufficiently long times and in a sufficiently large volume to allow experimental studies. While it is generally agreed that at high enough energies the new phase will be reached in the center of the collision system, the interpretation of possible experimental signals is still very much debated.

The interest in the quark–gluon plasma (QGP) phase transition is further increased by the fact that it is assumed to have played a crucial role in the early universe. As the universe cooled it was the last phase transition to occur and might therefore have left recognizable traces in the present day structure of the universe.

Another interesting point is that the QCD vacuum is a highly non-trivial state, as is the vacuum of the standard model in general. In fact, the existence of a complex vacuum state postulated by modern field theory with an energy density tens of orders of magnitude larger than any observable energy densities is probably one of the most interesting and least-tested features of modern particle theory. To understand some of the problems let us return to the concept of parton–hadron duality discussed in Example 5.10. We have argued that the description of hadronic reactions is in principle possible on the quark level as well as on the hadron level. These two descriptions are just based on a different set of basis states. Some processes like deep-inelastic scattering can be described very easily on the quark level but are extremely hard to treat on the hadron level and such processes therefore allow direct tests of QCD. On the other hand, properties which are easily described in terms of hadrons and their interactions are usually not a good test of QCD. Let us therefore start our detailed discussion by reviewing what hadron models can tell us about the high-temperature phase transition.

As early as in 1965 Hagedorn[3] observed that the experimentally known mass density of hadron states grows exponentially

$$\frac{d\rho}{dm} \sim m^a \mathrm{e}^{m/m_0} \quad , \quad m_0 \approx 200\,\mathrm{MeV} \tag{8.34}$$

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[3] See Hagedorn: Suppl. Nuov. Cim. **3**, 147 (1965).

and that this fact implies a limiting temperature. With $p^2\,\mathrm{d}p = p\,E\,\mathrm{d}E$ the number of states with energy between E and $E + \mathrm{d}E$ can be written as

$$\mathrm{d}n(E) \sim \mathrm{d}E \int\limits_0^E \mathrm{d}m\, \frac{\mathrm{d}\rho}{\mathrm{d}m}\, \mathrm{e}^{-E/kT}\, p\, E \quad , \quad p = \sqrt{E^2 - m^2} \tag{8.35}$$

implying for (8.34)

$$\begin{aligned} \mathrm{d}n(E) &\sim \mathrm{d}E \int\limits_0^E m^a\, \mathrm{e}^{m/m_0}\, \mathrm{e}^{-E/kT} \sqrt{E^2 - m^2}\, E\, \mathrm{d}m \\ &= \mathrm{d}E\, E^{a+3} \int\limits_0^1 z^a \sqrt{1 - z^2}\, \mathrm{e}^{Ez/m_0}\, \mathrm{d}z\, \mathrm{e}^{-E/kT} \quad , \end{aligned} \tag{8.36}$$

with $m = Ez$. Substituting $z = \cos(\varphi)$ we get

$$\mathrm{d}E\, E^{a+3} \mathrm{e}^{-E/kT} \int\limits_0^{\pi/2} \cos^a(\varphi) \sin^2(\varphi)\, \mathrm{e}^{E \cos(\varphi)/m_0}\, \mathrm{d}\varphi \quad . \tag{8.37}$$

Now we assume $E/m_0 \gg 1$. (This is the relevant limit for our discussion.) We can then approximate

$$\begin{aligned} \int\limits_0^{\pi/2} \cos^a(\varphi) \sin^2(\varphi)\, \mathrm{e}^{E \cos(\varphi)/m_0}\, \mathrm{d}\varphi &\approx \int\limits_0^{\pi} \cos^a(\varphi) \sin^2(\varphi)\, \mathrm{e}^{E \cos(\varphi)/m_0}\, \mathrm{d}\varphi \\ &\approx \int\limits_0^{\pi} \sin^2(\varphi)\, \mathrm{e}^{E \cos(\varphi)/m_0}\, \mathrm{d}\varphi = \sqrt{\pi}\, \frac{2m_0}{E}\, \Gamma\left(\frac{3}{2}\right) I_1\left(\frac{E}{m_0}\right) \\ &\approx \sqrt{\pi}\, \frac{2m_0}{E}\, \frac{\sqrt{\pi}}{2}\, \frac{\mathrm{e}^{E/m_0}}{\sqrt{2\pi\, E/m_0}} = \sqrt{\frac{\pi m_0^3}{2E^3}}\, \mathrm{e}^{E/m_0} \quad . \end{aligned} \tag{8.38}$$

Obviously the total energy density $\int_0^\infty E\, \mathrm{d}n(E)$ diverges for $kT > kT_0 = m_0 \approx$ 200 MeV, because the exponential factor $\exp(E/m_0 - E/kT)$ grows with energy, implying that either no higher temeratures are possible or that some new physics must become relevant. As in this treatment the finite size of the hadrons was neglected one expects the latter to be the case. Thus we have linked the existence of a phase transition to the fact that the density of states grows exponentially with energy. This behavior is, however, nearly universal to all composite models of the nucleon. It is found e.g. for the MIT bag model and is simply due to the fact that more and more angular momentum states become available with increasing energy. The phase transition temperature is simply determined by the level spacing between hadrons, and one has to conclude that the observation of a phase transition alone is not a very sensitive test of QCD.

In terms of quarks and gluons the phase transition can be linked to a reduction of the effective running coupling constant with increasing temperature. Finite temperature implies that the state in which a process has to be calculated is not the field theoretical vacuum state, but a state containing a thermal distribution of particles. This adds the graphs shown in Fig. 8.5 to the vacuum polarization tensor. It turns out that the resulting effective coupling constant α_{eff} becomes small enough for $T \sim \Lambda_{\text{QCD}}$ to suggest the insignificance of non-perturbative effects.

Another interesting fact is that theoretical studies show that one is really dealing with two phase transitions. The deconfinement phase transition is characterized by the fact that energy density ε and pressure P approach (although not very rapidly) the corresponding Stefan–Boltzmann limits for a free gas of quarks and gluons above the phase transition (see Figs. 7.18 and 7.19). Simultaneously one observes the chiral phase transition which describes the fact that chiral symmetry, which is violated at low temperatures, is conserved for $T > T_c$. According to our discussion in Sect. 7.3 this implies $\langle \bar{q}q \rangle \to 0$ which is in fact observed in lattice gauge calculations. The physical reason for the coupling of these two phase transitions is still disputed. The link between the two effects is the observation that the condensate can be expanded in terms of the density of 'zero modes' (i.e. modes with zero energy) which in turn depends on the long range properties of the interaction and thus on the presence or absence of confinement:

$$\langle \bar{q}q \rangle = -\frac{\pi}{V}\,\nu(0) \quad . \tag{8.39}$$

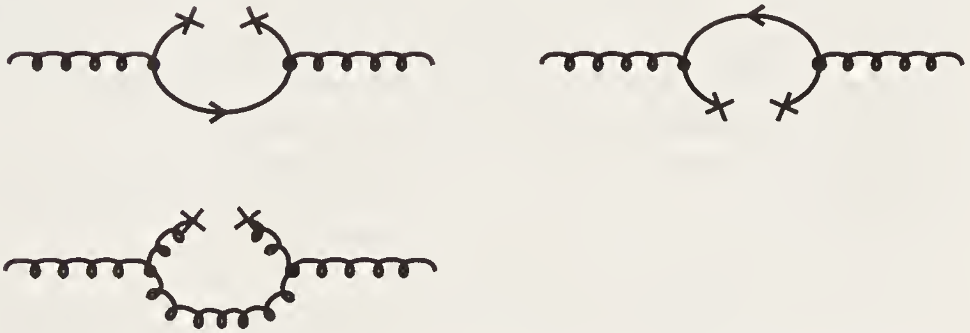

Fig. 8.5. Additional contributions to vacuum polarization in a thermal distribution of particles. Crosses mark a coupling to a thermal field

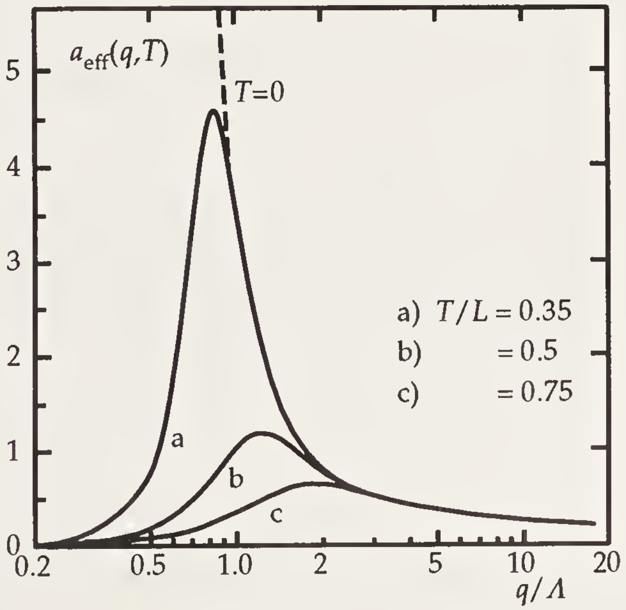

Fig. 8.6. Running coupling constant in a pure $SU(2)$ gauge theory at finite temperature (See B. Müller: *The Physics of the Quark–Gluon Plasma*, (Springer, Berlin, Heidelberg 1985), p. 81)

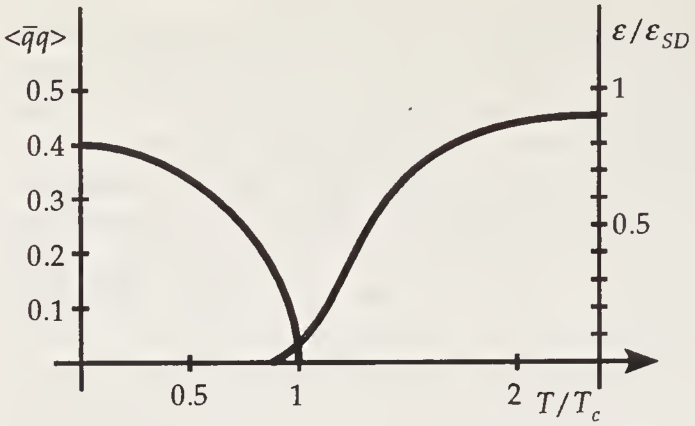

Fig. 8.7. Chiral phase transition $\langle \bar{q}q \rangle \to 0$ and confinement–deconfinement phase transition occur at the same temperature in lattice gauge simulations

Here $\nu(\Lambda)$ is the density of states which have the eigenvalue Λ with respect to the Dirac operator $-\mathrm{i}\gamma_\mu(\partial^\mu - \mathrm{i}gA^\mu)$. $\nu(0)$ is the density of states at $\Lambda = 0$. V is a normalization factor, namely the total space-time volume considered.

The properties of the QGP phase transition are further modified by the finite baryon density in a heavy ion collision, characterized by the chemical potential μ. Thus the phase transition is characterized by a line in the μ–T diagram, or even a surface in the μ–T–$\bar{s}s$ diagram if the further modifications due to a finite strangeness–antistrangeness content are added, see Fig. 8.8.

Let us briefly discuss how the μ and T dependence of the phase transition can be understood.

The energy density of free quarks and gluons is

$$\begin{aligned}\varepsilon = {} & 8 \times 2 \times \int_0^\infty \frac{\mathrm{d}^3 p}{(2\pi)^3}\, p \left(\mathrm{e}^{p/kT} - 1\right)^{-1} \\ & + 3 \times 2 \times N_f \times \int_0^\infty \frac{\mathrm{d}^3 p}{(2\pi)^3}\, p \left[\frac{1}{\mathrm{e}^{(p-\mu)/kT} + 1} + \frac{1}{\mathrm{e}^{(p+\mu)/kT} + 1}\right] \quad , \qquad (8.40)\\ & p = E \quad , \quad m_q = 0 \quad .\end{aligned}$$

The first term is the gluon contribution with 8 color states and 2 physical spin states. The second term is the quark and antiquark contribution for 3 color states, 2 spin states and N_f massless flavors.

Making use of the integral formula

$$\begin{aligned}\int_0^\infty \frac{x^{\nu-1}\mathrm{d}x}{\mathrm{e}^{\mu x} - 1} &= \frac{\Gamma(\nu)}{\mu^\nu}\, \xi(\nu) \quad , \quad (\mathrm{Re}\,\mu > 0, \mathrm{Re}\,\nu > 0) \\ \int_0^\infty \frac{x^{\nu-1}\mathrm{d}x}{\mathrm{e}^{\mu x} + 1} &= \frac{1 - 2^{1-\nu}}{\mu^\nu} \Gamma(\nu)\, \xi(\nu) \quad ,\end{aligned} \qquad (8.41)$$

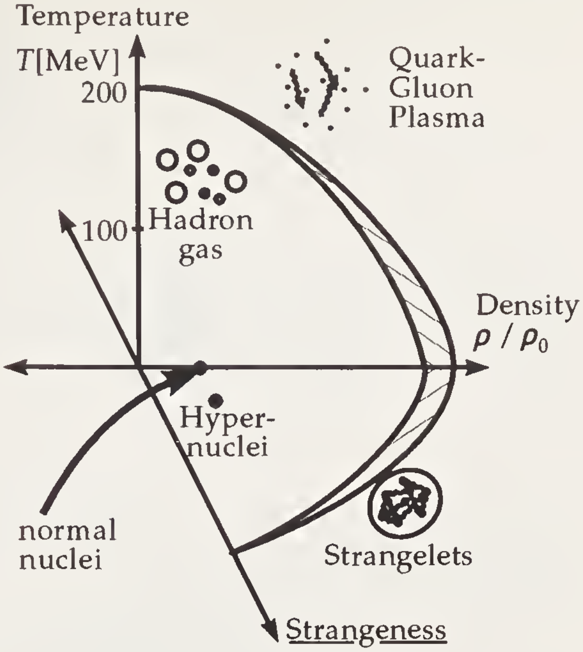

Fig. 8.8. The surface characterizing the transition between normal hadronic matter and the quark–gluon plasma as a function of μ, T, and $\bar{s}s$ content

the integrals can be performed analytically

$$\begin{aligned}
\varepsilon = {} & \frac{16 \times 4\pi}{(2\pi)^3} (kT)^4\, \Gamma(4)\, \xi(4) + \frac{6N_f}{(2\pi)^3} \left[\int\limits_{-\mu}^{\infty} \mathrm{d}p\; 4\pi\, (p+\mu)^3 \frac{1}{\mathrm{e}^{p/kT}+1} \right. \\
& \left. + \int\limits_{\mu}^{\infty} \mathrm{d}p\; 4\pi\, (p-\mu)^3 \frac{1}{\mathrm{e}^{p/kT}+1} \right] \\
= {} & \frac{8}{\pi^2} (kT)^4\, 6\, \frac{\pi^4}{90} + \frac{3N_f}{\pi^2} \left\{ \int\limits_0^{\infty} \mathrm{d}p\, \left[(p+\mu)^3 + (p-\mu)^3 \right] \frac{1}{\mathrm{e}^{p/kT}+1} \right. \\
& \left. + \int\limits_{-\mu}^{0} \mathrm{d}p\, (p+\mu)^3 \frac{1}{\mathrm{e}^{p/kT}+1} - \int\limits_{0}^{\mu} \mathrm{d}p\, (p-\mu)^3 \frac{1}{\mathrm{e}^{p/kT}+1} \right\} \\
= {} & \frac{8\pi^2}{15} (kT)^4 + \frac{3N_f}{\pi^2} \left[2(1-2^{-3})\, (kT)^4\, 6\, \frac{\pi^4}{90} + 2 \cdot 3\, (1-2^{-1})\, (kT)^2\, \mu^2 \cdot 1 \cdot \frac{\pi^2}{6} \right. \\
& \left. - \int\limits_0^{\mu} \mathrm{d}p\, (p-\mu)^3 \left(\frac{1}{\mathrm{e}^{-p/kT}+1} + \frac{1}{\mathrm{e}^{p/kT}+1} \right) \right] \\
= {} & \frac{8\pi^2}{15} (kT)^4 + \frac{3N_f}{\pi^2} \left[\frac{7}{8}\, \frac{2\pi^4}{15} (kT)^4 + \frac{\pi^2}{2}\, \mu^2\, (kT)^2 \right. \\
& \left. - \int\limits_0^{\mu} \mathrm{d}p\, (p-\mu)^3\, \frac{2 + \mathrm{e}^{p/kT} + \mathrm{e}^{-p/kT}}{2 + \mathrm{e}^{p/kT} + \mathrm{e}^{-p/kT}} \right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{8\pi^2}{15}(kT)^4 + 6N_f\left[\frac{7\pi^2}{120}(kT)^4 + \frac{\mu^2}{4}(kT)^2\right. \\
&\quad \left. - \frac{1}{8\pi^2}\mu^4 + \frac{3}{2\pi^2}\mu\frac{\mu^3}{3} - \frac{3}{2\pi^2}\mu^2\frac{\mu^2}{2} + \frac{1}{2\pi^2}\mu^3\cdot\mu\right] \\
&= \frac{8\pi^2}{15}(kT)^4 + 6N_f\left[\frac{7\pi^2}{120}(kT)^4 + \frac{\mu^2}{4}(kT)^2 + \frac{1}{8\pi^2}\mu^4\right] \quad . \qquad (8.42)
\end{aligned}
$$

The pressure $P = \varepsilon/3$ can now simply be equated to the bag constant to obtain an estimate for the phase-transition line

$$
B = \frac{8\pi^2}{45}(kT)^4 + 2N_f\left[\frac{7\pi^2}{120}(kT)^4 + \frac{\mu^2}{4}(kT)^2 + \frac{1}{8\pi^2}\mu^4\right] \quad . \qquad (8.43)
$$

In deriving this estimate all interaction effects were neglected such that the numerical values obtained from (8.43) cannot be trusted. However, the general form shown in Fig. 8.8 should be correct.

Having specified the transition region to the QGP let us now turn to the most important question, namely whether the new phase can be reached in a heavy-ion collision. To this end heavy-ion collisions were studied in great detail within many different models.[4] After much controversy it is now established that very high densities and enormous rates of particle production are reached. The CERN experiments with 200 GeV/A beams have produced a large amount of data testing the existing theoretical models and thus allowing a more trustworthy extrapolation to higher energies. The most important result of these and other studies is that the originally often advocated Bjorken-picture of the two nuclei passing through one another with only very little interaction is wrong. Instead enormous numbers of particles are produced, absorbing a large fraction of the original energy. Schematically the correct picture is given in Fig. 8.9.

Because so many particles take part in the interactions, it turns out that a hydroydnamic description is applicable except at the highest collision energies. This description was primarily developed by W. Greiner and collaborators in Frankfurt. In its language the crucial mechanism leading to the large energy deposition is the creation of nuclear shock waves, which had been predicted in a seminal paper by W. Scheid, H. Müller and W. Greiner.[5] Other effects predicted by the same model have also been observed, namely the fact that some of the projectiles are deflected for suitable impact parameters (sideways flow) and that greater than average number of particles are emitted perpendicular to the reaction plane (squeeze-out).

Based on all these investigations, one can be optimistic that a phase transition can indeed be reached, e.g. in a heavy-ion collision with 200 GeV/A at RHIC. The still unsolved problem, however, is whether there exists any signal that survives the subsequent hadronization. Until now no undisputed signal has been found. The hope rests therefore on very exotic objects which one cannot imagine to be produced in a normal nuclear surrounding. Probably the most interesting of these proposed signals are multi-baryon states with a very large strangeness content. B. Witten[6]

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[4] For a review see e.g. W. Greiner, H. Stöcker and A. Gallmann (eds.): *Hot and dense nuclear matter*, NATO-ASI SERIES B–Physics, Plenum Press (1994).

[5] See W. Scheid, H. Müller and W. Greiner: Phys. Lett. **13**, 741 (1974).

[6] See B. Witten: Phys. Rev. **D30**, 272 (1984).

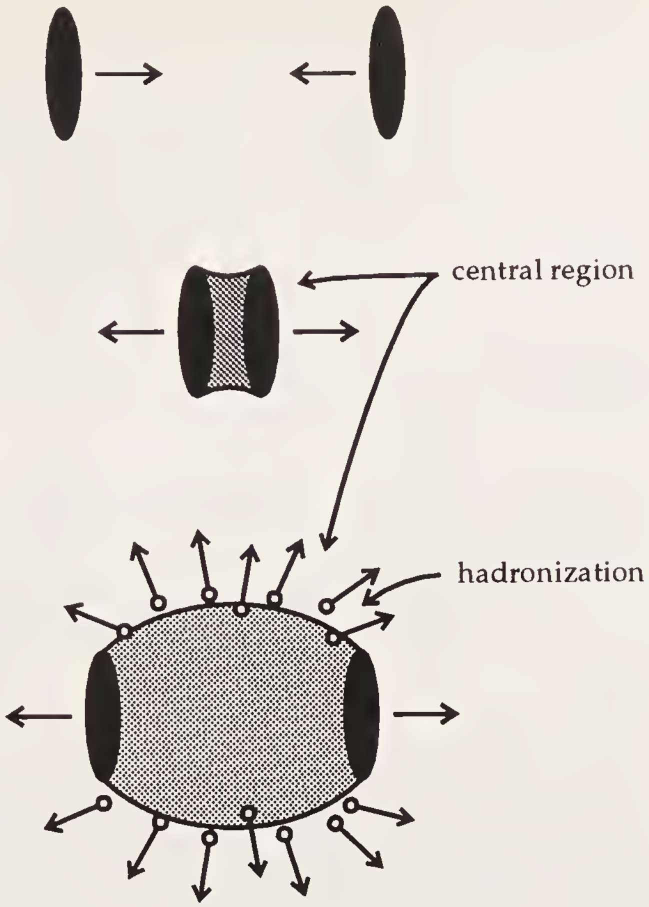

Fig. 8.9. Schematic description of an ultra-relativistic heavy-ion collision

and E. Farhi and R.L. Jaffe[7] had proposed that droplets of QGP could be stable or at least metastable if they contain a sufficient number of s-quarks. The question is thus whether the creation of such droplets, which are termed "strangelets", could be aided in a QGP. C. Greiner, P. Koch and H. Stöcker[8] have shown that this is indeed possible. Under the assumption that a mixed phase of QGP and ordinary nuclear matter is created in a sufficiently large region and lives long enough (10^{-22} s), an unmixing of s and $\bar{s}$ should happen where the s-quarks gather in the QGP while the $\bar{s}$ dominate the hadronic phase as kaons. If this mechanism is effective enough to collect sufficiently many s-quarks in the QGP, it would not decay and could then remain as a (experimentally easily detected) strangelet. Let us end this chapter by discussing some other rigorous results of QCD besides those of lattice-gauge calculations.

As sketched in Fig. 8.5, at high temperature the coupling to the thermal bath of particles reduces the effective coupling constant. For sufficiently large temperature the resulting theory should therefore again be tractable by perturbation theory. The most advanced formalism along these lines was developed by Braaten and Pisarski. It is based on earlier work by Kapusta. Let us just sketch some of the results obtained. By calculating the lowest order contributions to the gluon propagator in a thermal background Kapusta obtained an effective propagator which can be written in the form

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[7] See E. Farhi and R.L. Jaffe: Phys. Rev. **D30**, 2319 (1984).

[8] See C. Greiner, P. Koch and H. Stöcker: Phys. Rev. Lett. **58**, 1825 (1987).

$$D_L(\omega, k) = \frac{1}{\varepsilon_L(\omega, k)k^2} \quad , \tag{8.44}$$

$$D_T(\omega, k) = \frac{1}{\varepsilon_T(\omega, k)\omega^2 - k^2} \quad , \tag{8.45}$$

$$\varepsilon_L = 1 + \frac{g^2T^2}{k^2}\left[1 - \frac{\omega}{2k}\ln\left|\frac{\omega + k}{\omega - k}\right|\right] \quad , \tag{8.46}$$

$$\varepsilon_T = 1 - \frac{g^2T^2}{2k^2}\left[1 - \left(1 - \frac{k^2}{\omega^2}\right)\frac{\omega}{2k}\ln\left|\frac{\omega + k}{\omega - k}\right|\right] \quad , \tag{8.47}$$

with transverse and longitudinal color-dielectric functions. For a static source ($\omega \to 0$) these equations imply

$$\begin{aligned} \varepsilon_L &\to 1 + \frac{g^2T^2}{k^2}\left[1 - \frac{\omega}{2k}\frac{2\omega}{k}\right] \to 1 + \frac{g^2T^2}{k^2} \quad , \\ D_L &\to \frac{-1}{(-k^2) -- g^2T^2} \quad , \end{aligned} \tag{8.48}$$

$$\begin{aligned} \varepsilon_T &\to 1 - \frac{g^2T^2}{2k^2}\left[1 - \left(1 - \frac{k^2}{\omega^2}\right)\frac{\omega^2}{k^2}\right] = 1 - \frac{g^2T^2}{2k^2}\,2 \quad , \\ D_T &\to \frac{1}{-k^2} \quad . \end{aligned} \tag{8.49}$$

Thus only the longitudinal gluon field is screened, with a screening mass gT, and corresponding Debye length $\lambda_D = 1/gT$. The fact that the transverse propagator is not screened is one of the problems of this approach. This result of lowest order calculations implies the existence of relevant higher order terms, e.g. $m_T \sim g^2T$. Presently, great effort is being invested in calculating these terms reliably.

An interesting point is that the poles of D_L and D_T are the effective particles of the thermal state, the 'plasmons'. Some of their properties can easily be read off from (8.44-47). For the longitudinal case $\varepsilon_L = 0$ implies

$$0 = k^2 + g^2T^2\left[1 - \frac{\omega}{2k}\ln\left|\frac{\omega + k}{\omega - k}\right|\right] \quad . \tag{8.50}$$

For the transverse case $\varepsilon_T = k^2/\omega^2$ leads to

$$\frac{k^2}{\omega^2} = 1 - \frac{g^2T^2}{2k^2}\left[1 - \left(1 - \frac{k^2}{\omega^2}\right)\frac{\omega}{2k}\ln\left|\frac{\omega + k}{\omega - k}\right|\right] \quad . \tag{8.51}$$

To test whether this high energy phase is really deconfined one has to check the low k limit, i.e., the properties of the plasmon for large distances. For the longitudinal case one finds

$$\begin{aligned} 0 &= k^2 + g^2T^2\left[1 - \frac{\omega}{2k}\left(\frac{2k}{\omega} + \frac{2k^3}{3\omega^3}\right)\right] \\ &= k^2 + g^2T^2\left(-\frac{k^2}{3\omega^2}\right) \\ \Rightarrow \omega^2 &\approx \omega^2 - k^2 = \frac{g^2T^2}{3} \end{aligned} \tag{8.52}$$

$$m^L_{\mathrm{Plasmon}} = \frac{gT}{\sqrt{3}} \tag{8.53}$$

and for the transverse case

$$\begin{aligned}\frac{k^2}{\omega^2} &= 1 - \frac{g^2T^2}{2k^2}\left[1 - \left(1 - \frac{k^2}{\omega^2}\right)\left(1 + \frac{k^2}{3\omega^2}\right)\right] \\ &= 1 - \frac{g^2T^2}{2k^2}\left(\frac{2k^2}{3\omega^2}\right) = 1 - \frac{g^2T^2}{3\omega^2}\end{aligned} \tag{8.54}$$

$$\begin{aligned}\omega^2 - k^2 &= \frac{g^2T^2}{3} \\ m^T_{\mathrm{Plasmon}} &= \frac{gT}{\sqrt{3}} \quad .\end{aligned} \tag{8.55}$$

Thus both the longitudinal and the transverse gluons have, for large distances, the same effective mass $gt/\sqrt{3}$, implying that the color potentials are screened and confinement is no longer effective. Let us note that for realistically attainable temperatures $T \sim 250$ MeV, $g \sim 2$ one gets

$$m_{\mathrm{Plasmon}} \sim 300 \text{ MeV} \quad . \tag{8.56}$$

Obviously the simple idea of a free gas of massless gluons is completely ruled out by high temperature QCD, too.

Braaten and Pisarski pushed this kind of calculation much further. They distinguish three scales, namely $g^2T \ll gT \ll T$. Momenta of order gT can be treated by usual perturbative QCD, those of order T require a careful resummation of contributions of arbitrarily high order in perturbation theory. (This is achieved by deriving effective vertices and propagators.) Momenta of order g^2T cannot be treated. Within this approach a large number of properties were calculated, but the problem with all of this is that it only applies for $g \ll 1$, implying a temperature which cannot be reached in any realistic experiment. As stated above g is even larger than 1. The same criticism applies to approaches in which the QCD-field theory is treated as classical field theory. The classical Yang-Mills equations show chaotic behavior and B. Müller and collaborators showed that the corresponding leading Lyapunov exponent can actually be related to the gluon damping rate as obtained by Braaten and Pisarski, implying that the regime of asymptotically high temperatures seems to be really well understood. If one were to simply extrapolate the results of these two descriptions to realistic temperatures, they would imply very short thermalization times and consequently very good chances for quark-gluon plasma formation in heavy-ion collisions, but, as discussed above, such extrapolations are very problematic.

Here we have only been able to sketch some relevant ideas and developments in this very active field. At present it seems very probable that future heavy-ion experiments at RHIC and LHC will study hadronic matter under conditions where it will show exotic properties. Is is, however, still not clear whether it will be possible to link these experimental observations to basic properties of QCD in an undisputable manner.

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Quantum Chromodynamics is a thorough introduction for students of theoretical physics and scientists needing a reference and exercise book in this field. The book presents the necessary mathematical tools along with many examples and worked problems. In the introductory chapters the reader becomes familiar with the hadron spectrum, while the SU(N) symmetry groups and the relativistic field theory are briefly recapitulated; then a discussion of scalar quantum electrodynamics and scattering reactions follows before gauge quark-quark interactions, perturbational QCD, renormalization groups, and tests of perturbational QCD are all treated in detail. Chapters on nonperturbational QCD and quasi-phenomenological applications conclude the text.

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